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Partial two-particle relativistic scattering problems and superpositions of *δ*-shell potentials

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A method for approximate solution of two-particle relativistic equations with analytic potentials, which is based on smooth quasipotential replacing by superpositions of delta-potentials is discussed. Expressions for the scattering cross section calculation in the case of nonzero angular momentum of the two particle system are obtained. For a model potential a numerical analysis of *s*- and *p*- scattering states of two-particle systems is performed, and the partial cross section resonant behavior is investigated.
Keywords: quasipotential approach, delta-shell potentials, scattering cross section, resonance states.

Обсуждается метод приближенного решения двухчастичных релятивистских уравнений с аналитическими потенциалами, который основан на замене гладких квазипотенциалов суперпозициями дельта-потенциалов. Получены выражения для вычисления сечения рассеяния в случае ненулевых орбитальных моментов двухчастичной системы. На примере модельного потенциала проведен численный анализ *s*- и *p*-состояний рассеянных частиц, исследовано резонансное поведение парциальных сечений рассеяния.
Ключевые слова: квазипотенциальный подход, дельта-потенциалы, сечение рассеяния, резонансные состояния.

1. Introduction. In this paper we investigate the scattering states of two spinless particles of the same mass m each on the basis of three-dimensional quasipotential equations of quantum field theory [1], [2]. In this approach the interaction between two relativistic particles is defined by the quasipotential, which in the momentum representation (MR) depends on the initial and final relative momenta of the particles in the center-of-mass system and also on the two-particle energy $2E_q$ in the most general case. Since the problem of relativistic potentials finding in quantum field theory is complicated, various phenomenological potentials, not depending on energy, are frequently used by analogy with the non-relativistic theory. In addition to that the coordinate dependence of potentials in relativistic configurational representation (RCR) is often chosen to be similar to the coordinate dependence of non-relativistic potentials. Nevertheless, the number of relativistic problems having exact analytical solutions is much less than that in the non-relativistic theory. At the same time, the exactly solvable models allow us to draw important conclusions about the general properties of the quasipotential equations which contain more complex potentials, not allowing exact solutions.

In the non-relativistic theory much attention is being attracted by the potentials which are defined with the help of the Dirac's delta function, since scattering problems for such potentials can be solved exactly [3]–[7]. Similar problems have been considered in the relativistic theory as well. Exact solutions of the one-dimensional two-particle equations with d- potentials were obtained in [8], [9]. Exact solutions of the three-dimensional equations with a superposition of two d- shell potentials were investigated in detail for *s*-states ($l = 0$) of two spinless particles in [10].

The method presented below is for approximate solution of the relativistic two-particle scattering problems with quasipotentials of sufficiently general form, in the case of nonzero angular momentum l . It is based on the approximation of a smooth quasipotential function by a superposition of N d-shell potentials.

2. Relativistic equations in the coordinate representation. We consider the case when the quasipotential in the Lobachevsky momentum space is local and spherically symmetric [11, p. 677]. In such a case the relativistic integral equations of quantum field theory and the relativistic generalization of the Schrödinger equation in the integral form [12, p. 264] are analogous. In the relativistic configuration representation (RCR) these quasipotential equations have the form:

$$\psi(\vec{q},\vec{r}) = \xi(\vec{q},\vec{r}) + \int G_0(E_q;\vec{r},\vec{r}\,')\,V(E_q;\vec{r}\,')\,\psi(\vec{q},\vec{r}\,')\,d\vec{r}\,'. \tag{1}$$

Here the functions $\xi(\vec{q},\vec{r})$ are the solution of the free of interaction problem; they implement unitary irreducible representations of the Lorentz group [11, p. 659]:

$$\xi(\vec{p},\vec{r}) = \left(\frac{E_p - \vec{p}\vec{n}}{m}\right)^{-1-imr}. \tag{2}$$

The connection between the Green's functions in the RCR and the MR is determined by equation of the form [11, p. 660]:

$$G_0(E_q;\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\int \xi(\vec{p},\vec{r})\,G_0(E_q,E_p)\,\xi^*(\vec{p},\vec{r}\,')\frac{m}{E_p}\,d\vec{p}. \tag{3}$$

The functions $G_0(E_q,E_p)$ can take the form of the Green's functions of the Logunov–Tavkhelidze $(j=1)$, and the Kadyshevsky $(j=2)$ equations or their modified versions $(j=3,\ j=4)$ [10, p. 2]:

$$G_{0,1}(E_q,E_p) = \frac{1}{E_q^2 - E_p^2 + i0}; \qquad G_{0,2}(E_q,E_p) = \frac{1}{E_p(2E_q - 2E_p + i0)};$$
$$G_{0,3}(E_q,E_p) = \frac{E_p/m}{E_q^2 - E_p^2 + i0}; \qquad G_{0,4}(E_q,E_p) = \frac{1}{m(2E_q - 2E_p + i0)}. \tag{4}$$

