МАТЕМАТИКА

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О *SE*-ЯДРЕ ПОДГРУПП КОНЕЧНОЙ ГРУППЫ

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ON THE *SE*-CORE OF SUBGROUPS OF A FINITE GROUP

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Даются условия, при которых конечная группа принадлежит композиционной формации, содержащей все конечные сверхразрешимые группы, а также обобщаются основные результаты работ [20]–[32], [34]–[37].

Ключевые слова: *конечная группа, насыщенная формация, композиционная формация, локальная формация Бэра, p-нильпотентная группа, циклический главный фактор, сверхразрешимая группа, квазисверхразрешимая группа, S-квазинормальная подгруппа, S-добавляемая подгруппа, S-квазинормально вложенная подгруппа, SE-ядро подгруппы, силовская подгруппа, обобщенная подгруппа Фиттинга, максимальная подгруппа.*

We give some conditions under which a finite group belongs to a composition formation containing all finite supersoluble groups. The main results of the papers [20]–[32], [34]–[37] are generalized.

Keywords: *finite group, saturated formation, composition formation, Baer-local formation, p-nilpotent group, cyclic chief factor, supersoluble subgroup, quasisupersoluble group, S-quasinormal subgroup, S-supplemented subgroup, S-quasinormally embedded subgroup, the SE-core of subgroup, Sylow subgroup, the generalized Fitting subgroup, maximal subgroup.*

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Introduction

Throughout this paper, all groups are finite.

Let $\mathfrak{F}$ be any non-empty class of groups. Then we use $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of the group G with $G/N \in \mathfrak{F}$. A class $\mathfrak{F}$ of groups is said to be a formation if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. All applications of the formations theory are related to the so-called saturated and solubly saturated formations (see, for example, [1]–[6]).

Recall that a formation $\mathfrak{F}$ is said to be *saturated* if $\mathfrak{F}$ contains each group G with $G^{\mathfrak{F}} \le \Phi(G)$; $\mathfrak{F}$ is said to be *solubly saturated* or *Baer-local* (see [5, Chapter IV]) or *composition* [2] if $\mathfrak{F}$ contains each group G with $G^{\mathfrak{F}} \le \Phi(N)$ for some soluble normal subgroup N of G.

Many results of the theory of formations are related to the study of conditions under which a group belongs to a saturated formation. In this way a large number of criteria of solubility, supersolubility, p-nilpotency, nilpotency, and so on, as well as general criteria for membership of a group in a saturated formation have been found. Nevertheless, in this direction there are almost no results related to composition formations.

In this paper, we give conditions under which a group belongs to a composition formation containing all supersoluble groups.

1 Main definitions and results

Recall that a subgroup A of a group G is said to be S*-quasinormal*, S*-permutable*, or $\pi(G)$*-permutable* in G (Kegel [7]) if $AP = PA$ for all Sylow subgroups P of G. The S-quasinormal subgroups possess many interesting properties [7]–[9] and such subgroups are used for analysis of many questions of the group theory (see Section 5 in [10]). This circumstance was the main motivation for the introduction and study of various generalizations of the S-quasinormality. One of the most interesting generalizations of this concept was found by Ballester-Bolinches and Pedraza-Aguilera: A subgroup A of a group G is said to be S*-quasinormally embedded* or S*-permutably embedded* in G [11] if each Sylow subgroup of H is also a Sylow subgroup of some S-quasinormal subgroup of G. Based on the last concept, we introduce the following.

Let H be a subgroup of a group G, H_{seG} the subgroup generated by all subgroups of H which are S-quasinormally embedded in G. Then we say that H_{seG} is *the* SE*-core of* H in G.

***Definition* 1.1.** *Let H be a subgroup of a group G. Then we say that:*

(1) *H is SE-**quasinormal** in G if for some subnormal subgroup* T *of* G *we have* $HT = G$ *and* $H \cap T \le H_{seG}$.

(2) *H is SE-**supplemented** in G if for some subgroup T of G we have HT=G and* $H \cap T \le H_{seG}$.

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We have proved the following

***Theorem* 1.2.** *Let $\mathfrak{F}$ be a composition formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every non-cyclic Sylow p-subgroup P of E with $|P| = p^n$, there is an integer k such that $k < n$ and the subgroups of P or order p^k and order 4 are SE-quasinormal in G. Then $G \in \mathfrak{F}$.*

The proof of Theorem 1.2 relies on the following non-trivial observations.

