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On the Shemetkov – Schmid subgroup and related subgroups of finite groups

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В работе установлены свойства подгруппы Шеметкова – Шмида $\tilde{F}(G)$ и связанных с ней обобщенных подгрупп Фиттинга конечных групп. Мы называем подгруппу H R-субнормальной в группе G, если H субнормальна в $\langle H, R\rangle$. Изучены группы с заданными системами R-субнормальных подгрупп, если $R \in \{F(G), F^*(G), \tilde{F}(G)\}$. Найдены новые характеризации нильпотентных и сверхразрешимых групп.
Ключевые слова: конечная группа, подгруппа Фитинга, подгруппа Шеметкова – Шмида, квазинильпотентный радикал, нильпотентная группа, сверхразрешимая группа.

In this paper the properties of the Shemetkov-Schmidt subgroup as well as generalized Fitting subgroups related with it have been determined. We call a subgroup H R-subnormal in a group G, if H is subnormal in $\langle H, R\rangle$. Finite groups with given systems of R-subnormal subgroups have been studied for $R \in \{F(G), F^*(G), \tilde{F}(G)\}$. New characterizations of nilpotent and supersolvable groups have been obtained.
Keywords: finite group, the Fitting subgroup, the Shemetkov – Schmid subgroup, *R*-subnormal subgroup, nilpotent group, supersoluble group.

1 Introduction. All the considered groups are finite. In 1938 H. Fitting [1] showed that a product of two normal nilpotent subgroups is again nilpotent subgroup. It means that in every group there is the unique maximal normal nilpotent subgroup $F(G)$ called the Fitting subgroup. This subgroup has a great influence on the structure of a solvable group. For example Ramadan [2] proved the following result.

Theorem 1.1. *Let G be a soluble group. If all maximal subgroups of Sylow subgroups of* $F(G)$ *are normal in G then G is supersoluble.*

Analyzing proofs of such kind's theorems in solvable case one can note that the following properties of the Fitting subgroup $F(G)$ are often used:

(1) $C_G(F(G)) \subseteq F(G)$;

(2) $\Phi(G) \subseteq F(G)$ and $F(G/\Phi(G)) = F(G)/\Phi(G)$;

(3) $F(G)/\Phi(G) \leq Soc(G/\Phi(G))$.

But only (2) and (3) are held for the Fitting subgroup of an arbitrary group. Note that there are many groups G with $F(G)=1$. That is why there were attempts to generalize the Fitting subgroup.

In 1970 H. Bender [3] introduced the quasinilpotent radical $F^*(G)$. It can be defined by the formula $F^*(G)/F(G) = Soc(C_G(F(G))F(G)/F(G)$ and can be viewed as a generalization of the Fitting subgroup. For $F^*(G)$ the statements like (1) and (3) are held. This subgroup proved useful in the classification of finite simple groups. Also $F^*(G)$ was used by many authors in the study of nonsimple groups.

In 1985 P. Förster [4], [5] showed that there is the unique characteristic subgroup $\tilde{F}(G)$ ($F'(G)$ in Förster notation) in every group G which satisfies the statements like (1)-(3). Firstly subgroup with this properties was mentioned by P. Schmid [6] in 1972. It was defined in explicit form by L. Shemetkov in 1978 [7, p.79]. P. Schmid and L. Shemetkov used this subgroup in the study of stable groups of automorphisms for groups.

Definition 1.2. *The Shemetkov-Schmid subgroup* $\tilde{F}(G)$ *of group G is defined as follows*:

(1) $\tilde{F}(G) \supseteq \Phi(G)$;

(2) $\tilde{F}(G)/\Phi(G) = Soc(G/\Phi(G))$.

Proposition 1.3 [8], [9]. *$F^*(G) \trianglelefteq \tilde{F}(G)$ for any group G.*

The following example shows that in general case $F^*(G) \neq \tilde{F}(G)$.

