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On s-Semipermutable Subgroups[1]

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1. Introduction

A subgroup H of a group G is called quasinormal in G, if H permutable with all subgroups of G. A subgroup H of a finite group G is called s-quasinormal in G, if H is permutable with all Sylow subgroup of G. This concept was first introduced by Kegel [6]. As a generalization of s-quasinormality, the concept of s-semipermutability (or s-seminormality) was introduced in [1]. A subgroup G is called s-semipermutable in G, if H is permutable with every Sylow p-subgroup of G with $(|H|, p) = 1$. In this paper we use s-semipermutable subgroups to study the structure of finite groups.

All the groups considered in this paper are finite. All unexplained notations and terminologies are standard and can be found in Guo [3] and Shemetkov [9].

2. Elementary properties

A class of groups $\mathfrak{F}$ is called a formation, if $\mathfrak{F}$ satisfies the following two conditions:
(1) If $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$;
(2) If $G/N \in \mathfrak{F}$ and $G/M \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$.

It is clear that for a non-empty formation $\mathfrak{F}$, every group G has the smallest normal subgroup N (denoted by $G^{\mathfrak{F}}$) whose quotient group $G/N \in \mathfrak{F}$. The normal subgroup $G^{\mathfrak{F}}$ is called the $\mathfrak{F}$-residual of G. A formation $\mathfrak{F}$ is said to be a saturated formation if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. It is well known that the class $\mathfrak{N}$ of all nilpotent groups and the class $\mathfrak{U}$ of all supersoluble groups are both saturated formations.

The product of all nilpotent normal subgroups of a group G is also a nilpotent normal subgroup, denoted by $F(G)$, which is called the Fitting subgroup of G. Let $C = C_G(F(G))$, $Z = Z(F(G))$ and $M/Z = Soc(C/Z)$. We called $F^*(G) = MF(G)$ is the generalized Fitting subgroup of G (of [10, V, Def 4.9]). It is well known that if G is a soluble group, then $F^*(G) = F(G)$

A subgroup H of group G is called H a semipermutable subgroup if $HK = KH$ for every subgroup K such that $(|H|, |K|) = 1$. A subgroup H of G is called a s-semipermutable subgroup of G if $HP = PH$ for every Sylow p-subgroup such that $(p, |H|) = 1$ (see [1]).

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 [11]. *Let H be a s-semipermutable subgroup of G.*
(1) If $H \leqslant T \leqslant G$, then H is s-semipermutable in T;
(2) If H is a p-subgroup and $H \trianglelefteq G$, then HK/K is s-semipermutable in G/K.

Lemma 2.2 . *Let $\mathfrak{F}$ be be an S-closed saturated formation and H a subgroup of G. Then $H \cap Z_\infty^{\mathfrak{F}}(G) \subseteq Z_\infty^{\mathfrak{F}}(H)$.*

Lemma 2.3 [8, Lemma 3.9]. *Let $\mathfrak{F} = \mathfrak{G}_{q'}\mathfrak{N}_q$ is p-nilpotent class. If G is a minimal non-$\mathfrak{F}$-group, then the following statements hold:*

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(1) G is a Schmidt group;

(2) $G^{\mathfrak{F}}$ is a Sylow subgroup of G.

Lemma 2.4 [7, Lemma 2.8]. *Suppose G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$, then $C_G(P) \geqslant O^p(G)$, where $O^p(G)=\langle g \in G \mid p \nmid |g| \rangle$.*

3. Main Results

Theorem 3.1. *Let G be a group with $p \mid |G|$. Then G is p-nilpotent if and only if there exists a normal subgroup H of G such that G/H is p-nilpotent, every cyclic subgroup of order 4 of H is s-semipermutable in G and every subgroup of H of order p is contained in $Z_\infty^{\mathfrak{F}}(G)$, where $\mathfrak{F}$ is the formation of all p-nilpotent groups.*

Proof. The necessity part is obvious, we only need to prove sufficiency part. Assume that the claim is false and choose G to be a counterexample of minimal order.

Let K be a proper subgroup of G. Then $K/K \cap H \simeq KH/H \leqslant G/H$ implies that $K/K \cap H$ is p-nilpotent. Every cyclic subgroup of $K \cap H$ of order 4 is s-semipermutable in G and hence is s-semipermutable in K by Lemma 2.1. Every subgroup of $H \cap K$ of order p is contained in $Z_\infty^{\mathfrak{F}}(G) \cap K \subseteq Z_\infty^{\mathfrak{F}}(K)$ by Lemma 2.2. So K satisfies the hypotheses, and K is p-nilpotent by the choice of G. Thus G is a minimal non-p-nilpotent group. Now [10,VIII,3.4] implies that G is a Schmidt group, that is, G is not nilpotent but every proper subgroup of G is nilpotent. Then by [3, Theorem 3.4.11], G has a normal Sylow p-subgroup P and $G/P \simeq Q$, where Q is a non-normal cyclic Sylow q-subgroup of G, $P/\Phi(P)$ is a minimal subgroup of $G/\Phi(P)$. We consider the following cases:

Case 1. P is abelian. By [3, Theorem 3.4.11], P is an elementary abelian p-subgroup. Since G/H is p-nilpotent, we have $P \leqslant H$ by Lemma 2.3. By the hypotheses, $P \leqslant Z_\infty^{\mathfrak{F}}(G)$, hence G is p-nilpotent, a contradiction.

