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On τ-quasinormal and weakly τ-quasinormal subgroups of finite groups

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Throughout this paper, all groups are finite. All unexplained notations and rminologies are standard. The reader is refereed to [1], [3], [5] and [6] if necessary. G ways denotes a group. We use $\pi(G)$ to denote the set of all primes dividing $|G|$. For any bgroup H of G we put $\tau_G(H) = \{q \in \pi(G) \setminus \pi(H) \mid (|H|, |Q^G|) \neq 1$ for a Sylow q-subgroup of $G\}$, where Q^G is the normal closure of Q in G.

Recall that a subgroup A of a group G is said to *permute with a subgroup* B if $B = BA$. A subgroup H of a group G is said to be $\pi(G)$*-permutable* or $\pi(G)$*-quasinormal*). Kegel, [7]) in G if H permutes with every Sylow p-subgroup of G for all primes $p \in \pi$. subgroup H of G is said to be *c-normal* in G (Y. Wang, [10]) if G has a normal subgroup such that $HT = G$ and $T \cap H \leq H_G$, where H_G is the normal core of H in G.

In spite of the fact that the $\pi(G)$-quasinormality and the c-normality are quite fferent generalizations of normality, there are several analogous results which were obtained dependently for $\pi(G)$-quasinormal and c-normal subgroups (see Section 5 in [9]). In this aper we analyze some of these results on the base of the following more general concepts.

Definition. Let H be a subgroup of a group G. Then we say that:

(1) H is *τ-quasinormal in G* if H permute with all Sylow q-subgroups of G for all $\in \tau_G(H)$;

(2) H is *weakly τ-quasinormal in G* if G has a subnormal subgroup T such that $IT = G$ and $T \cap H \leq H_{\tau G}$.

In this definition $H_{\tau G}$ denotes the *τ-core* of H in G, that is, the subgroup generated y all those subgroups of H which are τ-quasinormal in G.

It is clear that every $\pi(G)$-quasinormal subgroup and every c-normal subgroup of G re weakly τ-quasinormal. The following simple example shows that in general the set of reakly τ-quasinormal subgroups is wider than the set of all $\pi(G)$-quasinormal subgroups nd the set of all c-normal subgroups.

Example. Let $p < r < q$ be odd primes, C_q be a group of order q and R a faithful rreducible C_q-module over a field $\mathbb{F}_r$. Let P be a faithful irreducible $([R]C_q)$-module over a ield $\mathbb{F}_p$ and $A = [P]([R]C_q)$. Let $M(r) = \langle\, x, y, z \mid x^r = y^r = z^r = 1,\ [x,z] = [y,z] = 1$ and $x, y] = z \,\rangle = [\langle x, z\rangle]\langle y\rangle$ (see p. 203 in [4]). Finally, let $G = A \wr M(r) = [K]M(r)$, where K s the base group of the regular wreath product of G. By [3, A, 18.5(b)], $L = P^\natural$ is the only ninimal normal subgroup of G. Note also that since by [3, A, 18.2], $G/P^\natural \simeq (A/P) \wr M(r)$ und $C_A(P) = P$, then $L \nsubseteq \Phi(G)$ by [6, III, 3.5]. Hence $L = C_G(L)$.

(1) *C_q is τ-quasinormal in A, but not $\pi(G)$-quasinormal in A* (this follows directly rom the fact that $R^A \leq PR$).

(2) *$H = \langle x, z\rangle$ is weakly τ-quasinormal in G and H is neither c-normal in G nor r-quasinormal in G.*

First suppose that H is τ-quasinormal in G. Let Q be a Sylow q-subgroup of K. It is not difficult to show that r divides $|Q^G|$. Besides, Q is a Sylow q-subgroup of G. Hence $HQ = QH$ is a subgroup of G. Note that since by [3, A, 18.2], $G/(PR)^\natural \simeq (A/PR) \wr M(r) \simeq C_q \wr M(r)$ and evidently $QH(PR)^\natural/(PR)^\natural \simeq QH$, $H \leq N_G(Q)$. For any $x \in G$, $H^xQ = QH^x$, which as above implies $H^x \leq N_G(Q)$. Hence $H^G \leq N_G(Q)$ and so $L \leq N_G(Q)$, which implies $Q \leq C_G(L)$. This contradiction shows that H is not τ-quasinormal in G. Now suppose that H is c-normal in G and let T be a normal subgroup of G such that $TH = G$ and $T \cap H \le H_G = 1$. Then $M(r) = M(r) \cap TH = H(T \cap M(r))$ is Abelian, a contradiction. Finally note that $T = K\langle y \rangle$ is subnormal in G, $TH = G$ and $T \cap H = 1$. Hence H is weakly τ-quasinormal in G.