***Theorem* 1.3.** *Let E be a normal subgroup of a group G, p a prime divisor of $|E|$. Let P be a Sylow p-subgroup of E and $|P| = p^n$. Suppose that either $n = 1$ or $n > 1$ and there is an integer k such that $k < n$ and the subgroups of P or order p^k and order 4 (if $p^k = 2$ and P is non-abelian) are SE-quasinormal in G.*

(I) *If $(p-1, |E|) = 1$, then each chief factor of G between E and $O_{p'}(E)$ is cyclic.*

(II) *If $(p-1, |G|) = 1$, then each chief factor H/K of G between E and $O_{p'}(E)$ is central, that is, $C_G(H/K) = G$.*

***Theorem* 1.4.** *Let E be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow p-subgroup P of E with $|P| = p^n$, there is an integer k such that $k < n$ and the subgroups of P or order p^k and order 4 (if $p^k = 2$ and P is non-abelian) are SE-quasinormal in G. Then each chief factor of G below E is cyclic.*

A chief factor H/K of a group G is called $\mathfrak{F}$-central provided $(H/K) \leftthreetimes (G/C_G(H/K)) \in \mathfrak{F}$ (see [3, p.127–128]).

***Theorem* 1.5** [13]. *Let $\mathfrak{F}$ be any formation and G a group. If E is a normal subgroup of G and each chief factor of G below $F^*(E)$ is $\mathfrak{F}$-central in G, then each chief factor of G below E is $\mathfrak{F}$-central in G as well.*

In this theorem $F^*(E)$ denotes the generalized Fitting subgroup of E, that is, the product of all normal quasinilpotent subgroups of E [12, Chapter X].

***Lemma* 1.6.** *Let $\mathfrak{F}$ be a composition formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

From Theorems 1.2 and 1.5 we get

***Theorem* 1.7.** *Let $\mathfrak{F}$ be a composition formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every non-cyclic Sylow p-subgroup P of $F^*(E)$ with $|P| = p^n$, there is an integer k such that $k < n$ and the subgroups of P or order p^k and order 4 (if $p^k = 2$ and P is non-abelian) are SE-quasinormal in G. Then $G \in \mathfrak{F}$.*

There are examples showing that we can not replace the condition «to be SE-quasinormal» in Theorems 1.2 and 1.7 by the weaker condition «to be SE-supplemented in G». Nevertheless, the following results are correct.

***Theorem* 1.8.** *Let $\mathfrak{F}$ be a composition formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every Sylow subgroup P of E, either the maximal subgroups of P or the cyclic subgroups of P of prime order and order 4 (if P is a non-abelian 2-group) are SE-supplemented in G. Then $G \in \mathfrak{F}$.*

***Theorem* 1.9.** *Let $\mathfrak{F}$ be a composition formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every Sylow subgroup P of $F^*(E)$, either the maximal subgroups of P or the cyclic subgroups of P of prime order and order 4 (if P is a non-abelian 2-group) are SE-supplemented in G. Then $G \in \mathfrak{F}$.*

The proofs of Theorems 1.8 and 1.9 are based on the following our results.

***Theorem* 1.10.** *Let E be a normal subgroup of a group G, $p_1 < p_2 < \ldots < p_n$ the set of all prime divisors of $|E|$ and $\pi_i = \{p_1, p_2, \ldots, p_i\}$. Suppose that for each $p \in \pi_i$ and for any Sylow p-subgroup P of E, the cyclic subgroups of P of prime order or order 4 are SE-supplemented in G. Then E has a normal Hall π_i-subgroup E_{π_i} and each chief factor of G between E and E_{π_i} is cyclic.*

***Theorem* 1.11.** *Let E be a normal subgroup of a group G, $p_1 < p_2 < \ldots < p_n$ the set of all prime divisors of $|E|$ and $\pi_i = \{p_1, p_2, \ldots, p_i\}$. Suppose that for each $p \in \pi_i$, the maximal subgroups of any Sylow p-subgroup of E are SE-supplemented in G. Then E has a normal Hall π_i-subgroup E_{π_i} and each chief factor of G between E and E_{π_i} is cyclic.*

Note that Theorems 1.2–1.5 and 1.7–1.11 cover main results of many papers. As an illustration of this, we discuss a small portion of them in Section 3. All unexplained notations and terminology are standard. The reader is referred to [5], [4] and [6] if necessary.

2 Some basic lemmas

Proofs of Theorems 1.2–1.5 and 1.7–1.11 consist of many stages and the following lemmas are some steps of them.