Example 1.4. Let $G \cong A_5$ be the alternating group on 5 letters and $\mathrm{K} = \mathrm{F}_3$. According to [10] there is a faithful irreducible Frattini KG-module A of dimension 4. According to known Gaschütz theorem [11], there exists a Frattini extension $A \to R \Rightarrow G$ such that A is G-isomorphic $\Phi(R)$ and $R/\Phi(R) \cong G$. From the properties of module A it follows that $\tilde{F}(R) = R$ and $F^*(R) = \Phi(R)$.

In this paper we continue to investigate the properties of the Shemetkov – Schmid subgroup and related subgroups. There is an overview of some applications of considered subgroups.

2 Preliminaries. We use standard notation and terminology, which can be found in [12], [13] if necessary. Recall that for a group G, $\Phi(G)$ is the Frattini subgroup of G; $\Delta(G)$ is the intersection of all maximal abnormal subgroups of G and G; $Z(G)$ is the center of G; $Z_\infty(G)$ is the hypercenter of G; $Soc(G)$ is the socle of G; G_{F} is the F-radical of G for a N_0-closed class F with 1; G^{F} is the F-residual of G for a formation F; N is the class of all nilpotent groups, N^* is the class of all quasinilpotent groups; F_p is a field composed by p elements.

A class of group F is said to be N_0-closed if $A, B \triangleleft G$ and $A, B \in \mathrm{F}$ imply $AB \in \mathrm{F}$.

A class of group F is said to be S_n-closed if $A \triangleleft G$ and $G \in \mathrm{F}$ imply $A \in \mathrm{F}$.

Lemma 2.1 [12, p. 127]. *Let G be a group. Then $C_G(F^*(G)) \leq F(G)$.*

Lemma 2.2 [7, p. 95], [14]. *Let G be a group. Then $\Delta(G/\Phi(G)) = \Delta(G)/\Phi(G) = Z_\infty(G/\Phi(G)) = Z(G/\Phi(G))$ and $\Delta(G)$ is nilpotent.*

3 Properties of $\tilde{F}(G)$ and related subgroups. It is well known that $F(F(G)) = F(G)$ and $F^*(F^*(G)) = F^*(G)$. In [4] P. Forster showed that there is a group G such that $\tilde{F}(\tilde{F}(G)) < \tilde{F}(G)$. He shows that there is a nonabelian simple group E which has F_pE-module V such that $R = Rad(V)$ is faithful irreducible F_pE-module and V/R is irreducible trivial F_pE-module. Let H be the semidirect product $V\lambda E$. Then $H' = RE$ is a primitive group and $|H : H'| = p$. There is F_qH-module W with $C_H(W) = H'$, where $q \neq p$. Let $G = W\lambda E$. Then $\Phi(G) = \Phi(H) = R$ and $Soc(G/R) = W \times ER/R$. So $\tilde{F}(G) = W \times ER$ and $\Phi(\tilde{F}(G)) = 1$. It means that $\tilde{F}(\tilde{F}(G)) = Soc(\tilde{F}(G)) = RW < \tilde{F}(G)$. This example led us to the following definition.

Definition 3.1. *Let G be a finite group. For any nonnegative integer n define the subgroup $\tilde{F}^n(G)$ by: $\tilde{F}^0(G) = G$ and $\tilde{F}^n(G) = \tilde{F}(\tilde{F}^{n-1}(G))$ for $n > 0$.*

It is clear that $\tilde{F}^i(G) = \tilde{F}^{i-1}(G) = \tilde{F}(\tilde{F}^{i-1}(G))$ for some i. So we can define the subgroup $\tilde{F}^\infty(G)$ as the minimal subgroup in the series $G = \tilde{F}^0(G) \supseteq \tilde{F}^1(G) \supseteq \ldots$. Now $\tilde{F}^\infty(G) = \tilde{F}^\infty(\tilde{F}^\infty(G))$.