Note that many results in which one of the conditions "to be a c-normal subgroup" or "to be a $\pi(G)$-quasinormal subgroup" is involved can be non-trivially generalized on the base of the weakly τ-quasinormality. The following our theorem serves a partial illustration for this.

Theorem. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-Abelian 2-group) are weakly τ-quasinormal in G. Then $G \in \mathcal{F}$.*

The proof of this theorem consists of a large number of steps. The sketch of this proof is concluded in following lemmas.

Lemma 1. *Let G be a group and $H \le K \le G$, $L \le G$. Then:*

(1) *If H is τ-quasinormal in G, then H is τ-quasinormal in K.*

(2) *Suppose that H is normal in G and $\pi(K/H) = \pi(K)$. If K is τ-quasinormal in G, then K/H is τ-quasinormal in G/H.*

(3) *Suppose that H is τ-quasinormal in G. If $HL = LH$ and $\pi(H \cap L) = \pi(H) \cap \pi(L)$, then $H \cap L$ is τ-quasinormal in L.*

(4) *Suppose that H is normal in G. Then the subgroup EH/H is τ-quasinormal in G/H for every τ-quasinormal in G subgroup E satisfying $(|H|, |E|) = 1$.*

(5) *If H is τ-quasinormal in G and $H \le O_p(G)$ for some prime p, then H is $\pi(G)$-quasinormal in G.*

(6) *Suppose that H and L are τ-quasinormal in G. If $HL = LH$ and $\pi(H \cap L) = \pi(H) = \pi(L)$, then $H \cap L$ is τ-quasinormal in G.*

Proof. (1) Let K_q be a Sylow q-subgroup of K such that $q \notin \pi(H)$ and $(|H|, |K_q^K|) \ne 1$. Then for some Sylow q-subgroup Q of G we have $K_q = Q \cap K$. Since $K_q^K \le Q^G \cap K \le Q^G$, $(|H|, |Q^G|) \ne 1$. Hence $q \in \tau_G(H)$ and so $HK_q = H(Q \cap K) = HQ \cap K = (Q \cap K)H = K_qH$.

(2) Let Q/H be a Sylow q-subgroup of G/H such that $q \notin \pi(K/H) = \pi(K)$ and $(|K/H|, |(Q/H)^{(G/H)}|) \ne 1$. Then for some Sylow q-subgroup G_q of G we have $Q = G_qH$. Since $(Q/H)^{(G/H)} = Q^G/H = (G_qH)^G/H = G_q^GH/H \simeq G_q^G/(G_q^G \cap H)$, $(|K|, |G_q^G|) \ne 1$. Hence $q \in \tau_G(K)$ and so $(K/H)(Q/H) = KQ/H = KG_qH/H = (Q/H)(K/H)$.

(3) By (1), H is τ-quasinormal in HL. Let Q be a Sylow q-subgroup of L such that $q \notin \pi(H \cap L)$ and $(|H \cap L|, |Q^L|) \ne 1$. Then $q \notin \pi(H)$, so Q is a Sylow q-subgroup of HL. Besides, $Q^L \le Q^{HL} \cap L \le Q^{HL}$. Hence $(|H|, |Q^{HL}|) \ne 1$ and so $q \in \tau_{HL}(H)$. Therefore, $Q(H \cap L) = QH \cap L = (H \cap L)Q$.

(4) Let Q/H be a Sylow q-subgroup of G/H such that $q \notin \pi(EH/H) = \pi(E)$ and $(|EH/H|, |(Q/H)^{(G/H)}|) \ne 1$. Then for some Sylow q-subgroup G_q of G we have $Q = G_qH$. Since $(Q/H)^{(G/H)} = Q^G/H = G_q^GH/H \simeq G_q^G/(G_q^G \cap H)$, $(|E|, |G_q^G|) \ne 1$. Hence $q \in \tau_G(E)$ and so $(EH/H)(Q/H) = EQ/H = EG_qH/H = (Q/H)(EH/H)$.

