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On $\mathfrak{F}$-covering subgroups of finite groups

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1. Introduction. Throughout this paper, all groups considered are finite groups. Recall that a subgroup H of a group G is said to be a $\mathfrak{X}$-covering subgroup for a set $\mathfrak{X}$ of groups if $H \in \mathfrak{X}$ and $T = KH$ whenever $H \leq T \leq G$ and $T/K \in \mathfrak{X}$. A subgroup H of a group G is called a Carter subgroup of G if H is a nilpotent group and $N_G(H) = H$. It is well known that a Carter subgroup of G is an $\mathfrak{N}$-covering subgroup of G where $\mathfrak{N}$ is the class of all nilpotent groups (cf. [4], Theorem 2.2.1). In 1961, Carter [1] proved that a soluble group has a Carter subgroup and any two Carter subgroups of a soluble group are conjugate. In 1963, Gaschütz [3] introduced the concept of an $\mathfrak{F}$-covering subgroup where $\mathfrak{F}$ is a class of groups. This is the concept which enables us to generalize the Carter's result. Gaschütz [3] proved that if $\mathfrak{F}$ is a nonempty saturated formation and G a soluble group, then G has an $\mathfrak{F}$-covering subgroup and any two $\mathfrak{F}$-covering subgroups of G are conjugate. Later on, Erickson [2], Schmid [6] and Shemetkov [7] weakened the solubility condition and proved that if the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is $\pi(\mathfrak{F})$-soluble, then G has an $\mathfrak{F}$-covering subgroup and any two $\mathfrak{F}$-covering subgroups are conjugate in G. Let π be a set of prime numbers. By Chunikhin [4], A group G is called a π-selected group, if $|\pi(H/K) \cap \pi| \leqslant 1$ for every nonabelian chief factor H/K of G. Obviously, a π-soluble group is a π-selected group, but the converse does not hold. In connection with the above results, the following open problem has been proposed by W. Guo in his monograph [4]:

Open Problem([4], Problem 3-4). Let $\mathfrak{F}$ be a nonempty saturated formation and the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of a group G be a $\pi(\mathfrak{F})$-selected group. 1) Has G an $\mathfrak{F}$-covering subgroup? 2) If G has $\mathfrak{F}$-covering subgroups, are any two $\mathfrak{F}$-covering subgroups of G conjugate ?

In this paper, we will prove that the answer of the problem is affirmative.

All unexplained notations and terminologies are standard. The reader is referred to the text of Guo [4] and Shemetkov [7].

2. Elementary Properties. Let π be a set of some prime numbers and π' the complement of π in the set of all prime numbers. For a group G, let $|G|$ be the order of G, $\pi(G)$ the set of all prime divisors of the order of G, $O_p(G)$ the maximal normal p-subgroup of G, $F(G)$ the Fitting subgroup of G, $\Phi(G)$ the Frattini subgroup of G. $H \leq G$ denotes that H is a subgroup of G; $H \trianglelefteq G$ denotes that H is a normal subgroup of G; $[N]H$ denotes the semidirect product of groups N and H. Let $H \leq G$. We denote by H_G the largest normal subgroup of G contained in H.

Let $\mathfrak{X}$ be a class of groups. We set $\pi(\mathfrak{X}) = \bigcup_{G \in \mathfrak{X}} \pi(G)$. A subgroup H of a group G is said to be an $\mathfrak{X}$-covering subgroup of G if the following conditions are satisfied: 1) $H \in \mathfrak{X}$; 2) if $H \leq T \leq G$, $K \trianglelefteq T$ and $T/K \in \mathfrak{X}$, then $T = KH$. A subgroup H of a group G is said to be an $\mathfrak{X}$-projector of G if HN/N is an $\mathfrak{X}$-maximal subgroup of G/N whenever $N \trianglelefteq G$.

A class of groups $\mathfrak{F}$ is called a formation if $\mathfrak{F}$ is closed under homomorphic images and subdirect products. It is clear that for every non-empty formation $\mathfrak{F}$, every group G has the smallest normal subgroup with quotient in $\mathfrak{F}$. We call such a smallest normal subgroup of G the $\mathfrak{F}$-residual of G, and denote by $G^{\mathfrak{F}}$. It is also clear that for every formation $\mathfrak{F}$, any $\mathfrak{F}$-covering subgroup of a group G is an $\mathfrak{F}$-projector, but the converse does not hold (cf.[4], p.67).

