Известия Гомельского государственного университета
имени Ф.Скорины, 3(36), 2006

UDC 512.542

Mutually permutable products of finite groups

A. Ballester-Bolinches[1]

Dedicated to the memory of Sergey Antonovich Chunikhin (1905–1985) and Klaus Doerk (1939–2004)

Factorised groups have played an important role in the theory of groups in the last sixty years. First contributions in this context are the Ito's Theorem on products of abelian groups or the one of Kegel-Wielandt stating th solubility of a product of two finite nilpotent groups. A well-known result by Fitting states that the product of two normal nilpotent subgroups is nilpotent. However, in general, the product of two normal supersoluble subgroups of a group is not supersoluble. To create intermediate situations it is usual to consider products of groups whose factors satisfy certain relations of permutability. Following Carocca [14] we say that a group $G = AB$ is the mutually permutable product of A and B if A permutes with every subgroup of B and viceversa. If, in addition, every subgroup of A permutes with every subgroup of B, we say that the group G is a totally permutable product of A and B.

In this context, we can consider as seminal the following results of Asaad and Shaalan.

Theorem A (Asaad and Shaalan [2]).

(i) *Assume that a group $G = AB$ is the mutually permutable product of A and B. Suppose that A and B are finite and supersoluble and that either A, B or G', the derived subgroup of G, is nilpotent. Then G is supersoluble.*

(ii) *If $G = AB$ is the totally permutable product of the finite supersoluble subgroups A and B, then G is supersoluble.*

These results were the beginning of an intensive research in the area of factorised groups. Many contributions are made in the context of formation theory in the finite universe (see [3], [4], [7], [15], [16] and [18]). Extensions of some of the above results in the non-finite universe are also considered (see [10], [11] and [12]).

From now on, all groups considered are finite.

Following Robinson [19], we say that a group is an SC-group if all of its chief factors are simple groups. Clearly a group is supersoluble if and only if it is a soluble SC-group. It is not difficult to see that the class of all SC-groups is a formation closed under taking normal subgroups. However it is neither an s-closed formation nor a saturated formation.

Robinson [19] describes the structure of SC-groups. They are characterized in the following way.

Proposition A. *A group G is a SC-group if and only if there is a perfect normal subgroup D such that G/D is supersoluble, $D/Z(D)$ is a direct product of G-invariant subgroups, and $Z(D)$ is supersolubly embedded in G, (that is, there is a G-admissible series of $Z(D)$ with cyclic factors).*

The relation between the soluble residual and the soluble radical, i.e. the product of all soluble normal subgroups, of a SC-group is shown in the following result.

[1]The author is supported by Proyecto MTM2004-08219-C02-02 (MCyT) and FEDER (European Union)

Lemma A ([19]). *If G is a SC-group and D is the soluble residual of G, then $C_G(D) = G_S$.*

The above facts allows us to give some information about totally permutable products of SC-groups.

Theorem 1 ([5]). *Let $G = AB$ the totally permutable product of the subgroups A and B. Then G is a SC-group if and only if A and B are SC-groups*

This work has been continued by Beidleman, Hauck and Heineken in [13].

On the other hand, it is clear that a mutually permutable product whose factors intersect trivially is a totally permutable one. Thus,in that case, if both factors are supersoluble, the group is also supersoluble. Taking this remark into account, Alejandre, Cossey and the author (see [1]) obtained that a mutually permutable product of supersoluble groups in which the intersection of the factors does not contain non-trivial normal subgroups is supersoluble.

Theorem 2 ([1]). *Let $G = AB$ be the mutually permutable product of the supersoluble subgroups A and B. If $Core_G(A \cap B) = 1$, then G is supersoluble.*

It could be natural to ask for a SC-version of the above result. This follows after the analysis of the behaviour of the soluble residuals of the factors in mutually permutable products. That is the main result in [6].

Theorem 3. *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. Then $[A, B^S]$ and $[A^S, B]$ are contained in $Core_G(A \cap B)$.*

Here A^S and B^S denote the soluble residuals of A and B, respectively.

Corollary 1 ([6]). *Let the group $G = AB$ be the mutually permutable of A and B. Then A^S and B^S are normal subgroups of G.*

As a consequence we have:

Theorem 4 ([6]). *Let the group $G = AB$ be the mutually permutable product of A and B. If A and B are SC-groups and $Core_G(A \cap B) = 1$, then G is an SC-group.*

The converse holds with no restrictions.

Theorem 5 ([6]). *Let $G = AB$ be the mutually permutable product of the subgroups A and B. If G is an SC-group, then A and B are SC-groups.*

Theorem 3 also applies to get results about mutually permutable products of SC-groups in which one of the factors is quasinilpotent. These are also contained in [6].

Theorem 6. *Let $G = AB$ be the mutually permutable product of A and B. Assume that A is an SC-group and B is quasinilpotent. Then G is an SC-group.*

Theorem 7. *Let $G = AB$ be the mutually permutable product of the subgroups A and B. If A and B are SC-groups and G', the derived subgroup of G, is quasinilpotent, then G is an SC-group.*

Notice that Theorem 6 and 7 are extension of the aforementioned results by Asaad and Shaalan.

