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The Influence of c-Permutable Subgroups on the Structure of Finite Groups[1]

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1. Introduction

Recall that a subgroup A of a group G is called permutable with a subgroup B if $AB = BA$. If A is permutable with all subgroups of G, then A is called a permutable subgroup [1] (or quasinormal subgroup [7]) of G.

The permutable subgroups have many interesting properties in the case when G is a finite group. For example, Ore [7] proved that every permutable subgroup of a finite group is subnormal. Itô and Szép [6] proved that for every permutable subgroup H of a finite group G, H/H_G is nilpotent.

However, in general, two subgroups H and T of a group G may not be permutable in G but G may contain an element x such that $HT^x = T^xH$. Base on the observations, recently, Guo, Shum and Skiba [2]introduce the concept of conditionally permutable subgroup and completely conditionally permutable subgroup.

Definition 1.1(cf.[2]). *Let H, T be subgroups of G. Then*

(1) *H and T are said to be conditionally permutable (or in brevity, c-permutable) in G if for some $x \in G$ we have $HT^x = T^xH$.*

(2) *H is said to be c-permutable in G if H is c-permutable with all subgroups of G.*

(3) *H and T are said to be completely c-permutable in G if H and T are c-permutable in $\langle H, T\rangle$.*

(4) *H is said to be completely c-permutable in G if for every subgroup T of G, the subgroups H and T are c-permutable in $\langle H, T\rangle$.*

Using this new concept Guo, Shum and Skiba have obtained some new elegant results on the structure of finite groups (cf. [2-3]).

As a continuation to [2-3], in this paper, we study the influence of c-permutable subgroups on the structure of finite groups. Some new sufficient and necessary conditions, under which a group is nilpotent, supersoluble and in general, in a given formation, are obtained.

All groups considered in this paper are finite. All unexplained notations and terminologies are standard. The reader is referred to the text of Shemetkov [10] or K.Doerk and T.Hawkes [1] for notations and terminologies not given in this paper.

2. Preliminary results

In this section, we give some basic results of c-permutable subgroups. We also cite some known results which will be often used in the later.

Lemma 2.1(cf.[2]). *Let G be a group, $K \trianglelefteq G$ and $H \leqslant G$. Then the following statements hold:*

(1) *if $K \leqslant T \leqslant G$ and H is (completely) c-permutable with T in G, then KH/K is (completely) c-permutable with T/K in G/K;*

(2) *if $K \leqslant H$, $T \leqslant G$ and H/K is (completely) c-permutable with KT/K in G/K, then H is (completely) c-permutable with T in G;*

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(3) *if* $T \leqslant M \leqslant G$, $H \leqslant M$ *and* H *is completely c-permutable with* T *in* M;

(4) *if* $T \leqslant G$ *and* H *is (completely) c-permutable with* T *in* G, *then for every* $x \in G$, *the subgroup* H^x *is (completely) c-permutable with* T^x *in* G.

Lemma 2.2 [8, Theorem 24.2]. *Let* $\mathcal{F}$ *be a local formation,* G *be a group and* $G^{\mathcal{F}}$ *is soluble. If* $G^{\mathcal{F}} \neq 1$ *and every* $\mathcal{F}$*-abnormal maximal subgroup of* G *belongs to* $\mathcal{F}$, *then the following statements hold:*

1) $G^{\mathcal{F}}$ *is a* p*-subgroup of* G *for some prime divisor* p *of* $|G|$.

2) $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ *is a minimal normal subgroup of* $G/\Phi(G^{\mathcal{F}})$.

3) *If* $G^{\mathcal{F}}$ *is abelian, then* $G^{\mathcal{F}}$ *is an elementary abelian* p*-group.*

4) *If* $p > 2$, *then the exponent of* $G^{\mathcal{F}}$ *is* p. *If* $p = 2$, *then the exponent of* $G^{\mathcal{F}}$ *is* 2 *or* 4.

5) $\Phi(G) \subseteq Z_{\mathcal{F}}(G)$.