Proposition 3.2. *Let n be a natural number, N and H be normal subgroups of a group G. Then*:

(1) *If $N \leq \Phi(\tilde{F}^{n-1}(G))$ then $\tilde{F}^n(G/N) = \tilde{F}^n(G)/N$;*

(2) $F^*(G) \subseteq \tilde{F}^n(G)$;

(3) *If $\Phi(\tilde{F}^{n-1}(G)) = 1$ then $\tilde{F}^n(G) = F^*(G)$;*

(4) $C_G(\tilde{F}^n(G)) \subseteq F(G)$;

(5) $\tilde{F}^n(N) \leq \tilde{F}^n(G)$;

(6) $\tilde{F}^n(G)N/N \leq \tilde{F}^n(G/N)$;

(7) *If $G = N \times H$ then $\tilde{F}^n(G) = \tilde{F}^n(H) \times \tilde{F}^n(N)$.*

Proof. (1) When $n=1$ it directly follows from the definition of $\tilde{F}(G)$ and $\Phi(G/N)=\Phi(G)/N$. By induction by n we obtain this statement.

(2) The proof was proposed by L. Shemetkov to the authors in case if $n=1$. Let a group G be the minimal order counterexample for (2). If $\Phi(G)\neq 1$ then for $G/\Phi(G)$ the statement is true. From $F^*(G)/\Phi(G)\subseteq F^*(G/\Phi(G))$ and $\tilde{F}(G/\Phi(G))=\tilde{F}(G)/\Phi(G)$ we have that $F^*(G)\subseteq \tilde{F}(G)$. It is a contradiction with the choice of G.

Let $\Phi(G)=1$. Now $\tilde{F}(G)=Soc(G)$. By 13.14.X [12] $F^*(G)=E(G)F(G)$. Note $\Phi(E(G))=1$. Since 13.7.X [12] $E(G)/Z(E(G))$ is the direct product of simple nonabelian groups, $Z(E(G))=F(E(G))$. From the theorem 10.6.A [15] we conclude that $E(G)=HZ(E(G))$, where H is the complement to $Z(E(G)$ in $E(G)$. Now H is the direct product of simple nonabelian groups. Since $H \operatorname{char} E(G)\lhd G$, we have $H\lhd G$. Note $H\subseteq Soc(G)$. Since $Z(E(G))\subseteq F(G)\subseteq \tilde{F}(G)$ and $H\subseteq Soc(G)$, it follows that $E(G)\subseteq \tilde{F}(G)$. Now $F^*(G)=E(G)F(G)\subseteq \tilde{F}(G)$. It is a contradiction with the choice of G.

Assume that $F^*(G)\subseteq \tilde{F}^n(G)$ for $n\geq 1$. It means that $F^*(\tilde{F}^n(G))=F^*(G)$. By induction $F^*(G)\subseteq \tilde{F}^{n+1}(G)$.

(3) If $\Phi(\tilde{F}^{n-1}(G))=1$ then $\tilde{F}^n(G)$ is the socle of $\tilde{F}^{n-1}(G)$ and hence $\tilde{F}^n(G)$ is quasinilpotent. From $F^*(G)\subseteq \tilde{F}^n(G)$ it follows that $\tilde{F}^n(G)=F^*(G)$.

(4) From $F^*(G)\leq \tilde{F}^n(G)$ it follows that $C_G(\tilde{F}^n(G))\leq C_G(F^*(G))$. Since $C_G(F^*(G))\leq F(G)$ by lemma 2.1, we see that $C_G(\tilde{F}^n(G))\leq F(G)$.

(5) Since $\Phi(N)\leq\Phi(G)$, we see that $\tilde{F}(G/\Phi(N))=\tilde{F}(G)/\Phi(N)$. Note that $\tilde{F}(G)/\Phi(N)$ is quasinilpotent. Hence $\tilde{F}(N)/\Phi(N)\subseteq F^*(G/\Phi(N))\subseteq \tilde{F}(G/\Phi(N))=\tilde{F}(G)/\Phi(N)$. Thus $\tilde{F}(N)\leq \tilde{F}(G)$. By induction $\tilde{F}^n(N)\leq \tilde{F}^n(G)$.