***Lemma* 2.1** [7]. *Let G be a group and $H \le K \le G$.*

(1) *If H is S-quasinormal in G, then H is S-quasinormal in K.*

(2) *Suppose that H is normal in G. Then K/H is S-quasinormal in G if and only if K is S-quasinormal in G.*

(3) *If H is S-quasinormal in G, then H is subnormal in G.*

(4) *If A and B are S-quasinormal in G, then $A \cap B$ and $\langle A, B \rangle$ are S-quasinormal in G.*

From Lemma 2.1 (3) we get

***Lemma* 2.2.** *If H is an S-quasinormal subgroup of a group G and H is a p-group for some prime p, then $O^p(G) \le N_G(H)$.*

***Lemma* 2.3** [8]. *Let G be a group and $A, B \le G$.*

(1) *If A is S-quasinormal in G, then $A \cap B$ is S-quasinormal in B.*

(2) *If A is S-quasinormal in G, then A/A_G is nilpotent.*

***Lemma* 2.4** [11]. *Let G be a group and $H \le K \le G$.*

(1) *If H is S-quasinormally embedded in G, then H is S-quasinormally embedded in K.*

(2) *If H is normal in G and E is an S-quasinormally embedded subgroup of G, then EH is S-quasinormally embedded in G and EH/H is S-quasinormally embedded in G/H.*

From Lemma 2.1 (4) we also get

***Lemma* 2.5.** *Let H be an S-quasinormally embedded subgroup of a group G. If $H \le O_p(G)$ for some prime p, then H is S-quasinormal in G.*

***Lemma* 2.6.** *Let G be a group, $H \le K \le G$ and N a normal subgroup of G.*

(1) $H_{seG} \trianglelefteq H$.

(2) $H_{seG} \le H_{seK}$.

(3) $H_{seG}N/N \le (HN/N)_{se(G/N)}$.

(4) *Suppose that* $(|N|, |H|) = 1$. *Then* $H_{seG}N/N = (HN/N)_{se(G/N)}$.

(5) $(H_{seG})^x = (H^x)_{seG}$ *for any* $x \in G$.

Proof. Let L be an S-quasinormally embedded subgroup of G contained in H, q a prime dividing $|L|$, Q a Sylow q-subgroup of L and E an S-quasinormal subgroup of G such that $Q \in \mathrm{Syl}_q(E)$.

(1) Let $x \in H$. Then $L^x \le H$ and if R is a Sylow q-subgroup of L^x, then $R = Q_1{}^x$ for some Sylow q-subgroup Q_1 of L. Without loss we may assume that $Q_1 = Q$. Note that $Q^x \in \mathrm{Syl}_q(E^x)$ and E^x clearly is an S-quasinormal subgroup of G. Hence L^x is an S-quasinormally embedded subgroup of G, so we have (1).

(2) $E \cap K$ is an S-quasinormal subgroup of K by Lemma 2.3 (1). Moreover, $Q \le E \cap K$ and hence Q is a Sylow q-subgroup of $E \cap K$. Therefore $L \le H_{seK}$, so $H_{seG} \le H_{seK}$.

(3) Clearly $LN/N \le HN/N$ and LN/N is an S-quasinormally embedded subgroup of G/N by Lemma 2.4 (2). Hence $H_{seG}N/N \le (HN/N)_{se(G/N)}$.

(4) In view of (3) we have only to prove that $(HN/N)_{se(G/N)} \le H_{seG}N/N$. Let V/N be an S-quasinormally embedded subgroup of G/N such that $V/N \le HN/N$. Then $V = V \cap HN = N(V \cap H)$. We shall show that $U = V \cap H$ is S-quasinormally embedded in G. Let p be any prime dividing $|U|$ and $P \in \mathrm{Syl}_p(U)$. Then $P \in \mathrm{Syl}_p(V)$ since $(|N|, |H|) = 1$. Hence $PN/N \in \mathrm{Syl}_p(V/N)$. Let W/N be an S-quasinormal subgroup of G/N such that $PN/N \in \mathrm{Syl}_p(W/N)$. Then $PN/N = W_pN/N$ for some Sylow p-subgroup W_p of E. Hence $PN = W_pN$. It is clear that $P, W_p \in \mathrm{Syl}_p(PN)$, so $P = (W_p)^n$ for some $n \in N$. The subgroup W is S-quasinormal in G by Lemma 2.1 (2), so W^n is S-quasinormal in G. Moreover, $P = (W_p)^n \in \mathrm{Syl}_p(W^n)$ and hence U is S-quasinormally embedded in G. Therefore $U \le H_{seG}$. Hence $V/N = UN/N \le H_{seG}N/N$ and so $(HN/N)_{se(G/N)} \le H_{seG}N/N$.

