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On Modular Identity for Local Fitting Classes[1]

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1. Introduction

As early as 1960s, Neumenn [8] and Shmel'kin [11] have studied the structure of the variety of groups. Later on, some of their results were extended to other classes of groups. For example, Barnes, Humphrey and Castinean-Hilles [1] proved that the lattice of all soluble Schunck classes of Lie algebras over some fixed field a distributive lattice; Sel'kin [9] proved that lattices of Schunck classes of finite groups is modular lattice; Skiba [12] proved that the lattice of all formations of finite groups and the lattice of all local formationss of finite groups are modular lattice. The author and Shum [4] proved that the lattice of universal algebras of Mal'cev algebras satisfying the maximal condition for subalgebras is distributive and its sublattice consisting of all Schunck classes of finite algebras is algebraic. The author and Shum [5] also proved that for every subsystem functor τ, the lattice of all τ-closed formations of finite algebras is modular ([5, Corollary 3.11]).

However, the problem of "whether the lattice of all soluble Fitting classes of finite groups is a modular or not" up to now is not resolved. The problem has been placed in Kourovka Notebook [13, Problem 14.47].

In this paper, we give a condition under which the modular law of the lattice of local Fitting classes holds.

All groups considered in this paper are finite. All unexplained notations and terminologies are standard. The reader is referred to the text of K.Doerk and T.Hawkes [3] or Guo [6] for notations and terminologies not given in this paper.

2. Preliminaries

A class of groups is called a Fitting class provided the following two conditions are satisfied:

(1) if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $N \in \mathcal{F}$.

(2) if $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 \in \mathcal{F}$, then $N_1N_2 \in \mathcal{F}$.

Condition (2) of the definition says that there is a unique maximal normal $\mathcal{F}$-subgroup of G which is called the $\mathcal{F}$-radical of G and denoted by $G_{\mathcal{F}}$.

Let $\mathcal{F}$ and $\mathcal{H}$ be Fitting classes. Then the class $\mathcal{F}\mathcal{H} = (G : G/G_{\mathcal{F}} \in \mathcal{H})$ is called the product of the Fitting classes $\mathcal{F}$ and $\mathcal{H}$. It is well known that the product of two Fitting classes is a Fitting class and the multiplication of Fitting classes satisfies associative law.

Recall that a class $\mathcal{F}$ of groups is called a formation if it is closed under homomorphic image and also subdirect product. It is clear that for a non-empty formation $\mathcal{F}$, every group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient is in $\mathcal{F}$. Let $\mathcal{M}$ and $\mathcal{H}$ be two non-empty formations. We define their product $\mathcal{M}\mathcal{H} = (G : G^{\mathcal{H}} \in \mathcal{M})$. It is well known that the product of two formations is again a formation. We denote by $\mathcal{E}_{p'}$ the formation of all finite p'-groups and $\mathcal{N}_p$ the formation of all p-groups. Let $F^p(G) = G^{\mathcal{N}_p\mathcal{E}_{p'}}$.

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A function f defined by $f : \mathbb{P} \longrightarrow \{\text{Fitting classes}\}$ is called a Hartley function (or in brevity, H-function) (see [10]). For a H-function f, let $LR(f) = (G : F^p(G) \in f(p)$ for all $p \in \in \pi(G))$. A Fitting class $\mathcal{F}$ is called a local [10], if there exists a H-function f such that $\mathcal{F} = LR(f)$. In this case, we say that $\mathcal{F}$ is locally defined by f or f is a H-function of $\mathcal{F}$.

We denote by $\mathcal{F} \vee \mathcal{H}$ the minimal Fitting containing $\mathcal{F} \cup \mathcal{H}$, and by $\mathcal{F} \vee_l \mathcal{H}$ the minimal local Fitting class containing $\mathcal{F} \cup \mathcal{H}$.