Lemma 2.3[5, Theorem VI, 5.4]. *Suppose that* G *is a group which is not* p*-nilpotent but whose proper subgroups are all* p*-nilpotent. Then* G *is a Schmidt group.*

Lemma 2.4 [4, Theorem 3.4.2]. *Suppose that* G *is a Schmidt group, that is* G *is not nilpotent but whose proper subgroups are all nilpotent. Then*

1) G *has a normal Sylow* p*-subgroup* P *for some prime* p *and* $G/P \cong Q$, *where* Q *is a non-normal cyclic* q*-subgroup for some prime* $q \neq p$.

2) $P/\Phi(P)$ *is a minimal normal subgroup of* $G/\Phi(P)$.

3) *If* P *is non-abelian and* $p \neq 2$, *then the exponent of* P *is* p.

4) *If* P *is non-abelian and* $p = 2$, *then the exponent of* P *is* 4.

5) *If* P *is abelian, then* P *is of exponent* p.

A group G is said to be A_4-free if every factor group of every subgroup of G is not isomorphic with A_4, the alternating group of degree 4 .

Lemma 2.5. *Let* G *be a finite group,* p *the smallest prime dividing the order of* G *and* G *is* A_4*-free. If* N *is a normal subgroup of* G *such that* G/N *is* p*-nilpotent and the order of* N *is not divisible by* p^3, *then* G *is* p*-nilpotent.*

Proof. Assume the lemma is false and let G be a counterexample of minimal order. Let M be an arbitrary proper subgroup of G. If $p \notin \pi(M)$, then M is p-nilpotent. Let $p \in \pi(M)$. Then M is A_4-free and $N \cap M \trianglelefteq M$ with $p^3 \nmid |N \cap M|$. Since $M/N \cap M \cong MN/N$ is p-nilpotent, M satisfies the conditions of the lemma, and hence M is p-nilpotent by the choice of G. This shows that G is a minimal non-p-nilpotent group. Then G is also a minimal non-nilpotent group by Lemma 2.3. In view of Lemma 2.4, G has a normal Sylow p-subgroup P for some prime p such that $G/P \cong Q$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. By [4, Theorem 3.4.7], we know that $P \subseteq N$. If $\Phi(P) \neq 1$, then, obviously, $G/\Phi(P)$ satisfied the conditions of the lemma and so $G/\Phi(P)$ is p-nilpotent. It follows that G is p-nilpotent since the class of all p-nilpotent groups is a saturation formation. Therefore, we may assume that $\Phi(P) = 1$, P is a minimal normal subgroup of G and P is a abelian group. Otherwise, $N = 1$ and G is nilpotent. Since $p^3 \nmid |N|$, $p^3 \nmid |P|$. Since $G/C_G(P) = N_G(P)/C_G(P)$ is isomorphic with some subgroup of $Aut(P)$, we have that $|G/C_G(P)|$ is a divisor of $p(p-1)(p+1)$. If $p \neq 2$, then the minimality of p shows that $G/C_G(P) = 1$ and hence by Burnside Theorem (cf[9, II, Theorem 5.4]), G is nilpotent, a contradiction. If $p = 2$, $|G/C_G(P)|$ is a divisor of 6. Since G is A_4-free, we have that $3 \nmid |G/C_G(P)|$. It is impossible that $|G/C_G(P)| = 2$ since P is the Sylow 2-subgroup of G and P is abelian. Therefore, $|G/C_G(P)| = |N_G(P)/C_G(P)| = 1$, and consequently, G is nilpotent. The final contradiction completes the proof.

Lemma 2.6. *Let* P *be a minimal normal* p*-subgroup of* G. *If every subgroup of order* p *of* P *is c-permutable in* G, *then* P *is a group of order* p.

Proof. Let D be a Sylow p-subgroup of G. Then $P \cap Z(D) \neq 1$. Let L be a subgroup of $P \cap Z(D)$ of order p. Since L is c-permutable in G, for every prime number $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $LQ = QL$. Since L is subnormal in LQ and L is a Sylow p-subgroup of LQ, we have L is normal in LQ. So $Q \leqslant N_G(L)$. On the other hand, $D \leqslant N_G(L)$. By the arbitrary choice of q, we see that $L \trianglelefteq G$. Since P is the minimal normal subgroup of G, we have $P = L$. Thus, P is a cyclic subgroup of order p.