(5) This is evident.

***Lemma* 2.7.** *Let G be a group, H an SE-quasinormal subgroup of G and N a normal subgroup of G.*

(1) *If $H \le K \le G$, then H is SE-quasinormal in K.*

(2) *If $N \le H$, then H/N is SE-quasinormal in G/N.*

(3) *If $(|N|, |H|) = 1$, then HN/N is SE-quasinormal in G/N.*

(4) *The subgroup H^x is SE-quasinormal in G for all $x \in G$.*

(5) *Suppose that H is a p-subgroup for some prime p and $H \ne H_{seG}$. Then G has a normal subgroup M such that $|G:M| = p$ and $G = HM$.*

Proof. Let T be a subgroup of G such that $HT = G$ and $H \cap T \le H_{seG}$. Then $K = H(T \cap K)$ and $(T \cap K) \cap H = T \cap H \le H_{seG} \le H_{seK}$ by Lemma 2.6 (2). Hence we have (1).

(2) Since $(H/N)(NT/N) = G/H$ and

$$(H/N) \cap (NT/N) =$$
$$= N(H \cap T)/N \le H_{seG}N/N \le (H/N)_{se(G/N)}$$

by Lemma 2.6 (3), H/N is SE-supplemented in G/N.

(3) Since $(|N|,|H|)=1$, $N \le T$. Now we have

$$(T/N)\cap(HN/N)=$$
$$=N(T\cap H)/N \le NH_{seG}/N \le (NH/N)_{se(G/N)}.$$

Hence HN/N is SE-supplemented in G/N.

(4) From $HT=G$ we have $H^xT^x=G$. On the other hand, from $H\cap T \le H_{seG}$ we get $H^x\cap T^x=(H\cap T)^x \le (H_{seG})^x=(H^x)_{seG}$ by Lemma 2.6 (5). Hence H^x is SE-supplemented in G.

(5) By hypothesis G has a subnormal subgroup T such that $HT=G$ and $T\cap H \le H_{seG} \ne H$. Since $H \ne H_{seG}$, $T \ne G$, so G has a proper normal subgroup K such that $T \le K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $HM=G$ and $|G:M|=p$.

We use $\mathcal{A}(p-1)$ to denote the formation of all abelian groups of exponent dividing $p-1$. The symbol $Z_{\mathfrak{U}}(G)$ denotes the largest normal subgroup of a group G such that every chief factor of G below $Z_{\mathfrak{U}}(G)$ is cyclic.

***Lemma* 2.8.** *Let G be a group, H an SE-subgroup of G and N a normal subgroup of G.*

(1) *If $H \le K \le G$, then H is SE-supplemented in K.*

(2) *If $N \le H$, then H/N is SE-supplemented in G/N.*

(3) *If $(|N|,|H|)=1$, then HN/N is SE-supplemented in G/N.*

(4) *The subgroup H^x is SE-supplemented in G for all $x \in G$.*

Proof. See the proof of Lemma 2.7.

***Lemma* 2.9** [13, Lemma 2.2]. *Let E be a normal p-subgroup of a group G. If $E \le Z_{\mathfrak{U}}(G)$, then*

$$(G/C_G(E))^{\mathcal{A}(p-1)} \le O_p(G/C_G(E)).$$

The following lemma may be proved based on some results in [14] on f-hypercentral action (see [2, Chapter II] or [5, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

Lemma 2.10. *Let P be a normal p-subgroup of a group G. Let D be a characteristic subgroup of a p-subgroup P such that every non-trivial p'-automorphism of P induces a non-trivial automorphism of D. Suppose that $D \le Z_{\mathfrak{U}}(G)$. Then $P \le Z_{\mathfrak{U}}(G)$.*

Proof. Let $C=C_G(P)$, H/K any chief factor of G below P. Then $O_p(G/C_G(H/K))=1$ by [15, Appendix C, Corollary 6.4]. Since $D \le Z_{\mathfrak{U}}(G)$, then $(G/C_G(D))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.9. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group. Thus $G/C_G(H/K) \in \mathcal{A}(p-1)$ and so $|H/K|=p$ by [15, Chapter 1, theorem 1.4]. Therefore $P \le Z_{\mathfrak{U}}(G)$.