It is easy to see that the set $\mathcal{L}$ of all Fitting classes is a partially ordered set by the inclusion relation. Moreover, any two-elements $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{L}$ have a least upper bound $\mathcal{F} \vee \mathcal{H}$ and a greatest lower bound $\mathcal{F} \cap \mathcal{H}$. Therefore, $\mathcal{L}$ is a lattice. It is also clear that the set of all soluble Fitting classes and the set of all soluble local Fitting classes are lattices.

For the set of all H-functions, we define a partial ordering: $f_1 \leqslant f_2$ if and only if $f_1(p) \subseteq f_2(p)$ for every prime p. Let $\mathcal{F}$ be a local Fitting class and f a H-function of $\mathcal{F}$. We say that f is a integrated if $f(p) \subseteq \mathcal{F}$ for all $p \in \mathbb{P}$.

Let Ω be the set of all H-functions of $\mathcal{F}$. Then, the least element in Ω is called the least H-function of $\mathcal{F}$.

Let $\mathcal{X}$ be a class of groups. Then $S_n(\mathcal{X})$ denotes the set of all normal subgroups of all groups in $\mathcal{X}$; Fit$\mathcal{X}$ denotes the least Fitting class containing $\mathcal{X}$, that is, Fit$\mathcal{X} = \cap \cap\{\mathcal{F} : \mathcal{F}$ is a Fitting class and $\mathcal{F} \supseteq \mathcal{X}\}$. and lFit$\mathcal{X}$ denotes the least local Fitting class containing $\mathcal{X}$, that is, lFit$\mathcal{X} = \cap\{\mathcal{F} : \mathcal{F}$ is a local Fitting class and $\mathcal{F} \supseteq \mathcal{X}\}$.

3. Lemmas

Lemma 3.1. Let $\mathcal{X}$ be a nonempty set of soluble groups and $\mathcal{F} = l\mathrm{Fit}\mathcal{X}$. Then the following statements hold.

(a) $\mathcal{F}$ has the least H-function f.

(b) For the least H-function f, we have

$$f(p) = \begin{cases} \mathrm{Fit}(\mathcal{X}(F^p)), & \text{if } p \in \pi(\mathcal{X}), \\ \varnothing, & \text{if } p \in \mathbb{P} \backslash \pi(\mathcal{X}), \end{cases}$$

where $\mathcal{X}(F^p) = \mathrm{Fit}(F^p(G) : G \in \mathcal{X})$.

Proof. (a) Let Ω be the set of all H-functions of the local Fitting class $\mathcal{F}$ and $f = \cap_{i \in I} f_i$, the intersection of all H-functions of $\mathcal{F}$. Then, obviously, $f \leqslant f_i$ for all $i \in I$ and $f_i \in \Omega$. We prove that $LR(f) = \mathcal{F}$. In fact, since $f \leqslant f_i$, $LR(f) \subseteq LR(f_i) = \mathcal{F}$. On the other hand, let $G \in \mathcal{F}$. Since $\mathcal{F} = LR(f_i) = (X : X^{\mathcal{N}_p \mathcal{E}_{p'}} \in f_i(p)$, for all $p \in \pi(X))$, for all $i \in I$, we have that $G \in \cap LR(f_i) = (Y : Y^{\mathcal{N}_p \mathcal{E}_{p'}} \in \cap_{i \in I} f_i(p) = f(p)) = LR(f)$. This shows that $\mathcal{F} \subseteq LR(f)$ and hence $LR(f) = \mathcal{F}$.

(b) It follows from [10, Lemma 22]. The proof is completed.

Let $\{f_i : i \in I\}$ be an arbitrary set of H-functions. Then, we denote by $\vee(f_i : i \in I)$ the H-function f such that $f(p) = \mathrm{Fit}(\cup_{i \in I} f_i(p))$ for all $p \in \mathbb{P}$, and denote by $\cap_{i \in I} f_i$ the H-function f such that $f(p) = \cap_{i \in I} f_i(p)$ for all $p \in \mathbb{P}$.