(5) Let Q be any Sylow q-subgroup of G, where $q \ne p$. Suppose that $HQ \ne QH$. Then by hypothesis, $Q^G \le O_{p'}(G)$. Hence $H \le O_p(G) \le C_G(Q)$, a contradiction.

(6) Let Q be a Sylow q-subgroup of G such that $q \notin \pi(H \cap L) = \pi(H) = \pi(L)$ and $(|H \cap L|, |Q^G|) \ne 1$. Then $(|H|, |Q^G|) \ne 1$ and $(|L|, |Q^G|) \ne 1$. Hence $q \in \tau_G(H)$ and $q \in \tau_G(L)$. Therefore, $Q(H \cap L) = QH \cap QL = (H \cap L)Q$.

From Lemma 1 we directly have

Lemma 2. *Let G be a group and $H \leq K \leq G$. Then the following statements hold:*

(1) *If H is a p-group for some prime p, then $H_{\tau G}$ is a τ-quasinormal subgroup of G and $H_G \leq H_{\tau G}$.*

(2) *$H_{\tau G} \leq H_{\tau K}$.*

(3) *Suppose that K is a p-group for some prime p and H is normal in G. Then $K_{\tau G}/H \leq (K/H)_{\tau(G/H)}$.*

(4) *Suppose that H is normal in G. Then $E_{\tau G}H/H \leq (EH/H)_{\tau(G/H)}$ for every subgroup E satisfying $(|H|, |E|) = 1$.*

Lemma 3. *Let G be a group and $H \leq K \leq G$. Then*

(1) *If H is τ-quasinormal in G, then H is weakly τ-quasinormal in G.*

(2) *Suppose that K is a p-group for some prime p and H is normal in G. If K is weakly τ-quasinormal in G, then K/H is weakly τ-quasinormal in G/H.*

(3) *If H is weakly τ-quasinormal in G, then H weakly τ-quasinormal in K.*

(4) *Suppose that H is normal in G. Then the subgroup EH/H is weakly τ-quasinormal in G/H for every weakly τ-quasinormal in G subgroup E satisfying $(|H|, |E|) = 1$.*

(5) *Suppose that H is a p-group for some prime p and H is not τ-quasinormal in G. Assume that H is weakly τ-quasinormal in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Proof. Statement (1) is evident.

(2) Assume that for some subnormal subgroup T of G we have $KT = G$ and $T \cap K \leq K_{\tau G}$. Then by [3, A, 14.1 and 14.4], TH/H is subnormal in G/H. Besides, $(TH/H)(K/H) = G/H$ and $(TH/H) \cap (K/H) = (TH \cap K)/H = (T \cap K)H/H \leq K_{\tau G}H/H = K_{\tau G}/H \leq (K/H)_{\tau(G/H)}$ by Lemma 2(3). Thus K/H is weakly τ-quasinormal in G/H.

(3) Let T be a subnormal subgroup of G such that $HT = G$ and $T \cap H \leq H_{\tau G}$. Then $K = K \cap HT = H(K \cap T)$. By [3, A, 14.1], $K \cap T$ is subnormal in K. By Lemma 2(2) we also know that $(K \cap T) \cap H \leq H_{\tau G} \leq H_{\tau K}$. Hence H is weakly τ-quasinormal in K.

(4) Assume that for some subnormal subgroup T of G we have $ET = G$ and $T \cap E \leq E_{\tau G}$. Clearly, $H \leq T$ and by [3, A, 14.1], T/H is subnormal in G/H. Besides, $(T/H)(EH/H) = G/H$ and $(T/H) \cap (EH/H) = (T \cap EH)/H = (T \cap E)H/H \leq E_{\tau G}H/H \leq (EH/H)_{\tau(G/H)}$ by Lemma 2(4). Thus EH/H is weakly τ-quasinormal in G/H.

(5) By hypothesis G has a subnormal subgroup T such that $HT = G$ and $T \cap H \leq H_{\tau G} \neq H$. Hence G has a proper normal subgroup V such that $T \leq V$. Since G/V is a p-group, G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$.

Recall that a formation is a homomorph $\mathcal{F}$ of groups such that each group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient is still in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if it contains each group G with $G/\Phi(G) \in \mathcal{F}$. In view of Lemma 1(5) and 2(1) the following two observations are corollaries of Lemmas 2.11 and 2.12 in [9] respectively.

