Boundedness of Hausdorff operators on Hardy spaces over homogeneous spaces of Lie groups

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Abstract. The aim of this note is to give the boundedness conditions for Hausdorff operators on Hardy spaces H^1 with the norm defined via $(1, q)$ atoms over homogeneous spaces of Lie groups with doubling property and to apply results we obtain to generalized Delsarte operators and to Hausdorff operators over multidimensional spheres.

Key words. Hausdorff operator, Lie group, homogeneous space, Hardy space, generalized shift operator of Delsarte.

MSC classes: 43A85, 47G10, 22E30

1 Introduction

One-dimensional Hausdorff operators were introduced by Hardy [1, Section 11.18] as a transformations of functions of a continuous variable analogous to the regular Hausdorff transformations for sequences and series. Although occasionally one-dimensional Hausdorff operators appeared before 2000 (see [2] and [3]), the modern development of this theory begins with the work of Liflyand and Moricz [4] where Hausdorff operators on one-dimensional Hardy space were considered. The multidimensional case was studied in [5]. For more details of the development of the theory of Hausdorff operators up to 2014 see [6], and [7].

Hausdorff operators on the Hardy space H^1 over homogeneous spaces of locally compact groups were first introduced by the author in [8] for the case of doubling measures, and in [9] for the case of locally doubling measures. The case of locally compact groups was considered earlier in [10]. The aim of this note is to improve and generalize results from [8] to the case of Hardy spaces $H^1(G/K)$ with the norm defined via $(1, q)$ atoms when G is a Lie group and to apply results we obtain to generalized Delsarte operators and to Hausdorff operators over multidimensional spheres.

2 The main result

Let G be a locally compact metrizable group with left invariant metric ρ and left Haar measure ν. We assume that the following *doubling condition* in a sense of [11] holds:

there exists a constant C such that

$$\nu(B(x, 2r)) \leq C\nu(B(x, r))$$

for each $x \in G$ and $r > 0$; here $B(x, r)$ denotes the ball of radius r around x.

The *doubling constant* is the smallest constant $C \geq 1$ for which the last inequality is valid. We denote this constant by C_ν. Then for each $x \in G, k \geq 1$ and $r > 0$

$$\nu(B(x, kr)) \leq C_\nu k^d \nu(B(x, r)), \qquad (D)$$

where $d = \log_2 C_\nu$ (see, e.g., [12, p. 76]). The number d takes the role of a "dimension" for a doubling metric measure space G.

Homogeneous group in a sense of Folland and Stein [13] (i.e., a connected simply connected Lie group G whose Lie algebra is equipped with dilations) enjoys the doubling condition and $C_\nu = 2^Q$, where Q stands for the homogeneous dimension of G [14, Lemma 3.2.12]. A complete Riemannian manifold, with Ricci curvature nonnegative outside a compact subset of the manifold, satisfies the doubling condition, as demonstrated in [15, Lemma 1.3]. Compact Lie groups endowed with Riemann metric and Haar measure satisfy the doubling condition, too [11, p. 588, Example (7)]. For complete noncompact manifolds with nonnegative Ricci curvature, the doubling property for the volume measure follows from the volume comparison inequality of Bishop and Gromov [16, Theorem 10.6.6].

We denote by $\mathrm{Aut}(G)$ the space of all topological automorphisms of G endowed with its natural topology $\mathcal{T}_\beta$ [17, Ch. X, §3, n 5], $\mathcal{L}(Y)$ denotes the space of linear bounded operators on a normed space Y.

Let K be a compact subgroup of G with normalized Haar measure β. Consider the quotient space G/K of left cosets $\dot{x} := xK = \pi_K(x)$ ($x \in G$) where $\pi_K : G \to G/K$ stands for a natural projection. We shall assume that the measure ν is normalized in such a way that (generalized) Weil's formula

$$\int_G g(x)dx = \int_{G/K} \left(\int_K g(xk)dk \right) d\lambda(\dot{x}) \qquad (1)$$

holds for all $g \in L^1(G)$, where λ denotes some left-G-invariant measure on G/K (see [18, Chapter VII, §2, No. 5, Theorem 2] and especially remark c) after this theorem or [19, Proposition 10.4.12]). Here G-left invariance of

2

λ means that $\lambda(xE) = \lambda(E)$ for every Borel subset E of G/K and for every $x \in G$. This measure is unique up to constant multiplier.