Lemma 3.2. Let f_i be the least H-function of Fitting class $\mathcal{F}_i$, $i \in I$. Then $\vee(f_i : i \in I)$ is the least H-function of the Fitting class $\mathcal{F} = \vee_l(\mathcal{F}_i : i \in I)$.

Proof. Let $\pi = \pi(\cup_{i \in I} \mathcal{F}_i) = \cup_{i \in I}(\pi(\mathcal{F}_i)) = \pi(\mathcal{F})$, $f = \vee(f_i : i \in I)$ and h be the least H-function of the Fitting class $\mathcal{F}$. We only need to prove that $h = f$.

Let $p \in \mathbb{P} \setminus \pi$. Then by Lemma 3.1, for every $i \in I$, $f_i(p) = \varnothing$ and $h(p) = \varnothing$. It follows that $f(p) = \varnothing = h(b)$. Now assume that $p \in \pi$. Then there exists $i \in I$ such that $f_i(p) \neq \varnothing$. By Lemma 3.1, we know that $h(p) = \mathrm{Fit}(\mathcal{F}(F^p))$, for all $p \in \pi$. However, since $\mathrm{Fit}(\mathcal{F}(F^p)) = \mathrm{Fit}(F^p(G) : G \in \cup_{i \in I} \mathcal{F}_i)$, we have $h(p) = \mathrm{Fit}(\cup_{i \in I} \mathrm{Fit}(F^p(G)) : G \in$

$\in \mathcal{F}_i) = \mathrm{Fit}(\cup_{i\in I} f_i(p)) = \vee(f_i : i \in I)(p) = f(p)$. This shows that $h = f$. The proof is completed.

The following lemma can be analogously verified (cf. [10, Lemma 21]).

Lemma 3.3. Let $\mathcal{F} = \cap_{i\in I}\mathcal{F}_i$, where $\mathcal{F}_i = LR(f_i)$. Then $\mathcal{F} = LR(f)$, where $f = \cap_{i\in I} f_i$.

4. The Main Results

Theorem 4.1. Let $\mathcal{X} = LR(x), \mathcal{Y} = LR(y)$ and $\mathcal{F} = LR(f)$ be local Fitting classes with least H-function x, y and f respectively, and $x \leqslant f$. If x and y satisfy that $x(p) \vee y(p) = S_n(G : G = G_{x(p)}G_{y(p)})$ for all $p \in \mathbb{P}$, where $x(p) \neq \varnothing$ and $y(p) \neq \varnothing$, then the following modular law hold:

$$(\mathcal{X} \vee_l \mathcal{Y}) \cap \mathcal{F} = \mathcal{X} \vee_l (\mathcal{Y} \cap \mathcal{F}).$$

Proof. We firstly prove that the modular law hold for the H-functions x, y and f.

Since $x \leqslant f$, we have that $x(p) \subseteq f(p)$ for all $p \in \mathbb{P}$. It is easy to see that $x \leqslant x \vee f$ and $x \leqslant x \vee y$. Hence $x \leqslant (x \vee f) \cap (x \vee y)$. Analogously, $y \cap f \leqslant f \leqslant x \vee f$ and hence $y \cap f \leqslant (x \vee y) \cap (x \vee f)$. This shows that $x \vee (y \cap f) \leqslant (x \vee y) \cap (x \vee f)$. By $x \vee f = f$, we have

$$x \vee (y \cap f) \leqslant (x \vee y) \cap f. \tag{1}$$

We now prove the reverse inclusion: $(x \vee y) \cap f \leqslant x \vee (y \cap f)$. It only need to prove that $((x \vee y) \cap f)(p) \subseteq (x \vee (y \cap f))(p)$, for every $p \in \mathbb{P}$. Obviously, if $f(p) = \varnothing$ or $x(p) \vee y(p) = \varnothing$, then $((x \vee y) \cap f)(p) \subseteq (x \vee (y \cap f))(p)$. Hence, we can assume that $x(p) \vee y(p)$ and $f(p)$ are all not empty.