Let p be a prime number. We denote by $\mathfrak{N}_p$ the formation of all p-groups. We call a formation $\mathfrak{F}$ p-local (or p-saturated), if $\mathfrak{N}_p\mathfrak{F}(p) \subseteq \mathfrak{F}$, i.e. $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}(p)} \subseteq \mathfrak{N}_p$, where

$$\mathfrak{F}(p) = \begin{cases} \varnothing, & \text{if } p \notin \pi(\mathfrak{F}), \\ form\{G/F_p(G) \mid G \in \mathfrak{F}\}, & \text{if } p \in \pi(\mathfrak{F}). \end{cases}$$

A formation $\mathfrak{F}$ is called a saturated formation (or local formation) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$.

A group G is called an E_π-group, if G has at least one Hall π-subgroup. A group is called a D_π-group if G has the Sylow π-property, i.e. G has Hall π-subgroups, any two Hall π-subgroups of G are conjugate in G and every π-subgroup of G is contained in a Hall π-subgroup of G. A group G is called an E_π^n-group, if G has a nilpotent Hall π-subgroup.

Lemma 2.1 (see [7], Lemma 1.2). $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$.

Lemma 2.2 (see [9]).*Let G be a π-selected group. Then every subgroup of G and every factor group of G is a π-selected groups.*

Lemma 2.3. (see [7], Lemma 4.2). *Let $\mathfrak{F}$ be a nonempty saturated formation. If $\mathfrak{N}_p \cap \cap \mathfrak{F} \neq 1$, then $\mathfrak{N}_p \subseteq \mathfrak{F}$.*

Lemma 2.4(see [4], Theorem 2.2.4). *Let $\mathfrak{F}$ be a nonempty saturated formation and G a group. Then the following statements hold.*

1) If H is an $\mathfrak{F}$-covering subgroups of G and $N \trianglelefteq G$, then HN/N is an $\mathfrak{F}$-covering subgroups of G/N;

2) If R/N is an $\mathfrak{F}$-covering subgroups of G/N and H is an $\mathfrak{F}$-covering subgroups of R, then H is an $\mathfrak{F}$-covering subgroup of G.

Recall that a group G is called a primitive group if it has a maximal subgroup M, such that $M_G = 1$. We say that a class of groups $\mathfrak{F}$ is primitively closed if the following condition is satisfied: if all primitive factor groups of a group G belong to $\mathfrak{F}$, then G also belongs to $\mathfrak{F}$.

Lemma 2.5 (Erickson [2]). *Let $\mathfrak{F}$ be a nonempty class of groups. Then every group has an $\mathfrak{F}$-projectors if and only if $\mathfrak{F}$ is closed under homomorphic images and $\mathfrak{F}$ is primitively closed.*

Lemma 2.6. *Let $\mathfrak{F}$ be a saturated formation. Then $\mathfrak{F}$ is primitively closed.*

Proof. Assume that all primitive factor groups of a group G belong to $\mathfrak{F}$. We prove that $G \in \mathfrak{F}$. If G is not primitive, then G/M_G is a primitive group for every maximal subgroup M of G. So, $G/M_G \in \mathfrak{F}$, for all maximal subgroups of G. It follows that $G/\cap M_G = G/\Phi(G) \in \mathfrak{F}$. Thus, $G \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated. This completes the proof.

Corollary 2.7. *If $\mathfrak{F}$ is a nonempty saturated formation, then every group G has an $\mathfrak{F}$-projector.*

Lemma 2.8 ([2]). *Let $\mathfrak{F}$ be a class of groups which is closed under homomorphic images and is primitively closed. If $G = F(G)E$ and $E \in \mathfrak{F}$, then G has an $\mathfrak{F}$-covering subgroup and E is contained in an $\mathfrak{F}$-covering subgroup of G.*

Lemma 2.9 ([5]). *Let G be a finite group and $H \trianglelefteq G$. If H is a E_π^n-group and G/H is a D_π-group, then G is a D_π-group.*

Lemma 2.10 ([8]). *A formation $\mathfrak{F}$ is saturated if and only if $\mathfrak{F}$ is p-local (p-saturated) for every prime p.*

Lemma 2.11. *Let $H \leq G$ and L be an abelian minimal normal subgroup of G. If $G = LH$, and $H \neq G$, then H is a maximal subgroup of G.*

Proof. Assume that $H \neq G$ and M is a maximal subgroup of G with $H \leq M$. Then $LM = LH = G$ and $L \nsubseteq M$. We claim that $L \cap M = L \cap H = 1$. In fact, it is clear that $M \cap L \trianglelefteq G$. But L is a minimal normal subgroup of G and $L \nsubseteq M$. Thus $M \cap L = 1$ and hence $H \cap L = 1$. It follows that $|H||L| = |HL| = |ML| = |M||L|$ and $|H| = |M|$. Therefore $H = M$ is a maximal subgroup of G. The lemma is proved.