Lemma 2.7 [3, Lemma 3.1]. *Let N and L be normal subgroups in G such that P/L is a Sylow p-subgroup in NL/L and M/L is a maximal subgroup in P/L. If P_p is a Sylow p-subgroup in $P \cap N$, then P_p is a Sylow p-subgroup in N such that $D = M \cap N \cap P_p$ is a maximal subgroup in P_p and $M = LD$.*

Lemma 2.8 [2, Theorem 3.8]. For a group G, the following statements are equivalent:

(i) G is supersoluble;

(ii) Every maximal subgroup of G is c-permutable in G;

(iii) G is soluble and every maximal subgoup of G is c-permutable with every maximal subgroup of every Sylow subgroup of G.

3. Main Results

Theorem 3.1. *Let G be a group. If every minimal subgroup of G is completely c-permutable in G and every Sylow 2-subgroup of G is abelian, then G is supersoluble.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order.

It is obvious that the hypotheses is inherited by all proper subgroups of G. So G is a minimal non-supersoluble group, by Lemma 2.2 and [4, Theorem 3.11.8], we have:

1) G has a normal Sylow p-subgroup P, for some prime divisor p of $|G|$.

2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

3) If P is abelian, then $\Phi(P) = 1$.

4) If $p > 2$, then the exponent of P is p. If $p = 2$, then the exponent of P is 2 or 4.

5) $P/\Phi(P)$ is not cyclic.

If $p = 2$, then by the hypothesis, P is an elementary abelian p-group. Therefore, in any cases, the exponent of P is p. Since every minimal subgroup of G is completely c-permutable in G. By Lemma 2.1(1), we see that every minimal subgroup of $P/\Phi(P)$ is completely c-permutable in $G/\Phi(P)$. Hence, $P/\Phi(P)$ is a cyclic subgroup of order p by Lemma 2.6. This contradicts to that $P/\Phi(P)$ is not cyclic. The proof is completed.

Theorem 3.2. *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups and G a group. Then, $G \in \mathcal{F}$ if and only if there exists a normal subgroup N of G such that $G/N \in \mathcal{F}$, every minimal subgroup of N is completely c-permutable in G and every Sylow 2-subgroup of N is abelian.*

Proof. The necessity part is obvious, we only need to prove the sufficiency part.

Assume that the theorem is false and let G be a counterexample of minimal order. Since $\mathcal{F}$ is a saturated formation, we have $G^{\mathcal{F}} \nsubseteq \Phi(G)$. Let M be a $\mathcal{F}$-abnormal maximal subgroup of G. Then, $G = MG^{\mathcal{F}}$. Since $G/G^{\mathcal{F}} \cong M/(M \cap G^{\mathcal{F}}) \in \mathcal{F}$, we obtain that $M^{\mathcal{F}} \subseteq G^{\mathcal{F}} \subseteq N$. By Lemma 2.1, we know that M and $M \cap G^{\mathcal{F}}$ satisfy the conditions of the theorem. Hence, by the choice of G, we have $M \in \mathcal{F}$. On the other hand, in view of theorem 3.1, we know that $G^{\mathcal{F}}$ is a soluble group. Thus, by using Lemma 2.2, we have:

1) $G^{\mathcal{F}}$ is a p-subgroup for some prime divisor p of $|G|$.

2) $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$.

3) If $G^{\mathcal{F}}$ is abelian, then $G^{\mathcal{F}}$ is an elementary abelian p-group.

4) If $p > 2$, then the exponent of $G^{\mathcal{F}}$ is p. If $p = 2$, then the exponent of $G^{\mathcal{F}}$ is 2 or 4.

5) $\Phi(G) \subseteq Z_{\mathcal{F}}(G)$.

We first show that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a cyclic subgroup of order p. In fact, by 2), $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal p-subgroup of $G/\Phi(G^{\mathcal{F}})$. By Lemma 2.1(1), we see that every minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is completely c-permutable in $G/\Phi(G^{\mathcal{F}})$. It follows from Lemma 2.6 that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a cyclic subgroup of order p.