Henceforth we write dx instead of $d\nu(x)$ and dk instead of $d\beta(k)$. We shall write also $d\dot{x}$ instead of $d\lambda(\dot{x})$.

The function $g : G \to \mathbb{C}$ is called *right-K-invariant* if $g(xk) = g(x)$ for all $x \in G$, $k \in K$. For such a function we put $\dot{g}(\dot{x}) := g(x)$. This definition is correct and for $g \in L^1(G)$ formula (1) implies that

$$\int_G g(x)dx = \int_{G/K} \dot{g}(\dot{x})d\dot{x} \tag{2}$$

(recall that $\int_K dk = 1$).

The map $g \mapsto \dot{g}$ is a bijection between the set of all right-K-invariant functions on G (all right-K-invariant functions from $L^1(G)$) and the set of all functions on G/K (respectively functions from $L^1(G/K, \lambda)$).

Let an automorphism $A \in \mathrm{Aut}(G)$ maps K onto itself. Since

$$A(\dot{x}) := A(xK) = \{A(x)A(k) : k \in K\} = A(x)K = \pi_K(A(x))$$

we get a homeomorphism $\dot{A} : G/K \to G/K$, $\dot{A}(\dot{x}) := \pi_K(A(x))$. Then for every right-$K$-invariant function g on G we have $\dot{g}(\dot{A}(\dot{x})) = g(A(x))$.

We put

$$\mathrm{Aut}_K(G) := \{\dot{A} : A \in \mathrm{Aut}(G), A(K) = K\}.$$

A ν-measurable function a on G is called an $(1, q)$-*atom* $(q \in (1, \infty])$ if
(i) the support of a is contained in a ball $B(x, r)$;
(ii) $\|a\|_\infty \leq \frac{1}{\nu(B(x,r))}$ if $q = \infty$, and
$\|a\|_q \leq \nu(B(x, r))^{\frac{1}{q}-1}$ if $q \in (1, \infty)$[1];
(iii) $\int_G a(x)d\nu(x) = 0$.

In case $\nu(G) < \infty$ we shall assume $\nu(G) = 1$; in this case the constant function having value 1 is also considered to be an atom.

Hereafter by atom we mean an $(1, q)$-atom on G.

Definition 1. [8], [9]. We define the *Hardy space* $H^1(G/K) = H^{1,q}(G/K)$[2] as a space of such functions f on G/K that admit an atomic decomposition of the form

$$f = \sum_{j=1}^\infty \lambda_j \dot{a}_j$$

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[1]As usual, $\| \cdot \|_q$ denotes the L^q norm.
[2]It is known that $H^{1,q}(G/K)$ does not depend on $q \in (1, \infty]$ [11, Theorem A, p. 592]. We write $H^{1,q}(G/K)$ instead of $H^1(G/K)$ in order to stress the fact that we use the norm $\| \cdot \|_{H^{1,q}(G/K)}$ described below.

3

where a_j are right-K-invariant $(1, q)$-atoms on G and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. In this case,

$$\|f\|_{H^{1,q}(G/K)} := \inf \sum_{j=1}^{\infty} |\lambda_j|,$$

and infimum is taken over all decompositions above of f.

In other words, $f = \dot{g}$ where $g = \sum_{j=1}^{\infty} \lambda_j a_j$, a_j are right-K-invariant $(1, q)$-atoms on G, and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Moreover, $\|f\|_{H^{1,q}(G/K)} = \|g\|_{H^{1,q}(G)}$.

Remark 1. Real Hardy spaces over compact connected (not necessary quasi-metric) Abelian groups were defined in [20].