If $x(p) = \varnothing$, then $(x(p) \vee y(p)) \cap f(p) = y(p) \cap f(p)$ and $x(p) \vee (y(p) \cap f(p)) = y(p) \cap f(p)$, and consequently, $((x \vee y) \cap f)(p) \subseteq (x \vee (y \cap f))(p)$.

Assume that $y(p) = \varnothing$. Then $(x(p) \vee y(p)) \cap f(p) = x(p) \cap f(p) = x(p)$. On the other hand, $x(p) \vee (y(p) \cap f(p)) = x(p)$. Hence, in this case, we also have that $((x \vee y) \cap f)(p) \subseteq (x \vee (y \cap f))(p)$.

Now, we assume that $x(p), y(p)$ and $f(p)$ are all not empty. Let $K \in (x(p) \vee y(p)) \cap f(p)$. Then, there exists a group $G = G_{x(p)}G_{y(p)}$ such that $K \trianglelefteq G$ and $K \in f(p)$. It follows that $K \trianglelefteq G_{f(p)} = G_{x(p)}G_{y(p)} \cap G_{f(p)} = G_{x(p)}(G_{y(p)} \cap G_{f(p)}) = G_{x(p)}G_{y(p)\cap f(p)}$. Therefore $K \in x(p) \vee (y(p) \cap f(p))$, and consequently, $((x \vee y) \cap f)(p) \subseteq (x \vee (y \cap f))(p)$, for every $p \in \mathbb{P}$. This shows that

$$(x \vee y) \cap f \leqslant x \vee (y \cap f). \tag{2}$$

By the equalities (1) and (2), we obtained that

$$(x \vee y) \cap f = x \vee (y \cap f). \tag{3}$$

By Lemma 3.2, we know that $\mathcal{X} \vee_l \mathcal{Y} = LR(x \vee y)$. Then, by Lemma 2.3, we have $(\mathcal{X} \vee_l \mathcal{Y}) \cap \mathcal{F} = LR((x \vee y) \cap f)$. On the other hand, $\mathcal{Y} \cap \mathcal{F} = LR(y \cap f)$ and $\mathcal{X} \vee_l (\mathcal{Y} \cap \mathcal{F}) = LR(x \vee (y \cap f))$. Thus, by using the equality (3), we obtain that

$$\mathcal{X} \vee_l (\mathcal{Y} \cap \mathcal{F}) = (\mathcal{X} \vee_l \mathcal{Y}) \cap \mathcal{F}.$$

This completes the proof.

It is well known that every nonempty Fitting class $\mathfrak{F}$ can be compared with the Fitting classes $\mathfrak{F}^*$ (cf. Lockett [7]), where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that the $\mathfrak{F}^*$-radical of the direct product $G \times H$ of the groups G and H is equal to the direct product of the $\mathfrak{F}^*$-radical of G and the $\mathfrak{F}^*$-radical of H, for all groups G and H. Let f be a H-function. We define a H-function f^* as follows: $f^*(p) = (f(p))^*$, for all $p \in \mathbb{P}$.

Abstract. Let $\mathcal{X} = LR(x), \mathcal{Y} = LR(y)$ and $\mathcal{F} = LR(f)$ be local Fitting classes with least H-function x, y and f respectively, and $x \leqslant f$. In this paper, we prove that if x and y satisfy that $x(p) \vee y(p) = S_n(G : G = G_{x(p)}G_{y(p)})$ for all $p \in \mathbb{P}$, where $x(p) \neq \varnothing$ and $y(p) \neq \varnothing$, then the following modular law hold: $(\mathcal{X} \vee_l \mathcal{Y}) \cap \mathcal{F} = \mathcal{X} \vee_l (\mathcal{Y} \cap \mathcal{F})$.

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