The problem of three-dimensional integral equation (1) solving can be greatly simplified by reducing it to the one-dimensional form. For convenience, we also use the rapidity variable (χ):

$$\vec{p} = m\,\mathrm{sh}\chi_p\,\vec{n}_p; \quad E_p = m\,\mathrm{ch}\chi_p; \quad \frac{m}{E_p}dp = m\,d\chi_p. \tag{5}$$

Scalar plane wave (2), the wave function $\psi(\vec{q},\vec{r})$ and the Green's function can be expanded in the basis of spherical harmonics [12, p. 38]:

$$\xi(\vec{p},\vec{r}) = \frac{4\pi}{pr}\sum_{lm} i^l\,\xi_l(\chi_p,r)\,Y_{lm}(\vec{n}_r)\,Y^*_{lm}(\vec{n}_p). \tag{6}$$

$$\psi(\vec{q},\vec{r}) = \frac{4\pi}{qr}\sum_{lm} i^l\,\psi_l(\chi_q,r)\,Y_{lm}(\vec{n}_r)\,Y^*_{lm}(\vec{n}_q); \tag{7}$$

$$G_0(E_q;\vec{r},\vec{r}\,') = \frac{1}{rr'}\sum_{lm} G_0^{(l)}(\chi_q;r,r')\,Y_{lm}(\vec{n}_r)\,Y^*_{lm}(\vec{n}_{r'}); \tag{8}$$

Substituting partial decompositions (6) and (8) into (3), it is necessary to execute the integration over the angles defining the vector $\vec{p}$ direction. Equating then the sums and using the spherical harmonic properties one can obtain the formula, connecting partial Green's functions in the RCR with partial Green's functions in the MR:

$$G_0^{(l)}(\chi_q;r,r') = \frac{2}{\pi}m\int_0^\infty \xi_l(\chi_p,r)\,G_0(E_q,E_p)\,\xi_l^*(\chi_p,r')\,d\chi_p; \tag{9}$$

Let us perform now the partial analysis of the equation (1). Having substituted expansions in spherical harmonics (6), (7), (8) into (1), and integrating over the angles defining the vector $\vec{r}\,'$ direction, and using the spherical harmonic properties again, we can find the one-dimensional partial differential equations in the *r*-representation:

$$\psi_l(\chi_q,r) = \xi_l(\chi_q,r) + \int_0^\infty G_0^{(l)}(\chi_q;r,r')\,V(E_q;r')\,\psi_l(\chi_q,r')\,dr'. \tag{10}$$

3. Partial waves and partial Green's functions in the RCR. Using formula (6) it is easy to obtain the partial decomposition of the relativistic plane wave in the basis of Legendre polynomials [13, p. 80]. Taking into account the explicit form of functions (2), one can easily obtain the integral expression for the determination of the explicit form of the partial components of the plane wave:

$$\xi_l(\chi,r) = \frac{(-i)^l}{2}\,m r\,\mathrm{sh}\chi\int_{-1}^{1}(\mathrm{ch}\chi - z\,\mathrm{sh}\chi)^{-1-imr}P_l(z)\,dz. \tag{11}$$

Performing the integration in (11) for $\ell = 0, 1$, one can write down the explicit form of the partial waves in the RCR:

$$\xi_0(\chi,\rho) = \sin(m\rho\chi); \tag{12}$$

$$\xi_1(\chi,\rho) = \frac{\operatorname{cth}\chi \sin(m\rho\chi) - m\rho\cos(m\rho\chi)}{m\rho + i}. \tag{13}$$

Having substituted (12) and (13) into (9), it can be shown that the partial Green's functions are represented by the following convenient formula ($\rho = m(r - r')$; $\rho' = m(r + r')$):

$$G_0^{(\ell)}(\chi_q;\rho,\rho') = \frac{\Gamma(im\rho - \ell)}{\Gamma(im\rho)} \times \frac{\Gamma(-im\rho' - \ell)}{\Gamma(-im\rho')}\left[\tilde{G}^{(\ell)}(\chi_q;\rho,\rho') - \tilde{G}^{(\ell)}(\chi_q;\rho',\rho)\right]. \tag{14}$$