Proposition 1. [9]. *Let $G \neq K$. Then the space $H^{1,q}(G/K)$ is nontrivial and Banach.*

Definition 2. [8]. Let (Ω, μ) be a measure space, $(\dot{A}(u))_{u \in \Omega} \subset \mathrm{Aut}_K(G)$ a family of homeomorphisms of G/K, and Φ a measurable function on (Ω, μ). For a Borel measurable function f on G/K we define a *Hausdorff operator on G/K* as follows

$$(\mathcal{H}_{\Phi, \dot{A}} f)(\dot{x}) := \int_{\Omega} \Phi(u) f(\dot{A}(u)(\dot{x})) d\mu(u).$$

For the proof of our main result the next two lemmas are crucial.

Lemma 1. [10]. *Let (Ω, μ) be σ-compact quasi-metric space with positive Radon measure μ, (X, m) be a measure space and $\mathcal{F}(X)$ be some Banach space of m-measurable functions on X. Assume that the convergence of a sequence strongly in $\mathcal{F}(X)$ yields the convergence of some subsequence to the same function for m-almost all $x \in X$. Let $F(u, x)$ be a function such that $F(u, \cdot) \in \mathcal{F}(X)$ for μ-almost all $u \in \Omega$ and the map $u \mapsto F(u, \cdot) : \Omega \to \mathcal{F}(X)$ is Bochner integrable with respect to μ. Then for m-almost all $x \in X$*

$$\left((B) \int_{\Omega} F(u, \cdot) d\mu(u) \right)(x) = \int_{\Omega} F(u, x) d\mu(u).$$

Lemma 2. *Let G be a (finite dimensional real or complex) connected Lie group with left invariant Riemann metric ρ. Then every automorphism $\varphi \in \mathrm{Aut}(G)$ is Lipschitz with Lipschitz constant $\|(d\varphi)_e\|$.*

Proof. Let $T_a(G)$ denotes the tangent space for G at the point $a \in G$. Let $L_a : x \mapsto ax$ be the left translation in G. Then the tangent map $l_a := (dL_a)_e : T_e(G) \to T_a(G)$ is a bijection. We fix the Euclidean norm $\| \cdot \|$ in $T_e(G)$ and introduce the norm in $T_a(G)$ by the rule $\|X_a\| := \|X_e\|$ if $X_a = l_a(X_e)$, $X_e \in T_e(G)$, $a \in G$.

As is well known, for every $p, q \in G$

$$\rho(p, q) = \inf_{\alpha} \int_0^1 \|\alpha'(t)\| \, dt$$

4

where infimum is taken over all piecewise smooth curves α from $[0,1]$ to G with $\alpha(0) = p$, $\alpha(1) = q$ ($\alpha'(t)$ stands, as usual, for the tangent vector to α at the point $\alpha(t)$). Since $\varphi \in \mathrm{Aut}(G)$, the formula $\beta = \varphi \circ \alpha$ gives the general form of all piecewise smooth curves in G with $\beta(0) = \varphi(p)$ and $\beta(1) = \varphi(q)$. Thus, by the chain rule

$$\rho(\varphi(p), \varphi(q)) = \inf_{\alpha} \int_0^1 \|(\varphi \circ \alpha)'(t)\| \, dt$$

$$= \inf_{\alpha} \int_0^1 \left\|(d\varphi)_{\alpha(t)} \alpha'(t)\right\| dt \le \inf_{\alpha} \int_0^1 \left\|(d\varphi)_{\alpha(t)}\right\| \|\alpha'(t)\| \, dt.$$

It is known (see, e.g., [21]) that for every left invariant vector field X on G (this means that $X_a = l_a(X_e)$ for all $a \in G$) the vector field $(d\varphi)(X)$[3] is left invariant, too. In other wards, $(d\varphi)_a(X_a) = l_a(d\varphi)_e(X_e)$, i.e., $(d\varphi)_a = l_a((d\varphi)_e)l_a^{-1}$ and therefore $\|(d\varphi)_a\| = \|(d\varphi)_e\|$ for all $a \in G$. The result follows.