Lemma 4. *Let N be an Abelian normal p-subgroup of a group G for some prime p. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly τ-quasinormal in G. Then some maximal subgroup of N is normal in G.*

Lemma 5. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and G a group with soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p and if every cyclic subgroup of P with prime order and order 4 (if $p = 2$ and P is non-Abelian) not having a p-nilpotent supplement in G is weakly τ-quasinormal in G, then $|P/\Phi(P)| = p$.*

Lemma 6. *Let $V = A \times B$ be a subgroup with order 4 of a group G such that $|A| = |B| = 2$ and A, V are τ-quasinormal in G. Then B is τ-quasinormal in G.*

Proof. Suppose that this lemma is false. Then G has a Sylow q-subgroup Q such that $q \neq 2$, $2 \mid |Q^G|$ and $BQ \neq QB$. By hypothesis, $VQ = QV$ and $AQ = QA$. Hence Q is normal in VQ. Therefore $BQ = QB$, a contradiction. This completes the proof.

Lemma 7. *Let $V = \langle x \rangle$ be a cyclic subgroup with order 4 of a group G and V is τ-quasinormal in G. Then $\langle x^2 \rangle$ is τ-quasinormal in G.*

Proof. See the proof of Lemma 6.

The following lemma is well known (see, for example, [8]).

Lemma 8. *Let H be a p-subgroup of a group G for some prime p. Then H is $\pi(G)$-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.*

In this paper we use $\mathcal{U}$ to denote the class of the supersoluble groups; $Z_{\mathcal{U}}(G)$ denotes the $\mathcal{U}$-hypercenter of a group G, that is, the product of all such normal subgroups of G whose G-chief factors have prime order.

Lemma 9. *Let G be a group, p an odd prime and $P \neq 1$ a normal p-subgroup of G. Suppose that every minimal subgroup of P is $\pi(G)$-quasinormal in G. Then $P \leq Z_{\mathcal{U}}(G)$.*

Proof. Suppose that this lemma is false and consider a counterexample (G, P) for which $|G||P|$ is minimal. First we show that $P \leq O^p(G)$. Indeed, assume $P \nleq O^p(G)$. Then $|P \cap O^p(G)| < |P|$, so $P \cap O^p(G) \leq Z_{\mathcal{U}}(G)$ by the choise of (G, P). On the other hand, from the G-isomorphism $PO^p(G)/O^p(G) \simeq P/P \cap O^p(G)$ we have $P/P \cap O^p(G) \leq Z_{\mathcal{U}}(G/P \cap O^p(G))$. Hence $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Therefore $P \leq O^p(G)$. Let G_p be a Sylow p-subgroup of G. Consider the series $1 \leq \Omega_1(P) \leq \Omega_2(P) \leq \ldots \leq \Omega_t(P) = P$. Since all members of this series are characteristic in P, the series may be refinemented to G_p-chief series of P

$$1 = P_0 \leq P_1 \leq \ldots \leq P_n = P. \qquad (*)$$

By Lemma 8 and [2, 1(i)] every factor $\Omega_i(P)/\Omega_{i-1}(P)$ is elementary and by [9, 2.4] every subgroup of $\Omega_i(P)/\Omega_{i-1}(P)$ is normalyzed by every p'-element of G. Hence the series $(*)$ is a chief series of G, so $P \leq Z_{\mathcal{U}}(G)$. This contradiction completes the proof.

Lemma 10. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-Abelian 2-group) not having a supersoluble supplement in G are weakly τ-quasinormal in G. Then $G \in \mathcal{F}$.*

Lemma 11. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-Abelian 2-group) not having a p-nilpotent supplement in G are weakly τ-quasinormal in G. Then G is p-nilpotent.*

Finally, note that main results of many papers are special cases of our Theorem (see Section 5 in [9]). In particular, Theorem 1.3 in [9] is a corollary of the theorem.

Abstract. Let G be a finite group and H a subgroup of G. We put $\tau_G(H) = \{q \in \pi(G) \setminus \pi(H) \mid (|H|, |Q^G|) \neq 1$ for a Sylow q-subgroup Q of $G\}$. We say that: (1) H is *τ-quasinormal in G* if H permute with all Sylow q-subgroups of G for all $q \in \tau_G(H)$; (2) H is *weakly τ-quasinormal in G* if G has a subnormal subgroup T such that $HT = G$ and $T \cap H \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ-quasinormal in G. Our main result here is the following theorem: *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that*

$1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-Abelian 2-group) are weakly τ-quasinormal in G. Then $G \in \mathcal{F}$.

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