Case 2. P is not abelian and $p > 2$. Then by [3, Theorem 3.4.11], the expotent of P is p. Since G/H is p-nilpotent, by Lemma 2.3, $P \subseteq H$. Hence, by the hypothesis, $P \leqslant Z_\infty^{\mathfrak{F}}(G)$. It follows that G is p-nilpotent, a contradiction.

Case 3. P is non-abelian and $p = 2$. Since $H \cap P \trianglelefteq G$, $(H \cap P)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. By [3, Theorem 3.4.11] $P/\Phi(P)$ is a minimal normal subgroup $G/\Phi(P)$, we have $(H \cap \cap P)\Phi(P) = \Phi(P)$ or $H \cap P = P$. If $H \cap P \subseteq \Phi(P)$, then $G/\Phi(P)$ is p-nilpotent, and so G is p-nilpotent, a contraction. If $H \cap P = P$, then $H \geqslant P$. If there exists $x \in P \backslash \Phi(P)$ and $|\langle x \rangle| = 4$, then by the hypotheses, $\langle x \rangle$ is s-semipermutable in G, that is $\langle x \rangle Q = Q \langle x \rangle$, where Q is a Sylow q-subgroup of G and $q \neq 2$. Obviously, $Q\langle x \rangle \Phi(P) \cap P = \langle x \rangle \Phi(P)$. For an arbitrary $q \in Q$, $(Q\langle x \rangle \Phi(P) \cap P)^q = Q\langle x \rangle q \Phi(P) \cap P = Q\langle x \rangle \Phi(P) \cap P$. This shows that Q normalizes $Q\langle x \rangle \Phi(P) \cap P = \langle x \rangle \Phi(P)$. Since $P/\Phi(P)$ is an elementary abelian p-subgroup, we have P normalizes $\langle x \rangle \Phi(P)$. So $1 \neq \langle x \rangle \Phi(P) \trianglelefteq [P]Q = G$. In view of minimal normality of $P/\Phi(P)$ we obtain that $\langle x \rangle \Phi(P) = P$, Therefore $P = \langle x \rangle$. The final contradiction completes the proof.

Theorem 3.2. *Suppose H is a normal subgroup of a group G such that G/H is nilpotent. If every cyclic subgroup of order 4 of $F^*(H)$ s-semipermutable in G. Then G is nilpotent if and only if every element of prime order of $F^*(H)$ is contained in $Z_\infty(G)$.*

Proof. If G is nilpotent, then $G = Z_\infty(G)$. Hence the necessity is obvious, we only need to prove the sufficiency.

Assume the claim is not true, and let G be a counterexample of minimal order. Then we prove it via the following steps.

(1) Every proper normal subgroup of G is nilpotent.

Let M is a maximal normal subgroup of G. Since $M/M \cap H \simeq MH/M \leqslant G/H$ is nilpotent, by [5,X,13.11] $F^*(M \cap H) \subseteq F^*(H)$ and by Lemma 2.2, we have $Z_\infty(G) \cap M \subseteq$ $\subseteq Z_\infty(M)$. Applying Lemma 2.1, we see that M and $M \cap H$ satisfies the hypotheses. The minimal choice of G implies that M is nilpotent. It follows that $F(G)$ is a unique maximal normal subgroup of G.

(2) $H = G = G'$ and $F^*(G) = F(G) < G$.

If $H < G$, then H is nilpotent by (1). Thus $F^*(H) = F(H) = H$. By Theorem 3.1 we have that G is nilpotent, a contradiction. Hence $H = G$, $F(G) < G$ and $G/F(G) = G/M$ is simple. If $G/M = G/F(G)$ is a cyclic subgroup of prime order, then G is nilpotent by Theorem 3.1, a contradiction. Hence we may assume that G/M is a non-abelian simple group. Then, obviously, we have $G' \nleq M$. It follows that $G = G'$. If $F(G) < F^*(G)$, then $F^*(H) = F^*(G) = G = H$. Again by Theorem 3.1 we have that G is nilpotent, a contradiction. Thus $F(G) = F^*(G)$.

(3) Final contradiction.