Let P be a p-group. If P is not a non-abelian 2-group we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P)=\Omega_2(P)$.

***Lemma* 2.11** [16]. *Let P be a p-group of class at most 2. Suppose that $\exp(P/Z(P))$ divides p.*

(1) *If $p>2$, then $\exp(\Omega(P))=p$.*

(2) *If P is a non-abelian 2-group, then $\exp(\Omega(P))=4$.*

Proof. See page 3 in [16].

***Lemma* 2.12** [17]. *Let G be a group and $A \le G$.*

(1) *If A is subnormal in G and A is a π-subgroup of G, then $A \le O_\pi(G)$.*

(2) *If A is subnormal in G and A is nilpotent, then $A \le F(G)$.*

Following Doerk and Hawkes [5], we use $C^p(G)$ to denote the intersection of the centralizers of all abelian p-chief factors of the group G ($C^p(G)=G$ if G has no such chief factors).

For every function f of the form

$$f:\mathbb{P}\cup\{0\}\to\{\text{group formations}\}, \qquad (*)$$

we put, following [18],

$$CLF(f)=\{G \text{ is a group} | G/G_{\mathcal{S}} \in f(0) \text{ and } G/C^p(G)\in f(p) \text{ for any prime } p\in\pi(Com(G))\}.$$

Here $G_{\mathcal{S}}$ denotes the $\mathcal{S}$-radical of G (i.e., the largest normal soluble subgroup of G); $Com(G)$ denotes the class of all abelian groups A such that $A \simeq H/K$ for some composition factor H/K of G.

Proof of Lemma 1.6. Without loss we may assume that E is a minimal normal subgroup of G, so $|E|=p$ for some prime p. By [18, Theorem 1], $F=CLF(f)$ for some function of the form (*). It is clear that the group $H=E\leftthreetimes(G/C_G(E))$ is supersoluble, so $H\in\mathfrak{F}$. Hence $G/C_G(E)\in f(p)$. Let $C^p/E=C^p(G/E)$. Then $C^p(G)=C^p\cap C_G(E)$ by [5, Chapter A, Theorem 3.2]. But since $G/E\in\mathfrak{F}$, $G/C^p \simeq (G/E)/C^p(G/E)\in f(p)$ and so $G\in\mathfrak{F}$.

***Lemma* 2.13** [2, Chapter I, Lemma 4.1]. *Let Q be a cyclic automorphism group of an elementary abelian p-group P of order p^n. If n is the smallest positive integer such that $|Q|$ divides p^n-1, then Q is irreducible.*

Proof of Theorem 1.7. Suppose that this assertion is false and consider a counterexample (G,E) for which $|G||E|$ is minimal. Let p be the smallest prime dividing $|E|$. Then E is p-nilpotent by Theorem 1.3. Let V be a Hall p'-subgroup of E.

Then V is characteristic in E, so V is normal in G. Moreover, the hypothesis holds for (G,V) and for $(G/V,E/V)$ by Lemma 2.7 (1) (3). Hence in the case where $V \neq 1$ we have $V \leq Z_{\mathfrak{U}}(G)$ and $E/V \leq Z_{\mathfrak{U}}(G/V)$ by the choice of (G,E), so $E \leq Z_{\mathfrak{U}}(G)$, a contradiction. Therefore $E = P$, so $E \leq Z_{\mathfrak{U}}(G)$ by Theorem 1.3.

3 Applications, examples, final remarks

I. In the literature one can find a large number of special cases of Theorems 1.2–1.5 and 1.7–1.11. In this section we discuss some of them.

A subgroup H of a group G is said to be c-normal in G [20] if G has a normal subgroup T such that $HT = G$ and $H \cap T \leq H_G$; H is said to be c^*-normal in G [21] if G has a normal subgroup T such that $HT = G$ and $H \cap T$ is S-quasinormally embedded in G.

Two open questions. Researches of many authors are connected with analysis of the following general question: *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Under what conditions on E then, does G belong to $\mathfrak{F}$?*

We recall a few typical results in this direction. If $\mathfrak{F}$ is a saturated formation containing all supersoluble groups and G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$, then the following results are true.

(1) If the cyclic subgroups of E of prime order and order 4 are either S-quasinormal (Ballester-Bolinches and Pedraza-Aguilera [22], Asaad and Csörgő [23]) or c-normal (Ballester-Bolinches and Wang [24]) in G, then $G \in \mathfrak{F}$.

