Известия Гомельского государственного университета
имени Ф.Скорины, 3(36), 2006

*UDC 519.41/47*

# Groups with finitely many nonnormal subgroups

N. S. Chernikov



Recently the complete constructive description of groups in which all nonnormal subgroups generate a proper subgroup was obtained [1]. Using [1], we obtain the following proposition which describes infinite groups with finitely many nonnormal subgroups.

**Theorem.** *For an infinite group $F$ the following statements are equivalent.*

(i) *The set of all nonnormal subgroups of $F$ is finite.*

(ii) *The set of all nonnormal cyclic subgroups of $F$ is finite.*

(iii) *All nonnormal subgroups of $F$ generate a finite subgroup.*

(iv) *All nonnormal cyclic subgroups of $F$ generate a finite subgroup.*

(v) *For some prime $p$,*

$$F = G \times D \tag{1}$$

*where $G$ is a nonabelian $p$-subgroup and $D$ is a finite Dedekind $p'$-subgroup, and also for some central quasicyclic subgroup $B$ of $G$, $G/B$ is finite abelian.*

Remind that Dedekind groups are groups in which all subgroups are normal. Dedekind's Theorem [2] gives the complete description of finite Dedekind groups.

*Proof.* Clearly, (i)$\longrightarrow$(ii).

(ii)$\longrightarrow$(i). Take any nonnormal subgroup $T$ of $F$. Put $S = \bigcap_{g\in F} T^g$. For any $a \in T\backslash S$, $\langle a\rangle \ntrianglelefteq F$. In view of Lemma 1 [1], $T = \langle T\backslash S\rangle$. Thus $T$ is generated by some nonnormal cyclic subgroups of $G$. Since the set of all such subgroups is finite, the set of all nonnormal subgroups of $F$ is finite too.

Also we conclude that the subgroup of $F$ generated by all its nonnormal subgroups coincides with the subgroup generated by all its nonnormal cyclic subgroups. Therefore (iii)$\longleftrightarrow$(iv).

Clearly, (iv)$\longrightarrow$(ii).

(ii)$\longrightarrow$(iv). Take any $\langle g\rangle \ntrianglelefteq F$ and any distinct primes $p, q$. Then $\langle g\rangle = \langle g^p\rangle\langle g^q\rangle$. Therefore $\langle g^p\rangle \ntrianglelefteq F$ or $\langle g^q\rangle \ntrianglelefteq F$. Consequently, the set of all primes $p$ for which $\langle g^p\rangle \ntrianglelefteq G$ is infinite. Therefore for some distinct primes $r$ and $s$, $\langle g^r\rangle = \langle g^s\rangle$. Consequently, $\langle g\rangle$ is finite. Therefore the set $M = \{a : \text{ for some } g \in F, a \in \langle g\rangle \ntrianglelefteq F\}$ is finite. In consequence of Dietzmann's Lemma (see, for instance, [3]), $\langle M\rangle$ is finite.

Thus (i)$\longleftrightarrow$(ii)$\longleftrightarrow$(iii)$\longleftrightarrow$(iv).

(iii)$\longrightarrow$(v). In view of Theorem 1 [1], for some prime $p$, (1) is valid where $G$ and $D$ are as above but $D$ is not necessarily finite. Also, by this theorem, for some locally cyclic subgroup $B \lhd G$ and subgroup $A$ of finite exponent, $G = AB$ and $A' \subseteq B$, and also in the case when $B$ is infinite (i.e. $B$ is quasicyclic), $B \subseteq Z(G)$.

Take any $\langle g\rangle \ntrianglelefteq F$. Then $\langle g\rangle = \langle u\rangle \times \langle v\rangle$ with $\langle u\rangle \subseteq G$ and $\langle v\rangle \subseteq B$. Clearly, $\langle u\rangle \ntrianglelefteq G$. Then, with regard to (1), for any $\langle w\rangle \subseteq D$, $\langle u\rangle\langle w\rangle \ntrianglelefteq F$. Consequently, $D$ belongs to the subgroup $T$ generated by all nonnormal subgroups of $F$. Since $T$ is finite, $D$ is finite too.

Further, in consequence of the statement 1 of Theorem 1 [1], the subgroup $S = T \cap G$ is generated by all nonnormal subgroups of $G$.