Green's functions (4) in rapidity space have the form:

$$G_{0,1}(E_q,E_p) = \frac{1}{m^2\operatorname{ch}^2\chi_q - m^2\operatorname{ch}^2\chi_p + i0}; \quad G_{0,2}(E_q,E_p) = \frac{1}{m\operatorname{ch}\chi_p(2m\operatorname{ch}\chi_q - 2m\operatorname{ch}\chi_p + i0)};$$
$$G_{0,3}(E_q,E_p) = \frac{\operatorname{ch}\chi_p}{m^2\operatorname{ch}^2\chi_q - m^2\operatorname{ch}^2\chi_p + i0}; \quad G_{0,4}(E_q,E_p) = \frac{1}{m(2m\operatorname{ch}\chi_q - 2m\operatorname{ch}\chi_p + i0)}. \tag{15}$$

The quantities that make up formula (14) can be calculated using methods of complex analysis; it is necessary to simply construct the contour integral and apply the residue theorem [14, p. 146]. Performing calculations at $\ell = 0$, we obtain:

$$\tilde{G}_1^{(0)}(\chi_q;\rho,\rho') = \frac{-i}{m} \times \frac{\operatorname{sh}[\rho(\pi/2 + i\chi_q)]}{\operatorname{sh}(\rho\pi/2)\operatorname{sh}(2\chi_q)}; \quad \tilde{G}_2^{(0)}(\chi_q;\rho,\rho') = \frac{-i}{m} \times \frac{\operatorname{sh}[\rho(\pi + i\chi_q)]}{\operatorname{sh}(\rho\pi)\operatorname{sh}(2\chi_q)} + \frac{(4m\operatorname{ch}\chi_q)^{-1}}{\operatorname{ch}(\rho\pi/2)};$$
$$\tilde{G}_3^{(0)}(\chi_q;\rho,\rho') = \frac{-i}{2m} \times \frac{\operatorname{ch}[\rho(\pi/2 + i\chi_q)]}{\operatorname{ch}(\rho\pi/2)\operatorname{sh}\chi_q}; \quad \tilde{G}_4^{(0)}(\chi_q;\rho,\rho') = \frac{-i}{2m} \times \frac{\operatorname{sh}[\rho(\pi + i\chi_q)]}{\operatorname{sh}(\rho\pi)\operatorname{sh}\chi_q}. \tag{16}$$

It should be noted that these partial components of the Green's functions (for *s*-states), calculated in this way, actually coincide with the results obtained in [8]–[10] by another method. Performing now calculations at $\ell = 1$, we obtain

$$\tilde{G}_1^{(1)}(\chi_q;\rho,\rho') = \tilde{G}_1^{(0)}(\chi_q;\rho,\rho')\left[\operatorname{cth}^2\chi_q + m\rho\frac{\rho' - \rho}{2} - i\rho\operatorname{cth}[\rho(\pi/2 + i\chi_q)]\operatorname{cth}\chi_q\right];$$
$$\tilde{G}_2^{(1)}(\chi_q;\rho,\rho') = \tilde{G}_2^{(0)}(\chi_q;\rho,\rho')\left[\operatorname{cth}^2\chi_q + m\rho\frac{\rho' - \rho}{2} - i\rho\operatorname{cth}[\rho(\pi + i\chi_q)]\operatorname{cth}\chi_q\right]$$
$$- \frac{(4m\operatorname{ch}\chi_q\operatorname{ch}(\rho\pi/2))^{-1}}{(m\rho + i)(m\rho' - i)}\left[\operatorname{cth}^2\chi_q - i\rho\operatorname{cth}[\rho(\pi + i\chi_q)]\operatorname{cth}\chi_q\right]; \tag{17}$$
$$\tilde{G}_3^{(1)}(\chi_q;\rho,\rho') = \tilde{G}_3^{(0)}(\chi_q;\rho,\rho')\left[\operatorname{cth}^2\chi_q + m\rho\frac{\rho' - \rho}{2} - i\rho\operatorname{th}[\rho(\pi/2 + i\chi_q)]\operatorname{cth}\chi_q\right];$$
$$\tilde{G}_4^{(1)}(\chi_q;\rho,\rho') = \tilde{G}_4^{(0)}(\chi_q;\rho,\rho')\left[\operatorname{cth}^2\chi_q + m\rho\frac{\rho' - \rho}{2} - i\rho\operatorname{cth}[\rho(\pi + i\chi_q)]\operatorname{cth}\chi_q\right].$$

These explicit forms of the partial Green's functions can be now used at partial equations solving.