3. Main Results. Theorem 3.1. *Let $\mathfrak{F}$ be a nonempty saturated formation and G a group. If $G^{\mathfrak{F}}$ is a $\pi(\mathfrak{F})$-selected group, then G has an $\mathfrak{F}$-covering subgroup.*

Proof. Assume that the theorem is false and G is a counterexample of minimal order. Then, obviously, $G \notin \mathfrak{F}$. Let $\pi = \pi(\mathfrak{F})$. If G is simple, then it is clear that $G = G^{\mathfrak{F}}$ and so $|\pi(G) \cap \pi| \leq 1$ since $G^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-selected. If $|\pi(G) \cap \pi| = 0$, then 1 is the $\mathfrak{F}$-covering subgroup of G. If $|\pi(G) \cap \pi| = 1$, i.e. $\pi(G) \cap \pi = \{p\}$, for some $p \in \pi(\mathfrak{F})$, then by Lemma 2.1, we see that Sylow p-subgroups of G are just the $\mathfrak{F}$-covering subgroups of G. This condridiction shows that G is not a simple group. Let N be a minimal normal subgroup of G. If $G/N \notin \mathfrak{F}$, then by Lemma 2.1, we have $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N \cong G^{\mathfrak{F}}/(G^{\mathfrak{F}} \cap N)$. Since $G^{\mathfrak{F}}$ is a $\pi(\mathfrak{F})$-selected group, by Lemma 2.2, $G^{\mathfrak{F}}/(G^{\mathfrak{F}} \cap N)$ is also a π-selected group. Thus, by the choice of G, G/N has an $\mathfrak{F}$-covering subgroup R/N and $R < G$. Since $R/(R \cap G^{\mathfrak{F}}) \cong RG^{\mathfrak{F}}/G^{\mathfrak{F}} = G/G^{\mathfrak{F}} \in \mathfrak{F}$, we have $R^{\mathfrak{F}} \leq R \cap G^{\mathfrak{F}}$. Then, by Lemma 2.2, $R^{\mathfrak{F}}$ is a π-selected group. This shows that R satisfies all conditions of the theorem. Hence, by the choice of G, R has an $\mathfrak{F}$-covering subgroup H. Then, by Lemma 2.4, we have that H is also an $\mathfrak{F}$-covering subgroup of G. This contradiction shows that $G/N \in \mathfrak{F}$. If G has another minimal normal subgroup $H \neq N$ with $G/H \in \mathfrak{F}$, then $G \cong G/H \cap N \in \mathfrak{F}$, a contradiction. Therefore, without loss of generality, we can assume that G has a unique minimal normal subgroup $N = G^{\mathfrak{F}}$. This implies that $|\pi(G^{\mathfrak{F}}) \cap \pi| \leq 1$.