Assume that there exists $h \in G\backslash S$ of order $p$. Since $\langle h\rangle \trianglelefteq G$ and $G$ is a $p$-group, obviously, $h \in Z(G)$. Take any $\langle g\rangle \ntrianglelefteq G$. Since $\langle g\rangle \subseteq S$, $gh \notin S$. So $\langle gh\rangle \trianglelefteq G$.

Further, take any $x \in G$ for which $g^x \notin \langle g \rangle$. For some $n \in \mathbb{N}$, $(gh)^x = (gh)^n$. So $g^x h^x = g^x h = g^n h^n$ and $g^{-n} g^x = h^{n-1}$. Since $g^{-n} g^x \in S$ and $S \cap \langle h \rangle = 1$, $h^{n-1} = 1$. Consequently, $g^x = g^n \in \langle g \rangle$, which is a contradiction.

Thus, all elementary abelian subgroups of $G$ belong to finite $S$ and, at the same time, are finite. Therefore, obviously, all abelian subgroups of $G$ are Chernikov, i.e. $G$ satisfies the minimal condition for abelian subgroups.

Since $A$ satisfies the minimal condition for abelian subgroups and, obviously, is a locally finite $p$-group, it is Chernikov (S.N.Chernikov's Theorem; see, for instance, [4], Theorem 4.1). But $A$ is of finite exponent. So $A$ is finite. Then, because of $G$ is infinite and $G = AB$, $B$ is infinite. Since $B$ is an infinite locally cyclic $p$-subgroup, it is quasicyclic.

Since $A$ is finite and $A' \subseteq B$ and $G = AB$, $G/B$ is finite abelian.

(v)⟶(iii). By O.J.Schmidt's Theorem (see, for instance, [3], Theorem 1.45), $G$ is locally finite. Take any finite subgroup $A$ of $G$ for which $G = AB$. Let $V < B$ and $|V| = \frac{|A'|}{p} t$ where $t$ is the exponent of the group $A/A'^p$. Then $AV$ is a finite subgroup of $G$. In consequence of Theorem 5 [1], $G$ is not Dedekind and $AV$ contains all nonnormal subgroups of $G$. By Lemma 4 [1], $AVD$ contains all nonnormal subgroups of $F$. Since $AVD$ is finite, (iii) is valid. Theorem is proven.

(Note that in the statement 3 of Theorem 5 [1] must be: "$B \supseteq A'$".)

***Remark.*** Obviously, $G$ from Theorem is locally finite. Take any finite subgroup $A$ of $G$ for which

$$G = AB. \tag{2}$$

It is easy to see that

$$A' = G' \subseteq B \cap A \tag{3}$$

and $A/B \cap A \simeq G/B$.

In view of Theorem 3 [1], for any finite $p$-group $A$ such that $A' \subseteq Z(A)$ and $A'$ is cyclic and for any cyclic subgroup $T$ of $A$ such that $A' \subseteq T \subseteq Z(A)$, there exists a $p$-group $G$ such that $A < G$ and for some central quasicyclic subgroup $B$ of $G$, relations (2), (3) are fulfilled and $B \cap A = T$. By Corollary 5 [1], if $G^* = A^* B^*$ and $B^*$ is a central quasicyclic $p$-subgroup of $G^*$ and also there exists an isomorphism $\psi$ of $A$ onto $A^*$ such that $(A \cap B)^\psi = A^* \cap B^*$, then $G^* \simeq G$.

**Abstract.** The complete description of infinite groups with the finite set of all nonnormal subgroups is obtained.

## References

1. Chernikov N.S., Dovzhenko S.A. Groups in which all nonnormal subgroups generate a proper subgroup // Sibirsk. Mat. Ž, 47, № 1 (2006), 211–235.

2. Dedekind R. Über Gruppen, deren sämmtliche Teiler Normalteiler sind // Math. Ann., 48 (1897), 548–561.

3. Robinson D.J.S. Finiteness conditions and generalized soluble groups. Pt 1., Berlin etc: Springer, 1972.

4. Chernikov S.N. Groups with prescribed properties of the system of subgroups, Moscow: Nauka, 1980.

Institut of Mathematics,
National Academy of Science,
Kiev, Ukraine

Received 4.04.06