(2) If the cyclic subgroups of every Sylow subgroup of $F^*(E)$ of prime order and order 4 are either S-quasinormal (Li and Wang [25]) or c-normal (Wei, Wang and Li [26]) or S-quasinormally embedded (Li and Wang [27]) in G, then $G \in \mathfrak{F}$.

(3) If the maximal subgroups of every Sylow subgroup of E are either S-quasinormal (Asaad [28]) or c-normal (Wei [29]) in G, then $G/E \in \mathfrak{F}$.

(4) If the maximal subgroups of every Sylow subgroup of $F^*(E)$ are either S-quasinormal (Li and Wang [30]) or c-normal (Wei, Wang and Li [31]) or c^*-normal (Wei and Wang, [21]) or S-quasinormally embedded (Li and Wang [27]) in G, then $G \in \mathfrak{F}$.

(5) If E is soluble and the maximal subgroups of every Sylow subgroup of $F(E)$ are S-quasinormally embedded in G, then $G \in \mathfrak{F}$ (Asaad and Heliel [31]).

Bearing in mind the above-mentioned results it is natural to ask:

(I) *Is it true that all the above-mentioned results can be strengthened by proving them for the case where $\mathfrak{F}$ is a composition formation?*

(II) *Is it true that all the above-mentioned results can be improved by of clarifying the character embedding of the subgroup E into the group G?*

Theorems 1.2, 1.4 and 1.7 show that the answers to both these questions are positive. Note that the methods of the above mentioned works [22]–[31] do not allow to answer to at least one of these questions. Note also that Theorems 1.2 and 1.7 not only strengthened all the results in [22]–[31] mentioned above but also give shorter proofs of many of them.

II. ***Further applications of the results.*** A subgroup H of a group G is said to be weakly S-permutable in G [10] if there are a subnormal subgroup T of G and an S-permutable subgroup H_s of G contained in H such that $HT = G$ and $H \cap T \leq H_s$. In the paper [32] the following concept which covers properly both S-permutable embedding property and weakly S-permutability was introduced.

***Definition* 3.1.** *Let H be a subgroup of a group G. Then H is said to be weakly S-permutable embedded in G* [32] *if there are a subnormal subgroup T of G and an S-permutably embedded subgroup H_{se} of G contained in H such that $HT = G$ and $H \cap T \leq H_{se}$.*

It is clear that every weakly S-permutable embedded subgroup is a SE-quasinormal subgroup. Hence main results in [32] are corollaries of Theorems 1.2–1.4 and Corollary 1.7.

The following example shows that in general the set of SE-quasinormal subgroups of a group is wider than the set of all its weakly S-permutable embedded subgroups.

Example. Let Ly be the Lyons simple group. Then in view of [33] there is a group D with minimal normal subgroup N such that $C_D(N) = N \leq O_{67}(D)$, $D/N \simeq Ly$ and $N \leq \Phi(D)$. Let Q be a group of order 17. Let $G = D \wr Q = K \rtimes Q$, where K is the base group of the regular wreath product G. Then $P = \Phi(K) = N^{\sharp}$. Since P is an elementary abelian 67-group, $P = P_1 \times P_2 \times \ldots \times P_t$, where P_i is a minimal normal subgroup of PQ for all $i = 1,2,\ldots,t$, and for some i we have $C_Q(P_i) = 1$. Hence $S = P_i \rtimes Q = Q^S$. It is not difficult to show that S is SE-quasinormal in G and it is not weakly S-permutable embedded in G.

Criteria of p-nilpotency. From Theorem 1.3 we get

Corollary 3.2. *Let E be a normal subgroup of a group G with p-nilpotent quotient G/E, p a prime dividing $|E|$ with $(p-1,|G|)=1$. Let P be a Sylow p-subgroup of E. Suppose that either the maximal subgroups of P or the cyclic subgroups of P of order p and order 4 (if P is non-abelian 2-group) are SE-quasinormal in G. Then G is p-nilpotent.*

Corollary 3.2 is independently interesting since this one generalizes many known results. Here we mention some recent theorems in this direction.