We now prove that M is a complement to $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ in G, that is, $G = MG^{\mathcal{F}}$ and $M \cap G^{\mathcal{F}} \leqslant \Phi(G^{\mathcal{F}})$. Indeed, since $G = MG^{\mathcal{F}}$, we only need to show that $M \cap G^{\mathcal{F}} \leqslant \Phi(G^{\mathcal{F}})$. For $M \cap G^{\mathcal{F}} \trianglelefteq M$, we know that $(M \cap G^{\mathcal{F}})\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}})$. By the minimality of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$, we have $M \cap G^{\mathcal{F}} = G^{\mathcal{F}}$ or $M \cap G^{\mathcal{F}} \leqslant \Phi(G^{\mathcal{F}})$. If $M \cap G^{\mathcal{F}} = G^{\mathcal{F}}$, then $G^{\mathcal{F}} \leqslant M$, this implies that $G = G^{\mathcal{F}}M = M$, a contradiction. So $M \cap G^{\mathcal{F}} \leqslant \Phi(G^{\mathcal{F}})$.

By [1, A, (15.5)] we know that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}}) \cong Soc(G/M_G) = \overline{H}$ and $C_{\overline{G}}(\overline{H}) = \overline{H}$, where $\overline{G} = G/M_G$. It follows that $\overline{H}$ is a cyclic subgroup of order p and $\overline{G}/C_{\overline{G}}(\overline{H}) = \overline{G}/\overline{H} \lesssim Aut(\overline{H})$ is a cyclic subgroup whose order divides $p-1$. Thus, $G/M_G \in \mathcal{U} \subseteq \mathcal{F}$, and so $G/M_G \cap G^{\mathcal{F}} \in \mathcal{F}$. This induces that $G \in \mathcal{F}$. This contradiction completes our proof.

Corollary 3.3. *A group G is supersoluble if and only if there exists a normal subgroup N of G such that G/N is supersoluble, every minimal subgroup of N is completely c-permutable in G and every Sylow 2-subgroup of N is abelian.*

Theorem 3.4. *Let p be the smallest prime dividing the order of a group G and G is A_4- free. If there exists a normal subgroup N of G such that G/N is p-nilpotent and every subgroup of N of order p^2 is completely c-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then we proceed the proof by providing the following claims:

(i) The hypothesis is inherited by all proper subgroups of G.

In fact, for all $H < G$, we have $H/H \cap N \cong HN/N \leqslant G/N$, thus $H/H \cap N$ is p-nilpotent. If $|H \cap N|_p \leqslant p^2$, then H is p-nilpotent by Lemma 2.5. So we can assume that $|H \cap N| > p^2$. Let P_1 be a subgroup of order p^2 of $H \cap N$. Then P_1 is also a subgroup of order p^2 of N, thus P_1 is completely c-permutable in G by the hypothesis. Then P_1 is completely c-permutable in N by Lemma 2.1. Hence, H satisfies the hypothesis of the theorem. The minimal choice of G implies that H is p-nilpotent. This shows that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. By Lemma2.3, G is a Schmidt group.

(ii) $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G. Furthermore, p^3 divides the order of P.

It follows directly from Lemma 2.4 and Lemma 2.5.

(iii) $P \leqslant N$.

Since $P \cap N \trianglelefteq G$, $(P \cap N)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. But $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$ by Lemma 2.5, we have $(P \cap N)\Phi(P) = \Phi(P)$ or $(P \cap N)\Phi(P) = P$. If $(P \cap N)\Phi(P) = \Phi(P)$, then $P \cap N \subseteq \Phi(P)$. For $G/P \cap N \lesssim G/P \times G/N$ is p-nilpotent, $G/\Phi(P) \cong (G/P \cap N)/(\Phi(P)/P \cap N)$ is p-nilpotent. This implies that G is p-nilpotent, a contradiction. Hence $(P \cap N)\Phi(P) = P$, and so $P \leqslant N$.

(iv) Final contradiction.