Since $F^*(G) = F(G)$ is not the identity group, we may choose a minimal prime divisor q of $|F(G)|$. Let Q be the Sylow q-subgroup of $F(G)$. Since Q char $F(G) \trianglelefteq G$, we have $Q \trianglelefteq G$. By the hypotheses, every element r of prime order of Q contains in $Z_\infty(G)$, thus $C_G(Q) \geqslant O^q(G)$ by Lemma 2.4. If $q \neq 2$, then $C_G(Q) \geqslant O^q(G)$ by [2, p.184, Theorem 3.10]. So $G/C_G(Q)$ is a q-group. If $q = 2$, for every subgroup $\langle x \rangle$ of order 4 of Q, by the hypotheses, $\langle x \rangle P$ is a subgroup of G, where P is a Sylow p-subgroup of G, and p is an arbitrary prime number with $p \neq 2$. Since $\langle x \rangle = \langle x \rangle (Q \cap P) = Q \cap \langle x \rangle P \trianglelefteq \langle x \rangle P$, we have $\langle x \rangle$ is normalized by P. Therefore $\langle x \rangle$ centralized by P by [2, p.178, Theorem 2.4]. By the arbitrary choice of P , we have $O^2(G) \leqslant C_G(Q)$ by [4, IV, Theorem 5.12]. So $G/C_G(Q)$ is a 2-group. Hence, in any case, we have $G/C_G(Q)$ is a q-group, where q is the smallest prime of $|F(G)|$.

We claim that $C_G(Q) = G$. In fact, if $C_G(Q) < G$, then by (1) $C_G(Q)$ is nilpotent, and $G/C_G(Q)$ is a q-group. It follows that G is soluble. But $G = G'$ implies that G is not soluble, a contraction. So $C_G(Q) = G$ and hence $Q \leqslant Z(G)$. It follows that $F^*(G/Q) =$ $= F^*(G)/Q$. Now consider the factor group $\overline{G} = G/Q$. Since Q is a Sylow q-subgroup of $F^*(G)$, and q is the smallest prime, every element $\overline{x}$ of prime order t in $F^*(\overline{G})$ can be view as the image of an element x of prime order t in $F^*(G)$, for every $t > q$. So x lies in $Z_\infty(G)$ by the hypotheses, and thus we have $\overline{x}$ lies in $Z_\infty(G/Q) = Z_\infty(G)/Q$. Obviously, $F^*(\overline{G})$ has no an element of order 2. It follows that G/Q satisfies the hypotheses. The minimal choice of G implies G/Q is nilpotent and so G is nilpotent, a contradiction. This completes the proof.

Corollary 3.2.1 [7,Theorem 4.5]. *Let H is a normal subgroup of a group G such that G/H is nilpotent. Suppose every cyclic subgroup of order 4 of $F^*(H)$ s-quasinormal in G. Then G is nilpotent if and only if all elements of prime order of $F^*(H)$ are contained in $Z_\infty(G)$.*

Corollary 3.2.2. *Let G be a group. Suppose that every cyclic subgroup of order 4 of $F^*(G')$ is s-semipermutable in G. Then G is nilpotent if and only if all elements of prime order of $F^*(G')$ are contained in $Z_\infty(G)$.*

Corollary 3.2.3. *Let G be a group. Suppose that every cyclic subgroup of order 4 of $F^*(G^{\mathfrak{N}})$ is s-semipermutable in G. Then G is nilpotent if and only if all elements of prime order of $F^*(G^{\mathfrak{N}})$ are contained in $Z_\infty(G)$.*

Abstract. A subgroup H of a finite group G is called s-semipermutable in G if H is permutable with every Sylow p-subgroup of G with $(p, |H|) = 1$. In this paper, we use s-semipermutable subgroups to study the structure of finite groups. In particular, we obtained

new criterions of p-nilpotency and nilpotency of a finite group. Some know results are generalized.

References

1. Chongmu Chen. Inner-outer $\sum$-groups and minimal non-$\sum$-groups [M]. Chongqing: Southwest Normal University Press, 1988.

2. D.Gorenstein. Finite Groups [M]. Chelsea, New York, 1968.

3. W.Guo. The Theory of Classes of Groups [M]. Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

4. B.Huppert. Endliche Gruppen I [M]. Berlin-Heidelberg-New York: Springer-Verlag, 1968.

5. B.Huppert. Blackburn N, Finite Group III [M]. Berliin-Heidelberg-New York, Springer- Verlag, 1982.

6. O.H.Kegel. Sylow-Gruppen and Subnormalteiler Endlicher Gruppen [J]. Math.Z, 1962, 78: 205-221.

7. Y.Li and Y.Wang. On π-quasinormally embedded subgroups of finite group [J]. J of Algebra, 2004, 281: 109-123.

8. L.Miao and W.Guo. The influence of c-normality of subgroups on the structure of finite groups [J]. Izvestiya of Gomel State University, Problems in Algebra, 2000, 3(16): 101-106.

9. L.A.Shemetkov. Formations of Finite Groups [M]. Moscow, Nauka, 1978.

10. M.Xu. A Introduction to Finite Groups [M]. Science Press, Beijing, 1999.

11. Q.Zhang and L.Wang. The influence of s-Semipermutable Subgroups on the Structure of Finite Groups [J]. J. of Math., 2005, 48(1): 81-88.

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