Now we are in a position to prove the next

Theorem 1. *Let G be a (finite dimensional real or complex) connected Lie group with left invariant Riemann metric ρ and left Haar measure ν such that the space (G, ρ, ν) is doubling. Let (Ω, μ) be σ-compact quasi-metric space with positive Radon measure μ, and let $q \in (1, \infty]$. If*

$$\|\Phi\|_{A,q} := \int_\Omega |\Phi(u)| (\mathrm{mod} A(u))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} d\mu(u) < \infty$$

where $k(u) := \|(d(A(u)^{-1}))_e\|$, then the operator $\mathcal{H}_{\Phi, \dot{A}}$ is bounded on the space $H^{1,q}(G/K)$ and

$$\|\mathcal{H}_{\Phi, \dot{A}}\|_{\mathcal{L}(H^{1,q}(G/K))} \le C_\nu^{1-\frac{1}{q}} \|\Phi\|_{A,q}.$$

Proof. If we set $X = G/K$ and $m = \lambda$ the pair (X, m) satisfies the conditions of Lemma 1 with $H^{1,q}(G/K)$ in place of $\mathcal{F}(X)$. Indeed, let $f_n = \dot{g}_n \in H^{1,q}(G/K)$, $f = \dot{g} \in H^{1,q}(G/K)$, and $\|f_n - f\|_{H^{1,q}(G/K)} \to 0$ $(n \to \infty)$. Since

$$\|f_n - f\|_{L^1(G/K)} = \int_{G/K} |\pi_K(g_n - g)| d\lambda$$

$$= \int_G |g_n(x) - g(x)| dx \le \|g_n - g\|_{H^{1,q}(G)} = \|f_n - f\|_{H^{1,q}(G/K)} \to 0$$

(by Hölder inequality $\|a\|_1 \le 1$ for each atom a), there is a subsequence of f_n that converges to f λ-a.e.

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[3]In [21] the map $d\varphi$ is denoted by $L(\varphi)$

Then Definition 2 and Lemma 1 imply for $f \in H^{1,q}(G/K)$ that

$$\mathcal{H}_{\Phi,\dot{A}}f = \int_\Omega \Phi(u) f \circ \dot{A}(u) d\mu(u),$$

the Bochner integral (recall that $H^{1,q}(G/K)$ is a subspace of $L^1(G/K, \lambda)$ [11, p. 592], and thus we identify functions that equal λ-a.e.).

Therefore (below $f = \dot{g}$)

$$\|\mathcal{H}_{\Phi,\dot{A}}f\|_{H^{1,q}(G/K)} \leq \int_\Omega |\Phi(u)| \|f \circ \dot{A}(u)\|_{H^{1,q}(G/K)} d\mu(u)$$

$$= \int_\Omega |\Phi(u)| \|g \circ A(u)\|_{H^{1,q}(G)} d\mu(u).$$

If $g = \sum_{j=1}^\infty \lambda_j a_j$ then

$$g \circ A(u) = \sum_{j=1}^\infty \lambda_j a_j \circ A(u). \tag{3}$$

We claim that

$$b_{j,u} := C_\nu^{\frac{1}{q}-1} (\mathrm{mod}(A(u))^{\frac{1}{q}} k(u)^{(\frac{1}{q}-1)s} a_j \circ A(u)$$

is an atom, too. Indeed, Lemma 2 implies that

$$A(u)^{-1}(B(x,r)) \subseteq B(x', k(u)r),$$

where $x' = A(u)^{-1}(x)$. If a_j is supported in $B(x_j, r_j)$ then $b_{j,u}$ is supported in $B(x'_j, k(u)r_j)$. So the condition (i) holds for $b_{j,u}$.