(6) Note that $\tilde{F}(G)N/N/\Phi(G)N/N\cong \tilde{F}(G)N/\Phi(G)N\cong \tilde{F}(G)/\tilde{F}(G)\cap\Phi(G)N$. From $\Phi(G)\subseteq \tilde{F}(G)\cap\Phi(G)N$ it follows that $\tilde{F}(G)N/N/\Phi(G)N/N$ is quasinilpotent. Since $\Phi(G)N/N\subseteq\Phi(G/N)$, we see that $\tilde{F}(G)N/N\leq\tilde{F}(G/N)$.

Assume that $\tilde{F}^n(G)N/N\leq\tilde{F}^n(G/N)$ for some $n\geq 1$. Now

$\tilde{F}^{n+1}(G/N)=\tilde{F}(\tilde{F}^n(G/N))\geq\tilde{F}(\tilde{F}^n(G)N/N)\geq\tilde{F}(\tilde{F}^n(G)N)N/N\geq\tilde{F}(\tilde{F}^n(G))N/N=\tilde{F}^{n+1}(G)N/N$

by the previous step and (5).

(7) Assume that the statement is false for $n=1$. Let a group G be a counterexample of minimal order. Assume that $\Phi(G)=1$. Then $G/\Phi(G)=H\Phi(G)/\Phi(G)\times N\Phi(G)/\Phi(G)$.

Note that $H\Phi(G)/\Phi(G)\cong H/H\cap\Phi(G)=H/H\cap(\Phi(H)\times\Phi(N))=H/\Phi(H)$. By analogy $N\Phi(G)/\Phi(G)\cong N/\Phi(N)$. So $\tilde{F}(G/\Phi(G))\cong\tilde{F}(H/\Phi(H))\times\tilde{F}(N/\Phi(N))$. From

$\tilde{F}(G/\Phi(G))=\tilde{F}(G)/\Phi(G)$, $\tilde{F}(N/\Phi(N))=\tilde{F}(N)/\Phi(N)$ and $\tilde{F}(H/\Phi(H))=\tilde{F}(H)/\Phi(H)$

it follows that $\tilde{F}(G)/\Phi(G)\cong\tilde{F}(H)/\Phi(H)\times\tilde{F}(N)/\Phi(N)$. Now

$$\frac{|\tilde{F}(G)|}{|\Phi(G)|}=\frac{|\tilde{F}(H)|}{|\Phi(H)|}\cdot\frac{|\tilde{F}(N)|}{|\Phi(N)|}$$

From $\Phi(G)=\Phi(N)\times\Phi(H)$ it follows that $|\tilde{F}(G)|=|\tilde{F}(N)||\tilde{F}(H)|$. From (5) it follows that $\tilde{F}(N)\leq\tilde{F}(G)$ and $\tilde{F}(H)\leq\tilde{F}(G)$. Thus $\tilde{F}(G)=\tilde{F}(H)\times\tilde{F}(N)$, a contradiction.

Now $\Phi(G)=1$. So $\tilde{F}(G)=F^*(G)$, $\tilde{F}(N)=F^*(N)$ and $\tilde{F}(H)=F^*(H)$. It is well known that $F^*(H\times N)=F^*(H)\times F^*(N)$, the final contradiction.

By induction we have that $\tilde{F}^n(G)=\tilde{F}^n(H)\times\tilde{F}^n(N)$. □

From proposition 3.2 properties of $\tilde{F}(G)$ follow.

Corollary 3.3. *Let* N *and* H *be normal subgroups of a group* G. *Then*:

(1) [9] *If* $N \le \Phi(G)$ *then* $\tilde{F}(G/N) = \tilde{F}(G)/N$;

(2) [8], [9] $F^*(G) \subseteq \tilde{F}(G)$;

(3) [4] *If* $\Phi(G) = 1$ *then* $\tilde{F}(G) = F^*(G)$;

(4) [6], [7] $C_G(\tilde{F}(G)) \subseteq \tilde{F}(G)$;

(5) [4] $\tilde{F}(N) \le \tilde{F}(G)$;

(6) [4] $\tilde{F}(G)N/N \le \tilde{F}(G/N)$;

(7) *If* $G = N \times H$ *then* $\tilde{F}(G) = \tilde{F}(H) \times \tilde{F}(N)$.