Since $\mathfrak{F}$ is a saturated formation, by Corollary 2.7, G has an $\mathfrak{F}$-projector H. Then, by the definition of a $\mathfrak{F}$-projector, we have $G = HG^{\mathfrak{F}}$ and H is an $\mathfrak{F}$-maximal subgroup of G. Now, we prove that H is an $\mathfrak{F}$-covering subgroup of G. Assume that $H \leq T \leq G$. We only need to prove that $T = HT^{\mathfrak{F}}$. If it is not, then the set $\{T \mid H \leq T \leq G \text{ and } T \neq HT^{\mathfrak{F}}\} \neq \varnothing$. Let T be a group of minimal order in this set. First, we prove that T is a D_π-group. In fact, because $T/T \cap G^{\mathfrak{F}} \cong TG^{\mathfrak{F}}/G^{\mathfrak{F}} = HG^{\mathfrak{F}}/G^{\mathfrak{F}} = G/G^{\mathfrak{F}} \in \mathfrak{F}$, we have $T^{\mathfrak{F}} \subseteq G^{\mathfrak{F}}$. It follows that $|\pi(T^{\mathfrak{F}}) \cap \pi| \leq 1$, and consequently $T^{\mathfrak{F}} \in E_\pi^n$. Since $T/T^{\mathfrak{F}} \in \mathfrak{F}$, $T/T^{\mathfrak{F}}$ is a π-group. By Lemma 2.9, we see that T is a D_π-group. Let H_1 be a Hall π-subgroup of T such that $H \subseteq H_1$. Then $T = H_1T^{\mathfrak{F}}$. Assume that $H_1 < T$. Since $H_1/H_1 \cap T^{\mathfrak{F}} \cong H_1T^{\mathfrak{F}}/T^{\mathfrak{F}} = T/T^{\mathfrak{F}} \in \mathfrak{F}$, we see $H_1^{\mathfrak{F}} \subseteq T^{\mathfrak{F}}$. On the other hand, by the choice of T, we have $H_1 = HH_1^{\mathfrak{F}} = H(H_1 \cap T^{\mathfrak{F}})$. Therefore, $T = H_1T^{\mathfrak{F}} = HT^{\mathfrak{F}}$, a contradiction. Now, assume that $T = H_1$, then T is a π-group. Since $|\pi(T \cap G^{\mathfrak{F}}) \cap \pi| \leq |\pi(G^{\mathfrak{F}}) \cap \pi| \leq 1$, we have $T \cap G^{\mathfrak{F}} = 1$ or $T \cap G^{\mathfrak{F}}$ is a group of prime order p, for some $p \in \pi$. If $T \cap G^{\mathfrak{F}} = 1$, then $T \in \mathfrak{F}$ and hence $T = H$, a contradiction. Hence $T \cap G^{\mathfrak{F}}$ is a p-group. Consequently, $T \cap G^{\mathfrak{F}} \subseteq F(T)$. On the other hand, $HF(T) = H(T \cap G^{\mathfrak{F}}) = T \cap HG^{\mathfrak{F}} = T \cap G = T$. By Lemma 2.8, T has an $\mathfrak{F}$-covering subgroup K which contains H. However, since H is an $\mathfrak{F}$-projector of G, we have that H is an $\mathfrak{F}$-maximal subgroup of T. Thus, $H = K$ is an $\mathfrak{F}$-covering subgroup of T. This leads to $T = HT^{\mathfrak{F}}$, a contradiction. The theorem is proved.

Theorem 3.2. *Let $\mathfrak{F}$ be a nonempty saturated formation and G a group. If $G^{\mathfrak{F}}$ is a π-selected group where $\pi = \pi(\mathfrak{F})$, then any two $\mathfrak{F}$-covering subgroups of G are conjugate in G.*

Proof. Let H_1 and H_2 be any two $\mathfrak{F}$-covering subgroups of G. We prove by induction on the order $|G|$ of G that H_1 and H_2 are conjugate in G.

If G is a simple group, then we can see that the result is trivial. Let G be not a simple group. Suppose that N is a minimal normal subgroup of G. If $G/N \notin \mathfrak{F}$, then by Lemma 2.4, we have H_1N/N and H_2N/N are $\mathfrak{F}$-covering subgroups of G/N. Since $G^{\mathfrak{F}}$ is a π-selected group, by Lemma 2.2, we see that $G^{\mathfrak{F}}N/N$ is also a π-selected group. Hence, the factor group G/N satisfies the conditions of the theorem. By induction, there is an element $x \in G$ such that $H_1N/N = H_2^xN/N$, and consequently $H_1N = H_2^xN$. By the definition of $\mathfrak{F}$-covering subgroups, we can see that H_2^x is also an $\mathfrak{F}$-covering subgroup of G. Then H_1 and H_2^x are $\mathfrak{F}$-covering subgroups of H_1N. Since $G/N \notin \mathfrak{F}$, we have $H_1N < G$. By induction, there exists an element y in NH_1 such that $H_1 = (H_2^x)^y = H_2^{xy}$. This shows that H_1 and H_2 are conjugate in G.