Let a be an element of $Z(P)$ of order p. By Lemma 2.4, $exp\ P = p$ or 4. If $exp\ P = p$, then for any element $x \in P \setminus \langle a \rangle$, the subgroup $\langle x \rangle \langle a \rangle$ is of order p^2. By (iii), $\langle x \rangle \langle a \rangle$ is completely c-permutable in G, thus, $\langle x \rangle \langle a \rangle Q_1 < G$ by (ii), where $Q_1 \in Syl_q(G)$. By (i), thus, $Q_1 \leqslant N_G(\langle x \rangle \langle a \rangle)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P/\Phi(P)$ is an elementary abelian p-group. Therefore, $\langle x \rangle \langle a \rangle \Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$. This implies that $\langle x \rangle \langle a \rangle \Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. If $\langle x \rangle \langle a \rangle \Phi(P) = P$, then $\langle x \rangle \langle a \rangle = P$, that is $|P| = p^2$, which contradicts to (ii). If $\langle x \rangle \langle a \rangle \Phi(P) = \Phi(P)$, then $\langle x \rangle \langle a \rangle \subseteq \Phi(P)$. By the arbitrary choice of x, we have $P \subseteq \Phi(P)$, a contradiction. Now assume $exp\ P = 4$. Let $x \in P$ be an element of order 4. Using the same proof of above, we have that $\langle x\rangle\Phi(P) \trianglelefteq G$ and hence $P = \langle x\rangle$. It follows from Lemma 2.5 that G is 2-nilpotent, a contradiction. This completes the proof of the theorem.

Theorem 3.5. *Let p be a prime number dividing the order of a group G. If there exists a normal p-soluble subgroup H of G such that G/H is p-nilpotent, every maximal subgroup of every Sylow subgroup of H is c-permutable in G and $(|G|, p-1) = 1$, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) If N is a minimal normal subgroup of G, then G/N is p-nilpotent.

In fact, NH/N is a p-nilpotent normal subgroup of G/N such that the factor group $(G/N)/(NH/N) \cong G/NH \cong (G/H)/(NH/H)$ is p-nilpotent. Let P/N be a Sylow p-subgroup in NH/N and M/N be a maximal subgroup in P/N. If P_p is a Sylow p-subgroup in $P \cap H$, then, by Lemma 2.7, P_p is a Sylow p- subgroup in H such that $L = M \cap H \cap P_p$ is a maximal subgroup in P_p and $M = NL$. Thus, by the hypothesis, L is c-permutable in G. By Lemma 2.1, we have $M/N = LN/N$ is c-permutable in G/N. This shows that the conditions of the theorem are inherited by G/N. Hence, the choice of G implies G/N is p-nilpotent.

(2) G has a unique minimal normal subgroup N and $N = C_G(N) = O_p(G) = F(G) \nsubseteq \Phi(G)$.

If N is a p'-group, then G is p-nilpotent, a contradiction. So N is a p-group. Since the class of all p-nilpotent groups is a saturated formation, by (1) we see that (2) holds.

(3) $|N| = p$.

If $|N| = p^\alpha$, for some natural number $\alpha > 1$. Let P be a Sylow p-subgroup of G. Since $N \nsubseteq \Phi(G)$, we have $N \nsubseteq \Phi(P)$. Hence, there exists a maximal subgroup P_1 of P such that $N \nsubseteq P_1$. For $N \subseteq H$, we have $P_1 \cap H$ is a maximal subgroup of some Sylow p-subgroup of H. Thus, by the hypothesis, for every Sylow q-subgroup Q of G, there exists an element $x \in G$ such that $(P_1 \cap H)Q^x = Q^x(P_1 \cap H)$. Let $Q_1 = Q^x$. Then, $(P_1 \cap H)Q_1 = Q_1(P_1 \cap H)$ and $P_1 \cap H$ is a Sylow q-subgroup of $(P_1 \cap H)Q_1$. Hence, $N \cap P_1 = N \cap (P_1 \cap H)Q_1 \lhd (P_1 \cap H)Q_1$. We also have $N \cap P_1 \lhd P$. This shows that $N \cap P_1 \lhd G$. But, N is a minimal normal subgroup of G, so we have $N \cap P_1 = 1$. Thus $|N| = p$.

(4) Final contradiction.

Since $N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$, we see that the order of $N_G(N)/C_G(N)$ must divide $(|G|, p-1) = 1$. So $N_G(N) = C_G(N)$. By Burnside Theorem, G is p-nilpotent. The final contradiction completes the proof.