Also we obtain new properties of $\tilde{F}^{\infty}(G)$.

Corollary 3.4. *Let* N *and* H *be normal subgroups of a group* G. *Then*

(1) $\tilde{F}^{\infty}(G)/\Phi(\tilde{F}^{\infty}(G))$ *is quasinilpotent*;

(2) $F^*(G) \subseteq \tilde{F}^{\infty}(G) \subseteq \tilde{F}(G)$;

(3) *If* $\Phi(\tilde{F}^{\infty}(G)) = 1$ *then* $\tilde{F}^{\infty}(G) = F^*(G)$;

(4) $C_G(\tilde{F}^{\infty}(G)) \subseteq F(G)$;

(5) $\tilde{F}^{\infty}(N) \le \tilde{F}^{\infty}(G)$;

(6) $\tilde{F}^{\infty}(G)N/N \le \tilde{F}^{\infty}(G/N)$;

(7) *If* $G = N \times H$ *then* $\tilde{F}^{\infty}(G) = \tilde{F}^{\infty}(H) \times \tilde{F}^{\infty}(N)$.

In [4] Förster introduced a class $\hat{\mathfrak{N}} = E_{\Phi}\mathfrak{N}^* = (G \mid \tilde{F}(G) = G)$ and showed that $\hat{\mathfrak{N}}$ is N_0-closed Shunck class that is neither formation nor s_n-closed. Note that $\hat{\mathfrak{N}} = (G \mid \tilde{F}^{\infty}(G) = G)$.

Proposition 3.5. *Let* G *be a group. Then* $G_{\hat{\mathfrak{N}}} = \tilde{F}^{\infty}(G)$, *i.e.* $\tilde{F}^{\infty}(G)$ *is the maximal among normal subgroups* N *of* G *such that* $N/\Phi(N)$ *is quasinilpotent.*

Proof. From $G_{\hat{\mathfrak{N}}}/\Phi(G_{\hat{\mathfrak{N}}}) \in \mathfrak{N}^*$ and $\Phi(G_{\hat{\mathfrak{N}}}) \le \Phi(G)$ it follows that $G_{\hat{\mathfrak{N}}} \le \tilde{F}(G)$. By induction $G_{\hat{\mathfrak{N}}} = \tilde{F}(G_{\hat{\mathfrak{N}}}) \le \tilde{F}^{\infty}(G)$. By proposition 3.2 and the definition of $\hat{\mathfrak{N}}$ we obtain $G_{\hat{\mathfrak{N}}} = \tilde{F}^{\infty}(G)$. □

Problem 3.6. Let $\mathfrak{F}$ be an N_0-closed class of groups and $1 \in \mathfrak{F}$. Then there is the maximal normal $\mathfrak{F}$-subgroup $G_{\mathfrak{F}}$ in any group G. In the context of our work the following general problem appears: to describe all N_0-closed classes (formations, Fitting classes, Shunck classes) $\mathfrak{F}$ with 1 for which one of the following statements holds:

(1) $F(G) \subseteq G_{\mathfrak{F}} \subseteq F^*(G)$ for any group G;

(2) $F^*(G) \subseteq G_{\mathfrak{F}} \subseteq \tilde{F}(G)$ for any group G;

(3) $F(G) \subseteq G_{\mathfrak{F}} \subseteq \tilde{F}(G)$ for any group G.

Theorem 3.7. *Let* $\mathfrak{F}$ *be a* N_0*-closed formation. Then:*

(1) *If* $\mathfrak{F}$ *is a saturated formation and* $F(G) \subseteq G_{\mathfrak{F}} \subseteq \tilde{F}(G)$ *for any group* G *then* $\mathfrak{F} = \mathfrak{N}$.