4. Approximate solution method. Let us consider the solution of relativistic partial equations (10) containing the superposition of δ- potentials, each of which is localized on a sphere of a finite radius $r_k > 0$. According to papers [8]–[10] we use the analogy with the non-relativistic theory for the determination of the relativistic scattering amplitude for the relativistic scattering amplitude determination. In the non-relativistic theory the following property is used for finding the scattering amplitude: the asymptotic wave function expression at $r \to \infty$ can be represented as the sum of the incident plane wave and the outgoing spherical wave [15, c. 186]. In turn, the asymptotic expression of the partial wave function at $r \to \infty$ has the form [15, p. 187]:

$$\psi_\ell(p,r)\big|_{r\to\infty} \approx \hat{j}_\ell(pr) + p\, f_\ell(p)\, e^{i(pr - \ell\pi/2)}, \tag{18}$$

here $f_\ell(p)$ are the partial scattering amplitudes via which the total scattering cross section can be determined as the partial scattering cross sections sum [15, c. 89]:

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell = 4\pi\sum_{\ell=0}^{\infty}(2\ell + 1)\left|f_\ell(p)\right|^2. \tag{19}$$

Using this approach, we consider the method of approximate solution of relativistic problems with smooth potentials, similar to how it was done in the non-relativistic theory [7, c. 534]. This method is based on using superpositions of N δ-shell potentials instead of the analytical (smooth) potential:

$$V(r) \to \sum_{k=1}^{N} V_k \delta(r - r_k); \qquad V_k = V\left(\frac{r_k + r_{k-1}}{2}\right)\cdot\frac{r_k - r_{k-1}}{2}. \tag{20}$$

Substituting (20) into (10) and integrating with respect to r', one obtains

$$\psi_\ell(\chi_q, r) = \xi_\ell(\chi_q, r) + \sum_{k=1}^{N} G_0^{(\ell)}(\chi_q; r, r_k) V_k \psi_\ell(\chi_q, r_k). \tag{21}$$

Taking equality (20) at points r_k one can obtain the following linear algebraic equations system (LAES):

$$\begin{pmatrix} 1 - G_0^{(\ell)}(\chi_q; r_1, r_1)V_1 & -G_0^{(\ell)}(\chi_q; r_1, r_2)V_2 & \ldots & -G_0^{(\ell)}(\chi_q; r_1, r_N)V_N \\ -G_0^{(\ell)}(\chi_q; r_2, r_1)V_1 & 1 - G_0^{(\ell)}(\chi_q; r_2, r_2)V_2 & \ldots & -G_0^{(\ell)}(\chi_q; r_2, r_N)V_N \\ \vdots & \vdots & \ddots & \vdots \\ -G_0^{(\ell)}(\chi_q; r_N, r_1)V_1 & -G_0^{(\ell)}(\chi_q; r_N, r_2)V_2 & \ldots & 1 - G_0^{(\ell)}(\chi_q; r_N, r_N)V_N \end{pmatrix} \begin{pmatrix} \psi_\ell(\chi_q, r_1) \\ \psi_\ell(\chi_q, r_2) \\ \vdots \\ \psi_\ell(\chi_q, r_N) \end{pmatrix} = \begin{pmatrix} \xi_\ell(\chi_q, r_1) \\ \xi_\ell(\chi_q, r_2) \\ \vdots \\ \xi_\ell(\chi_q, r_N) \end{pmatrix}. \tag{22}$$

The asymptotic behavior of the wave function (21) at $r \to \infty$ will be as follows:

$$\psi_\ell(\chi_q, r)\Big|_{r\to\infty} = \xi_\ell(\chi_q, r) + \left(\sum_{k=1}^{N} V_k \frac{-\xi_\ell^*(\chi_q, r_k)}{m\,\mathrm{sh}\chi_q\,\mathrm{ch}\tilde{\chi}_{(j)}} \psi_\ell(\chi_q, r_k)\right) e^{i(mr\chi_q - \pi\ell/2)}. \tag{23}$$

According to formula (18) one can obtain from (23) the relativistic scattering amplitude:

$$f_\ell(\chi_q) = \frac{-1}{m^2\,\mathrm{sh}^2\chi_q\,\mathrm{ch}\tilde{\chi}_{(j)}} \cdot \sum_{k=1}^{N} \xi_\ell^*(\chi_q, r_k) V_k \psi_\ell(\chi_q, r_k), \tag{24}$$

where $\psi_\ell(\chi_q, r_k)$ are solutions of LAES (22) and $\tilde{\chi}_{(j)} = (\chi_q, \chi_q, 0, 0)$ in the four cases under consideration. The total scattering cross section, in accordance with (19), can be then written as

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1)\left|f_\ell(\chi_q)\right|^2. \tag{25}$$

Thus, substituting the parameters of potential (20) and the explicit form of the partial waves (12), (13) and the Green's functions (16) and (17) into (22), one can easily calculate the scattering amplitude (24) and scattering cross section (25).