***Corollary* 3.4** (Assad and Heliel, [31, Theorem 3.1]). *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If all maximal subgroups of P are S-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 3.5 (Li at all, [35, Theorem 3.1]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor $|G|$ with $(p-1,|G|)=1$. If all maximal subgroups of P are S-quasinormally embedded in G, then G is p-nilpotent.*

***Corollary* 3.6** (Radaman, Azzat Mohamed and Heliel [36]). *Let G be a group, p the smallest prime dividing $|G|$ and P a Sylow subgroup of G. If the cyclic subgroups of P of prime order or order 4 are c-normal in G, then G is p-nilpotent.*

***Corollary* 3.7** (Wei and Wang, [21]). *Let E be a normal subgroup of a group G with p-nilpotent quotient G/E, p a prime dividing $|E|$ with $(p-1,|G|)=1$. Let P be a Sylow p-subgroup of E. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

***Corollary* 3.8** (Li at all [37]). *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are weakly S-permutable in G, then G is p-nilpotent.*

Criteria of quasisupersolubility. A group G is said to be quasisupersoluble [38] if for every its non-cyclic chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K. It is clear that every supersoluble group is quasisupersoluble. Moreover, in [38] it is proved that the class of all quasisupersoluble groups is a Baer-local formation. Hence from Theorems 1.2 and 1.7 we get

***Theorem* 3.9.** *Let G a group with a normal subgroup E such that G/E is quasisupersoluble. If the cyclic subgroups of $F^*(E)$ of prime order or order 4 are SE-supplemented in G, then G is quasisupersoluble.*

***Theorem* 3.10.** *Let G be a group with normal subgroups $X \le E$ such that G/E is quasisupersoluble. Suppose that the maximal subgroups of every Sylow subgroup of X are SE-supplemented in G. If either $X=E$ or $X=F^*(E)$, then G is quasisupersoluble.*

A large number of other known results (see Section 5 in [10]) are special cases of these two theorems as well.

REFERENCES

1. *Huppert, B.* Endliche Gruppen I / B. Huppert. – New York: Springer, 1967. – 793 p.
2. *Shemetkov, L.A.* Formations of finite groups / L.A. Shemetkov. – Moscow : Nauka, Main Editorial Board for Physical and Mathematical Literature, 1978. – 272 p.
3. *Shemetkov, L.A.* Formations of Algebraic Systems / L.A. Shemetkov, A.N. Skiba. – Moscow : Nauka, 1989. – 256 p.
4. *Ballester-Bolinches, A.* Classes of Finite groups / A. Ballester-Bolinches, L.M. Ezquerro. – Dordrecht : Springer, 2006. – 367 p.
5. *Doerk, K.* Finite Soluble Groups / K. Doerk, T. Hawkes. – Berlin-New York : Walter de Gruyter, 1992. – 889 p.
6. *Wenbin, Guo.* The Theory of Classes of Groups / Wenbin Guo. – Beijing-New York-Dordrecht-Boston-London : Science Press-Kluwer Academic Publishers, 2000. – 325 p.
7. *Kegel, O.* Sylow-Gruppen and Subnormalteiler endlicher Gruppen / O. Kegel // Math. Z. – 1962. – Vol. 78. – P. 205–221.
8. *Deskins, W.E.* On quasinormal subgroups of finite groups / W.E. Deskins // Math. Z. – 1963. – Vol. 82. – P. 125–132.
9. *Schmid, P.* Subgroups Permutable with All Sylow Subgroups / P. Schmid // J. Algebra. – 1998. – Vol. 82. – P. 285–293.
10. *Skiba, A.N.* On weakly s-permutable subgroups of finite groups / A.N. Skiba // J. Algebra. – 2007. – Vol. 315. – P. 192–209.
11. *Ballester-Bolinches, A.* Sufficient conditions for supersolvability of finite groups / A. Ballester-Bolinches, M.C. Pedraza-Aguilera // J. Pure Appl. Algebra. – 1998. – Vol. 127. – P. 113–118.
12. *Huppert, B.* Finite Groups III / B. Huppert, N. Blackburn. – Berlin–New-York : Springer-Verlag, 1982. – 745 p.
13. *Skiba, A.N.* A characterization of the hypercyclically embedded subgroups of finite groups / A.N. Skiba // Journal of Pure and Apllied Algebra []. – 2010. – doi:10.1016 / j.jpaa. – 17.04.2010.
14. *Laue, R.* Dualization for saturation for locally defined formations / R. Laue // J. Algebra. – 1978. – Vol. 52. – P. 347–353.
15. *Weinstein, M.* Between Nilpotent and Solvable / M. Weinstein, N.J. Passaic. – Polygonal Publishing House, 1982. – 367 p.

16. *Bercovich, Ya.* Indices of elements and normal structure of finite groups / Ya. Bercovich, L. Kazarin // J. Algebra. – 2005. – Vol. 283, №1. – P. 564–583.