(2) *If* $F^*(G) \subseteq G_{\mathfrak{F}} \subseteq \tilde{F}(G)$ *for any group* G *then* $\mathfrak{F} = \mathfrak{N}^*$.

Proof. Let prove (1). From $F(G) \subseteq G_{\mathfrak{F}}$ it follows that $\mathfrak{N} \subseteq \mathfrak{F}$. Assume that the set $\mathfrak{F} \setminus \mathfrak{N}$ is not empty and we choose a minimal order group G from it. Since $\mathfrak{F}$ and $\mathfrak{N}$ are both saturated formations, from minimality of G we may assume that $\Phi(G) = 1$ and there is only one minimal normal subgroup of G. From $G_{\mathfrak{F}} \subseteq \tilde{F}(G)$ it follows that $G = Soc(G)$ is nonabelian simple group. From [10] it follows that for prime p dividing $|G|$ there exist faithful irreducible $\mathbb{F}_pG$-module A admitting a group extension $A \to E \Rightarrow G$ with $A \subseteq \Phi(G)$. Since $\mathfrak{F}$ is a saturated formation so $E \in \mathfrak{F}$ and $A/1$ is chief factor of E. According to [15, p. 335] we see that $H = A \lambda (E/A) \in \mathfrak{F}$. Note that $\tilde{F}(H) = A$, a contradiction. Thus $\mathfrak{N} = \mathfrak{F}$.

Let prove (2). From $F^*(G) \subseteq G_{\mathfrak{F}}$ it follows that $\mathfrak{N}^* \subseteq \mathfrak{F}$. Assume that the set $\mathfrak{F} \setminus \mathfrak{N}^*$ is not empty and G is a group a minimal order from it. Since $\mathfrak{F}$ and $\mathfrak{N}^*$ are both formations, from minimality of G

we may assume that there is only one minimal normal subgroup N of G. If $\Phi(G) = 1$ then $G = Soc(G) \in$N*, a contradiction. So $N \subseteq \Phi(G)$.

Now N is a normal elementary abelian p-subgroup G. By our assumption $G/N \in$N*. Assume that $C_G(N) = G$. Now G acts as inner automorphisms on $N/1$ and on every chief factor of G/N. By definition of quasinilpotent groups $G \in$N*, a contradiction. Hence $C_G(N) \neq G$. Note that N is the unique minimal subgroup of $H = N\lambda(G/C_G(N)) \in$F by [15, p. 335] and $\Phi(H) = 1$. So $\tilde{F}(H) = N$ and $H_{\mathrm{F}} = H$, a contradiction. Thus $\mathrm{N}^* = \mathrm{F}$. □

Let consider another direction of generalization of the Fitting subgroup. A subgroup functor τ is called m-functor if $\tau(G)$ contains G and some maximal subgroups of G for every group G. Recall [16, p. 198] that $\Phi_\tau(G)$ is the intersection of all subgroups from $\tau(G)$.

Definition 3.8. *Let τ be m-functor. For every group G subgroup $\tilde{F}_\tau(G)$ is defined as follows*:

1) $\Phi_\tau(G) \subseteq \tilde{F}_\tau(G)$;

2) $\tilde{F}_\tau(G)/\Phi_\tau(G) = Soc(G/\Phi_\tau(G))$.

If $\tau(G)$ is the set of all maximal subgroups of G for any group G then we obtain the definition of $\tilde{F}(G)$. If $\tau(G)$ is the set of all maximal abnormal subgroups and G for any group G then $\Phi_\tau(G) = \Delta(G)$. Subgroup $\tilde{F}_\tau(G) = \tilde{F}_\Delta(G)$ was introduced by M. Selkin and R. Borodich [17].