5. Numerical analysis example of the scattering problem with a model potential. Consider the numerical analysis algorithm on the model potential example [7, c. 538], allowing the existence of resonance states:

$$V(r) = A r^2 e^{-r}. \tag{26}$$

First, we approximate this smooth potential by a superposition of N δ-shell potentials. The choice of the approximation method determines how quickly the numerical solution approaches the exact solution. We used several methods and identified how features of the potential define the best. Then we construct the mesh of the potential and the resulting coordinate array and the potential value array and use them for the scattering amplitude (24) calculating and the partial cross section (25) finding.

Figure 1 shows the dependence of the partial cross sections σ_ℓ on the two-particle energy E_q for all Green's functions (j=1–4). The left picture shows the results in the $\ell = 0$ case (s–states), the right one – in the $\ell = 1$ case (p–states). For these calculations we used the superposition of fifty δ-shell potentials (N=50). Analysis of the partial cross sections shows that their dependence on the angular momentum ℓ has many similarities with according results in the relativistic one-particle scattering problem case [16, c. 187]. Partial cross sections have the following feature: in the $\ell = 0$ case the cross section has its maximum value at $E_q = m$, on the other hand for all other values of ℓ the partial cross sections tend to zero at $E_q = m$. Partial cross sections are significantly different from zero only in a certain energy range, with the l - increasing this range moves to the larger energy area. It gives excellent opportunity to exactly construct the total cross section for some energy interval using formula (25).

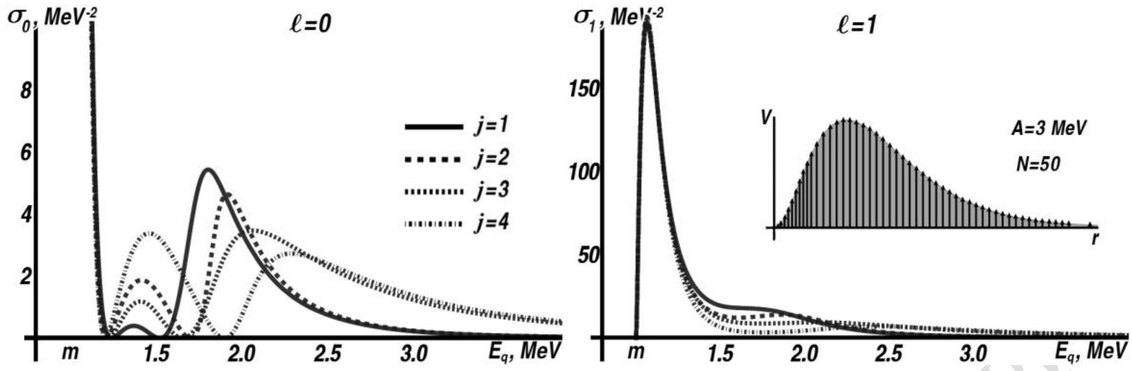

Figure 1 – The dependence of the partial cross sections s_0 and s_1 on energy E_q (m = 1 MeV)

The obtained dependence of partial cross sections on energy E_q has resonant character [15, p. 238]. At the resonance energy one can see a sharp increase of the scattering cross section. The number of resonant states increases with the parameter A increasing. The partial cross section energy dependence tends to a certain limit (exact solution) with the parameter N increasing. The positions of the resonance peaks (especially narrow ones) on the energy scale is determined quite accurately already with a small number of potentials. The maximum errors of solutions are observed near the resonance energies. The relative error in the $l = 0$ case does not exceed 1,5 % (in the $l = 1$ case – 10 %), for the numerical calculation parameters taken. It is important to note that even with a small amount of d- potentials it is possible to obtain qualitatively correct description of the physical process of particle interaction.

6. Conclusions. In this paper it is shown that the method, based on the analytical potential replacing by superpositions of delta-potentials is effective in solving the two-particle equations of quantum field theory for the scattering states. Even for small number of delta-functions ($N = 50$) the method allows us to obtain sufficiently accurate results for the partial and total cross sections and predict with good accuracy the positions and nature of the resonances. The study of the provisions of the resonances in the complex energy plane at non-zero l - values, which will be discussed in a separate paper, is of great interest.

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