17. *Wielandt, H.* Subnormal subgroups and permutation groups / H. Wielandt. – Columbus, Ohio : Lectures given at the Ohio State University, Columbus, 1971. – 275 p.

18. *Skiba, A.N.* Multiply $\mathfrak{L}$-Composition Formations of Finite Groups / A.N. Skiba, L.A. Shemetkov // Ukrainsk. Math. Zh. – 2000. Vol. 52, №6. – P. 783–797.

19. *Gorenstein, D.* Finite Groups / D. Gorenstein. – New York-Evanston-London : Harper & Row Publishers, 1968. – 463 p.

20. Wang, Y. *c*-normality of groups and its properties / Y. Wang // J. Algebra. – 1996. – Vol. 180. – P. 954–965.

21. *Wei, H.* On c^*-normality and its properties / H. Wei, Y. Wang // J. Group Theory. – 2007. – Vol. 10. – P. 211–223.

22. *Ballester-Bolinches, A.* On minimal subgroups of finite groups / A. Ballester-Bolinches, M.C. Pedraza-Aguilera // Acta Math. Hungar. – 1996. – Vol. 7. – P. 335–342.

23. *Asaad, M.* Influence of minimal subgroups on the structure of finite group / M. Asaad, P. Csörgő // Arch. Math. (Basel). – 1999. – Vol. 72. – P. 401–404.

24. *Ballester-Bolinches, A.* Finite groups with some C-normal minimal subgroups / A. Ballester-Bolinches, Y. Wang // J. Pure Appl. Algebra. – 2000. – Vol. 153. – P. 121–127.

25. *Li, Y.* The influence of minimal subgroups on the structure of a finite group / Y. Li, Y. Wang // Proc. Amer. Math. Soc. – 2002. – Vol. 131. – P. 337–341.

26. *Wei, H.* On *c*-normal maximal and minimal subgroups of Sylow subgroups of finite groups, II / H. Wei, Y. Wang, Y. Li // Comm. Algebra. – 2003. – Vol. 31. – P. 4807–4816.

27. *Li, Y.* On π-quasinormally embedded subgroups of finite groups / Y. Li, Y. Wang // J. Algebra. – 2004. – Vol. 281, №8. – P. 109–123.

28. *Asaad, M.* On maximal subgroups of finite group / M. Asaad // Comm. Algebra. – 1998. – Vol. 26. – P. 3647–3652.

29. *Wei, H.* On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups / H. Wei // Comm. Algebra. – 2001. – Vol. 29. – P. 2193–2200.

30. *Li, Y.* The influence of π-quasinormality of some subgroups of a finite group / Y. Li, Y. Wang // Arch. Math. (Basel). – 2003. – Vol. 81. – P. 245–252.

31. *Asaad, M.* On *S*-quasinormally embedded subgroups of finite groups / M. Asaad, A.A. Heliel // J. Pure and Applied Algebra. – 2001. – Vol. 165. – P. 129–135.

32. *Li, Y.* On Weakly *s*-permutably Embedded Subgroups of Finite Groups / Y. Li, Sh. Qiao, Y. Wang // Comm. Algebra. – 2009. – Vol. 37. – P. 1086–1097.

33. *Griess, R.* The Frattini module / R. Griess, P. Schmid // Arch. Math. – 1978. – Vol. 30. – P. 256–266.

34. *Li, S.* On *S*-quasinormal and *c*-normal subgroup of a finite group / S. Li, Y. Li // Czech. Math. J. – 2008. – Vol. 58, №4. – P. 1083–1095.

35. *Li, Y.* On p-nilpotency of finite groups with some subgroups S-quasinormally embedded / Y. Li, Y. Wang, H. Wei // Acta Math. Hungar. – 2005. – Vol. 108, № 4. – P. 283–298.

36. *Radaman, M.* On *c*-normaliti of certain subgroups of prime power order of finite groups / M. Radaman, M.A. Mohamed, A.A. Heliel // Arch. Math. – 2005. – Vol. 85. – P. 203–210.

37. *Li, Y.* A note on a result of Skiba / Y. Li, S. Qiao, Y. Wang // Siberian Math. Journal. – 2009. – Vol. 50, № 3. – P. 467–473.

38. *Guo, W.* On some classes of finite quasi-$\mathfrak{F}$-groups / W. Guo, A.N. Skiba // J. Group Theory. – 2009. – Vol. 12. – P. 407–417.

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