Proposition 3.9. *Let G be a group. Then $\Delta(G) \subseteq \tilde{F}(G)$ and $\tilde{F}(G/\Delta(G)) = \tilde{F}(G)/\Delta(G)$.*

Proof. From lemma 2.2 it follows that $\Delta(G) \subseteq \tilde{F}(G)$. Let a group G be a counterexample of minimal order to the second statement of proposition. Assume that $\Phi(G) \neq 1$. By inductive hypothesis

$$\tilde{F}((G/\Phi(G))/\Delta(G/\Phi(G))) = \tilde{F}(G/\Phi(G))/\Delta(G/\Phi(G))$$

Now $\Delta(G/\Phi(G)) = \Delta(G)/\Phi(G)$ by lemma 2.2 and $\tilde{F}(G/\Phi(G)) = \tilde{F}(G)/\Phi(G)$ by corollary 3.3 (1).

$$\tilde{F}((G/\Phi(G)/\Delta(G/\Phi(G))) = \tilde{F}(G/\Phi(G)/\Delta(G)/\Phi(G)) \simeq \tilde{F}(G/\Delta(G))$$

$$\tilde{F}(G/\Phi(G))/\Delta(G/\Phi(G)) = \tilde{F}(G)/\Phi(G)/\Delta(G)/\Phi(G) \simeq \tilde{F}(G)/\Delta(G)$$

Hence $\left|\tilde{F}(G)/\Delta(G)\right| = \left|\tilde{F}(G/\Delta(G))\right|$. By corollary 3.3 (6) it follows that $\tilde{F}(G)/\Delta(G) \le \tilde{F}(G/\Delta(G))$. Thus $\tilde{F}(G)/\Delta(G) = \tilde{F}(G/\Delta(G))$, a contradiction.

Now $\Phi(G) = 1$. It means that $\Delta(G) = Z(G) \le Soc(G)$. So $Z(G)$ is complemented in G. Let M be a complement of $Z(G)$ in G. Hence $M \lhd G$ and $G = M \times Z(G) = M \times \Delta(G)$. Note that $M \simeq G/\Delta(G)$. From corollary 3.3 (7) it follows that $\tilde{F}(G) = \tilde{F}(M) \times \tilde{F}(\Delta(G)) = \tilde{F}(M) \times \Delta(G)$. Thus $\tilde{F}(G/\Delta(G)) \simeq \tilde{F}(M) \simeq \tilde{F}(G)/\Delta(G)$. From corollary 3.3 (6) we obtain the final contradiction. □

Corollary 3.10. *Let G be a group. Then $\tilde{F}_\Delta(G) = \tilde{F}(G)$.*

Proposition 3.11. *Let G be a group. Then $\tilde{F}_\tau(G) \ge \tilde{F}(G)$. In particular $C_G(\tilde{F}_\tau(G)) \le \tilde{F}_\tau(G)$.*

Proof. From corollary 3.3 it follows that if $N \lhd G$ and $\Phi(G) \subseteq N$ then $\tilde{F}(G)N/N$ is quasinilpotent. Also note that $\Phi(G/\Phi_\tau(G)) = 1$. By corollary 3.3 $F^*(G/\Phi_\tau(G)) = Soc(G/\Phi_\tau(G))$. Hence $\tilde{F}(G)\Phi_\tau(G)/\Phi_\tau(G) \le Soc(G/\Phi_\tau(G))$. Thus $\tilde{F}_\tau(G) \ge \tilde{F}(G)$. The second statement follows from (4) of corollary 3.3. □

4. Applications of Shemetkov-Schmid subgroup and related subgroups. In [8] A. Vasil'ev and etc. proved the following theorem.

Theorem 4.1. *The intersection of all maximal subgroups M of a group G such that $M\tilde{F}(G) = G$ is the Frattini subgroup $\Phi(G)$ of G.*

Another direction of applications of generalizations of the Fitting subgroup is connected with the following concept. Recall [16], that a subgroup functor is a function τ which assigns to each group G a possibly empty set $\tau(G)$ of subgroups of G satisfying $f(\tau(G)) = \tau(f(G))$ for any isomorphism $f: G \to G^*$.