Next, by the property (D) we have

$$\nu(B(x_j, k(u)r_j)) \leq C_\nu k(u)^d \nu(B(x_j, r_j)).$$

This estimate yields in view of (ii) and the left invariance of ρ and ν that

$$\|a_j \circ A(u)\|_q = \left(\int_G |a_j \circ A(u)| d\nu \right)^{\frac{1}{q}} = (\mathrm{mod}(A(u))^{-\frac{1}{q}} \|a_j\|_q$$

$$\leq (\mathrm{mod}(A(u))^{-\frac{1}{q}} (\nu(B(x_j, r_j)))^{\frac{1}{q}-1} \leq (\mathrm{mod}(A(u))^{-\frac{1}{q}} \left(\frac{\nu(B(x_j, k(u)r_j))}{C_\nu k(u)^d} \right)^{\frac{1}{q}-1}$$

$$= \left(C_\nu^{\frac{1}{q}-1} (\mathrm{mod}(A(u))^{\frac{1}{q}} k(u)^{(\frac{1}{q}-1)d} \right)^{-1} (\nu(B(x'_j, k(u)r_j)))^{\frac{1}{q}-1}.$$

Thus, condition (ii) holds for $b_{j,u}$, too. Finally, the validity of (iii) follows from [18, VII.1.4, formula (31)].

Since formula (3) can be rewritten in the form

$$g \circ A(u) = \sum_{j=1}^{\infty} \left(\lambda_j C_\nu^{1-\frac{1}{q}} (\operatorname{mod}(A(u))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} \right) b_{j,u},$$

we have

$$\|g \circ A(u)\|_{H^{1,q}(G)} \leq C_\nu^{1-\frac{1}{q}} (\operatorname{mod}(A(u))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} \sum_{j=1}^{\infty} |\lambda_j|.$$

It follows that (recall that $f = \dot{g}$)

$$\|g \circ A(u)\|_{H^{1,q}(G)} \leq C_\nu^{1-\frac{1}{q}} (\operatorname{mod}(A(u)))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} \|g\|_{H^{1,q}(G)}$$

$$= C_\nu^{1-\frac{1}{q}} (\operatorname{mod}(A(u)))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} \|f\|_{H^{1,q}(G/K)}.$$

Therefore

$$\|\mathcal{H}_{\Phi,\dot{A}}\|_{\mathcal{L}(H^{1,q}(G/K))} \leq C_\nu^{1-\frac{1}{q}} \int_\Omega |\Phi(u)| (\operatorname{mod} A(u))^{-\frac{1}{q}} k(u)^{d(1-\frac{1}{q})} d\mu(u)$$

and the proof is complete.

Setting in Theorem 1 $\Omega = \mathbb{Z}_+$ with counting measure μ we have the next result for discrete Hausdorff operators.

Corollary 1. *Let* (G, ρ, ν) *and* K *be as in the Theorem 1,* $(\dot{A}(n))_{n \in \mathbb{Z}_+} \subset \operatorname{Aut}_K(G)$, *and* $q \in (1, \infty]$. *If* $\Phi : \mathbb{Z}_+ \to \mathbb{C}$ *be such that*

$$\|\Phi\|_{A,q} := \sum_{n=0}^{\infty} |\Phi(n)| (\operatorname{mod} A(n))^{-\frac{1}{q}} k(n)^{(1-\frac{1}{q})d} < \infty,$$

then the discrete Hausdorff operator

$$\mathcal{H}_{\Phi,\dot{A}} f(\dot{x}) := \sum_{n=0}^{\infty} \Phi(n) f(\dot{A}(n)(\dot{x}))$$

is bounded on the space $H^{1,q}(G/K)$ *and*

$$\|\mathcal{H}_{\Phi,\dot{A}}\|_{\mathcal{L}(H^{1,q}(G/K))} \leq C_\nu^{1-\frac{1}{q}} \|\Phi\|_{A,q}.$$

As a special case of Theorem 1 for $K = \{e\}$ (e denotes the unit of G) one has the