Now, we may assume that $G/N \in \mathfrak{F}$ and $N = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G. Since $G^{\mathfrak{F}}$ is a π-selected group, we have $|\pi(G^{\mathfrak{F}}) \cap \pi| \leq 1$, and hence $G^{\mathfrak{F}}$ is an E_π^n-group. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$, $G/G^{\mathfrak{F}}$ is π-group. Thus, by Lemma 2.9, we see that G is a D_π-group. Therefore, there is a Hall π-subgroup T of G and an element $x \in G$ such that $H_1 \subseteq T$ and $H_2^x \subseteq T$. Obviously, H_1 and H_2^x are $\mathfrak{F}$-covering subgroups of T. Assume that $T < G$. It is clear that $G = TG^{\mathfrak{F}}$. Since $G/G^{\mathfrak{F}} = TG^{\mathfrak{F}}/G^{\mathfrak{F}} \cong T/T \cap G^{\mathfrak{F}} \in \mathfrak{F}$, we have that $T^{\mathfrak{F}} \subseteq \subseteq T \cap G^{\mathfrak{F}}$. Since $|\pi(G^{\mathfrak{F}}) \cap \pi| \leq 1$, we have $|\pi(T^{\mathfrak{F}}) \cap \pi| \leq 1$ and $T^{\mathfrak{F}}$ is a π-selected group. Thus, by induction, H_1 and H_2^x are conjugate in T. It follows that H_1 and H_2 are conjugate in G. Assume that $G = T$. Then G is a π-group and hence $G^{\mathfrak{F}}$ is an elementary abelian p-group, for some prime $p \in \pi(\mathfrak{F})$ because $|\pi(G^{\mathfrak{F}}) \cap \pi| \leq 1$. By the definition of $\mathfrak{F}$-covering subgroups, we have that $H_1G^{\mathfrak{F}} = H_2G^{\mathfrak{F}} = G$. If $G = H_i$, for some $i \in \{1, 2\}$, then $G \in \mathfrak{F}$ and $G = H_2 = H_1$. If $G \neq H_1$ and $G \neq H_2$, then by Lemma 2.11, we see that H_1 and H_2 are maximal subgroups of G. If $H_1 = H_2$, then H_1 and H_2 are already conjugate in G. Assume that $H_1 \neq H_2$. Then $H_1 \cap G^{\mathfrak{F}} = H_2 \cap G^{\mathfrak{F}} = 1$, and hence $[G^{\mathfrak{F}}]H_2 = [G^{\mathfrak{F}}]H_1 = G$. Let $R/G^{\mathfrak{F}}$ be a minimal normal subgroup of $G/G^{\mathfrak{F}}$. If $R/G^{\mathfrak{F}}$ is p'-group, then R is $D_{p'}$-group by Schur–Zassenhaus Theorem. Hence R has a unique conjugate class of Hall p'-subgroups. Obviously, $H_{11} = H_1 \cap R$ and $H_{21} = H_2 \cap R$ are Hall p'-subgroups of R, $H_1 \subseteq N_G(H_{11})$ and $H_2 \subseteq N_G(H_{21})$. If $H_{11} \trianglelefteq G$, then $G^{\mathfrak{F}} \subseteq H_{11}$ since $G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G, a contradiction. This shows that $N_G(H_{11}) < G$ and hence $H_1 = N_G(H_{11})$ since H_1 is a maximal subgroup of G. Analogously, $H_2 = N_G(H_{21})$. Because H_{11} and H_{21} are conjugate in R, we see that H_1 and H_2 are conjugate in G. If $R/G^{\mathfrak{F}}$ is a p-group, then R is a p-group. Thus, $R \subseteq F(G) = C_G(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$, a contradiction. Finally, we assume that $F_p(G/G^{\mathfrak{F}}) = 1$. Then, $(G/G^{\mathfrak{F}})/F_p(G/G^{\mathfrak{F}}) = G/G^{\mathfrak{F}} \in \mathfrak{F}(p)$. Since $G^{\mathfrak{F}}$ is a p-group, we have $G \in \mathfrak{N}_p\mathfrak{F}(p)$. Hence by Corollary 2.10, we obtain that $G \in \mathfrak{F}$, a contradiction. This completes the proof of the theorem.

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Abstract. Let $\mathfrak{F}$ be a nonempty saturated formation and G a finite group. In this paper, we proved that if the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is $\pi(\mathfrak{F})$-selected, then G has a unique conjugacy class of $\mathfrak{F}$-covering subgroups. This answers one open problem proposed by W. Guo.

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