Definition 4.2. *Let θ be a subgroup functor and R be a subgroup of a group G. We will call a subgroup H of G the R-θ-subgroup if $H \in \theta(\langle H, R\rangle)$.*

Let θ be the **P**-subnormal subgroup functor. Recall [18] that a subgroup H of a group G is called **P**-subnormal in G if $H = G$ or there is a chain of subgroup $H = H_0 < H_1 < ... < H_n = G$ where $|H_i : H_{i-1}|$ is a prime for $i = 1, ..., n$. O. Kramer's theorem [19, p.12] states

Theorem 4.3. *If every maximal subgroup of a soluble group G is $F(G)$-**P**-subnormal then G is supersoluble.*

This theorem was generalized by Yangming Li, Xianhua Li in [9].

Theorem 4.4. *A group G is supersoluble if and only if every maximal subgroup of G is $\tilde{F}(G)$-**P** -subnormal.*

It is well known that a group *G* is nilpotent if and only if every maximal subgroup of *G* is normal in *G***.**

Theorem 4.5. *A group G is nilpotent if and only if every maximal subgroup of G is $\tilde{F}(G)$-subnormal.*

Let θ be the conjugate-permutable subgroup functor. Recall [20] that a subgroup H of a group G is called conjugate-permutable if $HH^x = H^xH$ for all $x \in G$.

Corollary 4.4 [21]. *A group G is nilpotent if and only if every maximal subgroup of G is $\tilde{F}(G)$-conjugate-permutable.*

Corollary 4.7 [8], [21]**.** *If G is a non-nilpotent group then there is an abnormal maximal subgroup M of G such that G=M$\tilde{F}(G)$.*

Theorem 4.8 *The following statements for a group G are equivalent***:**

(1) *G is nilpotent*;

(2) *Every abnormal subgroup of G is $F^*(G)$-subnormal subgroup of G*;

(3) *All normalizers of Sylow subgroups of G are $F^*(G)$-subnormal subgroups of G*;

(4) *All cyclic primary subgroups of G are $F^*(G)$-subnormal subgroups of G*;

(5) *All Sylow subgroups of G are $F^*(G)$-subnormal subgroups of G.*

Corollary 4.9. *A group G is nilpotent if and only if the normalizers of all Sylow subgroups of G contain $F^*(G)$.*

Theorem 4.10 [22]. *If a group G is a product of two F(G)-subnormal nilpotent subgroups A and B then G is nilpotent.*

Theorem 4.11. *Let a group G=AB, where A and B are F(G)-subnormal supersoluble subgroups of G. If [A,B] is nilpotent, then G is supersoluble.*

The well known result states that a group *G* is supersoluble if it contains two subnormal supersoluble subgroups with coprime indexes in *G*.

The following example shows this result will fail if we replace «normal» by «*F*(*G*)-subnormal». Let *G* be the symmetric group on 3 letters. By theorem 10.3B [158, p. 173] there is a faithful irreducible F_7G-module *V* and the dimension of *V* is 2. Let *R* be the semidirect product of *V* and *G*. Let $A = VG_3$ and $B = VG_2$ where G_p is a Sylow *p*-subgroup of *G*, $p \in \{2, 3\}$. Since $7 \equiv 1 \pmod p$ for $p \in \{2, 3\}$, it is easy to check that subgroups *A* and *B* are supersoluble. Since *V* is faithful irreducible module, *F*(*R*) = *V*. Therefore *A* and *B* are the *F*(*R*)-subnormal subgroups of *G*. Note that *R* = *AB* but *R* is not supersoluble.

Theorem 4.12. *Let A, B and C be a F(G)-subnormal supersoluble subgroups of a group G. If indexes of A, B and C in G are pairwise coprime then G is supersoluble***.**

Corollary 4.13. *Let A, B and C be a supersoluble subgroups of a group G. If indexes of A***,** *B and C in G are pairwise coprime and F(G) $\subseteq A \cap B \cap C$ then G is supersoluble.*

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