7

Corollary 2. *Let Let (G, ρ, ν) and (Ω, μ) be as in the Theorem 1, $(A(u))_{u \in \Omega} \subset \mathrm{Aut}(G)$, and $q \in (1, \infty]$. If $\|\Phi\|_{A,q} < \infty$ then the operator $\mathcal{H}_{\Phi,A}$ is bounded on $H^{1,q}(G)$ and*

$$\|\mathcal{H}_{\Phi,A}\|_{\mathcal{L}(H^{1,q}(G))} \leq C_\nu^{1-\frac{1}{q}} \|\Phi\|_{A,q}.$$

Remark 2. The condition $\|\Phi\|_{A,q} < \infty$ is not necessary for boundedness of $\mathcal{H}_{\Phi,A}$ in Hardy space as the following simple example shows[4]. Consider the Hausdorff operator

$$(\mathcal{H}_1 f)(x) := \int_\Omega f(u_1 x_1, \ldots, u_n x_n) du$$

in $H^1(\mathbb{R}^n)$. Here $G = \mathbb{R}^n$, $\Omega = \{u \in \mathbb{R}^n : u_j \neq 0 \text{ for } j = 1, \ldots, n\}$, μ and ν are Lebesgue measures on Ω and $\mathbb{R}^n$ respectively, $K = \{0\}$, $A(u)(x) = A_u x$, where $A_u = \mathrm{diag}\{u_1, \ldots, u_n\}$ ($x \in \mathbb{R}^n$ a column vector, $u \in \Omega$), $\Phi = 1$, $d = n$. The necessary moment condition $\int_{\mathbb{R}^n} f(u) du = 0$ for functions from $H^1(\mathbb{R}^n)$ yields that $\mathcal{H}_1 f = 0$ for all $f \in H^1(\mathbb{R}^n)$. On the other hand, here $\mathrm{mod} A(u) = |\det A_u| = |u_1 \ldots u_n|$ [23, Subsection VII.1.10, Corollary 1], $(dA(u)^{-1})_0 X = A_u^{-1} X$ ($X \in \mathbb{R}^n$), $k(u) = \|A_u^{-1}\| = (\sum_{j=1}^n u_j^{-2})^{1/2} \geq n^{1/2} |u_1 \ldots u_n|^{-1/n}$. Then

$$\|\Phi\|_{A,q} = \int_\Omega (\mathrm{mod} A(u))^{-\frac{1}{q}} k(u)^{(1-\frac{1}{q})d} du \geq n^{\frac{1}{2}(1-\frac{1}{q})n} \int_\Omega \frac{du}{|u_1 \ldots u_n|} = \infty.$$

3 Examples

3.1 Generalized shift operator of Delsarte

Let G be as above and $\mathfrak{A}$ a compact subgroup of $\mathrm{Aut}(G)$ with normalized Haar measure m. Recall that the generalized shift operator of Delsarte [24], [25, Ch. I, §2] (also the terms "generalized translation operator of Delsarte", or "generalized displacement operator of Delsarte" are used) is defined to be

$$T^x f(h) = \int_{\mathfrak{A}} f(ha(x)) dm(a) \quad (x, h \in G).$$

Since the group G acts on G/K, one can define a generalization of this operator to G/K as follows. Let $\Omega := \{u \in \mathfrak{A} : u(K) = K\}$. Then Ω is a

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[4]The sufficient boundedness conditions from [5] and [22] are also not met in this example.

compact subgroup of $\mathfrak{A}$. We denote by μ the normalized Haar measure of Ω and put for a Borel measurable function f on G/K

$$T^{\dot{x}}f(h) := \int_\Omega f(h\dot{u}(\dot{x}))d\mu(u) \quad (\dot{x} \in G/K, h \in G).$$