Coming back to totally permutable products, in [8], Cossey and Esteban-Romero and the author studied the behaviour of totally permutable products of two nilpotent groups proving that totally permutable products of nilpotent groups are extensions of an abelian by a nilpotent group. In [9], we are interested to find more information about the structure of the product of two nilpotent mutually permutable subgroups. In particular, if G is the mutually permutable product of two nilpotent subgroups of odd order, then we obtain that G is abelian by nilpotent.

As a first step, the particular case when $G^{\mathcal{N}}$, the nilpotent residual of the group G, is a p-group contained in one of the factors is analysed (recall that $G^2 = < g^2 : g \in G >$):

Lemma 1. Let $G = AB$ be the mutually permutable product of the nilpotent subgroups A and B. Assume that $G^{\mathcal{N}}$ is a p-group and that $G^{\mathcal{N}} \leqslant A$. Then G satisfies one of the following statements: (a) G^2 is nilpotent; (b) $G^{\mathcal{N}}$ is abelian.

Theorem 8. Let $G = AB$ be the mutually permutable product of the nilpotent subgroups A and B. Then $(G^2)^{\mathcal{N}}$ is abelian. In particular, if G is of odd order we obtain $(G^{\mathcal{N}})$ is abelian.

Abstract. The author gives a survey of results on mutually permutable products of finite groups.

References

1. Manuel J. Alejandre, A. Ballester-Bolinches and John Cossey, *"Permutable products of supersoluble groups"* J. Algebra, 276 (2004), 453-461.

2. M. Asaad and A. Shaalan, *"On the supersolvability of finite groups"* Arch. Math., 53 (1989), 318-326.

3. A. Ballester-Bolinches and M.C. Pedraza-Aguilera and M. D. Pérez-Ramos, *"Totally and mutually permutable products of finite groups"* Proc. Groups Bath/St. Andrews (1997). London Math. Soc. Lecture Notes 260, Vol I (1999), 65-68.

4. A. Ballester-Bolinches and M. D. Pérez-Ramos, *"A Question of R. Maier concerning formations"* J. Algebra, 182 (1996), 738-747.

5. A. Ballester-Bolinches and John Cossey, *"Totally permutable products of finite groups satisfying SC or PST"* Monatshefte für Mathematik, to appear.

6. A. Ballester-Bolinches, John Cossey and M.C. Pedraza-Aguilera, *"On mutually permutable products of finite groups"* J. Algebra, 294 (2005), 127-135.

7. A. Ballester-Bolinches and M.C. Pedraza-Aguilera, *"Mutually permutable products of finite groups II"* J. Algebra, 218 (1999), 563-572.

8. A. Ballester-Bolinches, John Cossey and R. Esteban-Romero, *"On totally permutable products of finite groups"* J. Algebra, 293(2005), 269-278.

9. A. Ballester-Bolinches, J. C. Beidleman, John Cossey, H. Heineken and M. C. Pedraza-Aguilera, *Mutually permutable products of two nilpotent groups* Accepted in Rendiconti Sem. Mat. Padova.

10. J. C. Beidleman, A. Galoppo, H. Heineken and M. Manfredino, *"On certain products of soluble groups"* Forum Mathematicum, to appear.

11. J.C. Beidleman and H. Heineken, *"Survey of mutually and totally permutable products in infinite groups"* Topics in infinite groups, 45-62, Quad. Mat., 8 Dept. Math., Seconda Univ. Napoli, Caserta, 2001.

12. J.C. Beidleman and H. Heineken, *"Totally permutable torsion groups"* J. Group Theory, 2 (1999), 377-392.

13. J.C. Beidleman, P. Hauck and H. Heineken, *"Totally permutable products of certain classes of finite groups"* J. Algebra, 276 (2004), 826-835.

14. A. Carocca, *"p-supersolvability of factorizedf finite groups"* Hokkaido Math. J. 21 (1992), 395-403.

15. A. Carocca, *"A note on the product of $\mathcal{F}$-subgroups in a finite group"* Proc. Edinburgh Math. Soc., 39 (1996), 37-42.

16. A. Carocca and R. Maier, *"Theorems of Kegel-Wielandt type"* Proc. Group Bath/St. Andrews(1997). London Math. Soc. Lecture Notes Series 260, Vol. I, (1999), 195-201.

17. K. Doerk and T. Hawkes, *"Finite Soluble Groups"* , Walter De Gruyter, Berlin / New York, (1992).

18. R. Maier, *"A completeness property of certain formations"* Bull. London Math. Soc., 24, (1992) 540-544.

19. D.J.S. Robinson, *"The structure of finite groups in which permutability is a transitive relation"* J.Austral. Math. Soc. 70 (2001), 143-149.

Universitat de València

Received 17.04.06