Corollary 3.6. *Let p be the smallest prime number dividing the order of a p-soluble group G. If every maximal subgroup of every Sylow subgroup of G is c-permutable in G, then G is p-nilpotent.*

Corollary 3.7. *Let G be a p-soluble group. If every maximal subgroup of every Sylow subgroup of G is c-permutable in G, then G is a Sylow tower group of supersoluble type.*

Proof. Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. By Corollary 3.6, G is p-nilpotent. Let N be a normal p-complement of G. Clearly, N satisfies the hypotheses of G. therefore, by induction, N is a Sylow tower group of supersoluble type. This proves that G is a Sylow tower group of supersoluble type.

Theorem 3.8. *Let G be a soluble group. If for every $p \in \pi(G)$, there exists $P \in Syl_p(G)$ such that*

(i) $N_G(P)/C_G(P)$ is a p-group,

(ii) every maximal subgroup of P is c-permutable with every maximal subgroup of G.

Then G is nilpotent.

Proof. Assume the result is false and let G be a counterexample of minimal order. Then, we have:

(1) For every Sylow p-subgroup P^* of G, both (i) and (ii) hold.

In fact, by Sylow Theorem, there exists an element $x \in G$ such that $P^* = P^x$. Since $N_G(P^*) = N_G(P^x) = N_{G^x}(P^x) = (N_G(P))^x$ and $C_G(P^*) = C_G(P^x) = C_{G^x}(P^x) = (C_G(P))^x$, $N_G(P^*)/C_G(P^*) \cong N_G(P)/C_G(P)$ is a p-group. Assume $P_1^* < \cdot P^* = P^x$, then $(P_1^*)^{x^{-1}} < < \cdot P$. Let M be a maximal subgroup of G. By the hypotheses of the theorem, $(P_1^*)^{x^{-1}}$ is c-permutable with M , so P_1^* is c-permutable with M^x by Lemma 2.1(4).

(2) G is supersoluble by Lemma 2.8. Hence, if q is the largest prime factor of $|G|$, $Q \in Syl_q(G)$, then $Q \trianglelefteq G$.

(3) If N is a minimal normal subgroup of G, then G/N satisfies the hypothesis and so G/N is nilpotent.

Consider $\overline{G} = G/N$. Let $\overline{P} \in Syl_p(\overline{G})$, then there exists a Sylow p-subgroup P in G such that $\overline{P} = PN/N$. Then $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$. Since $C_{\overline{G}}(\overline{P}) \geqslant C_G(P)N/N$, we see that $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is a p-subgroup. Suppose that N is a r-group and $P_1/N < \cdot PN/N$. If $r = p$, then $N \leqslant P$. So P_1/N is a maximal subgroup of $PN/N = P/N$, and consequently, $P_1 < \cdot P$. Thus, by (ii), P_1 is c-permutable with every maximal subgroup M of G. By Lemma 2.1, we know that P_1/N is c-permutable with M/N in G/N. If $r \neq p$, then $P_1 = = P_1 \cap PN = (P_1 \cap P)N$. It is easy to see that $P_1 \cap P < \cdot P$. So $P_1 \cap P$ is c-permutable with M, and consequently, P_1/N is c-permutable with M/N in G/N by Lemma 2.1. Therefore, G/N satisfies the conditions of the hypothesis. By the choice of G, we have that G/N is nilpotent.

(4) $N = Q$.

Since the class of all nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. It follows that $F(G) = N = C_G(N)$. Since $Q \leqslant \leqslant F(G) = N$ and N is the unique minimal normal subgroup of G, we see that $N = Q$ and so $Q \leqslant C_G(Q)$. In view of the condition (i), $N_G(Q)/C_G(Q)$ is a q-group, we have $N_G(Q) = C_G(Q) = G$. It follows that $Q \leqslant Z(G)$. Since G/Q is nilpotent. We obtain that $G/Z(G) \cong (G/Q)/(Z(G)/Q)$ is nilpotent. Hence G is nilpotent. This contradiction completes the proof of the theorem.

Abstract. A subgroup is called conditionally permutable(or in brevity, c-permutable) in G if for every subgroup T of G there exists an element $x \in G$ such that $HT^x = T^xH$. We obtain some results about the c-permutable subgroups and use them to determine the structures of some finite groups.

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