Let h be fixed and $L^h f(\dot{x}) := T^{\dot{x}}f(h)$. Then $L^h = \mathcal{H}_1\tau_h$, where

$$\mathcal{H}_1 f(\dot{x}) := \int_\Omega f(\dot{u}(\dot{x}))d\mu(u)$$

is a Hausdorff operator on G/K with $\Phi(u) = 1$ and $A(u) = u$, and $\tau_h f(\dot{x}) := f(h\dot{x})$. Note that mod is a continuous homomorphism from $\mathrm{Aut}(G)$ to the multiplicative group $(0, \infty)$. Since Ω is a compact group, it follows that $\mathrm{mod}(\Omega) = \{1\}$. Assume that the doubling conditions for the Lie group G holds and Ω is quasi-metric. Then the operator $\mathcal{H}_1$ is bounded on $H^{1,q}(G/K)$ by Theorem 1 and

$$\|\mathcal{H}_1\| \le C_\nu^{1-\frac{1}{q}} \int_\Omega k(u)^{(1-\frac{1}{q})s}d\mu(u)$$

where $k(u) = \|(d(u^{-1}))_e\|$. Since τ_h is an isometry of $H^{1,q}(G/K)$, we conclude that the operator L^h is bounded on $H^{1,q}(G/K)$ and

$$\|L^h\| \le C_\nu^{1-\frac{1}{q}} \int_\Omega k(u)^{(1-\frac{1}{q})d}d\mu(u).$$

3.2 Hausdorff operators on the unit sphere in $\mathbb{R}^n$

Consider the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ (the case $n = 3$ was considered in [9]).

The compact group $G = SO(n)$ acts on $\mathbb{S}^{n-1}$ transitively by restriction of the natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. It is known that the isotropy subgroup K of the point $e_n := (0, \ldots, 1) \in \mathbb{S}^{n-1}$ consists of all elements in $SO(n)$ of the form

$$\widetilde{a} := \begin{pmatrix} a & \mathbf{0}^\top \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^{n-1}$, $a \in SO(n-1)$. Hence we identify $\mathbb{S}^{n-1}$ with the homogeneous space $SO(n)/K$. Let $s \in \mathbb{S}^{n-1}$. If a matrix $x(s) \in SO(n-1)$ satisfies $s = x(s)e_n^\top$ we can identify the point s with the coset $\dot{x}(s) := x(s)K$.

Consider the set of automorphisms of $G = SO(n)$ of the form

$$A(u)(x) = \widetilde{u}^{-1}x\widetilde{u}, \quad u \in O(n-1).$$

9

Since every mapping $x \mapsto u^{-1}xu$ with $u \in O(n-1)$ maps $SO(n-1)$ onto itself (being a connected component of unit in $O(n-1)$ the group $SO(n-1)$ is a normal subgroup of $O(n-1)$), we have in our case that all automorphisms $A(u)$ where $u \in O(n-1)$ map K onto K. Then by definition the coset

$$\dot{A}(u)(\dot{x}(s)) = \pi_K(\widetilde{u}^{-1}x(s)\widetilde{u})$$

can be identified with the point

$$\widetilde{u}^{-1}x(s)\widetilde{u}e_n^\top = \widetilde{u}^{-1}x(s)e_n^\top = \widetilde{u}^{-1}s = (u^{-1}s', s_n)$$

$(s' := (s_1, \ldots, s_{n-1}))$ of $\mathbb{S}^{n-1}$.

Thus, Definition 2 takes the form (we put $x = x(s)$ in this definition and identify the coset $\dot{x}(s)$ with a column vector $s \in \mathbb{S}^{n-1}$)

$$(\mathcal{H}_{\Phi,\mu}f)(s) = \int_{O(n-1)} \Phi(u)f(u^{-1}s', s_n)d\mu(u) \tag{4}$$

where μ stands for a (regular Borel) measure on $O(n-1)$ and f is a Borel measurable function on $\mathbb{S}^{n-1}$.

Note that the point $(u^{-1}s', s_n)$ runs over the cross-section of $\mathbb{S}^{n-1}$ by the hyperplane $\{x = s_n\} \subset \mathbb{R}^n$ (which contains s) orthogonal to the last coordinate axis when u runs over $O(n-1)$. So (4) looks as a "horizontal slice transform" on $\mathbb{S}^{n-1}$ and the function $\mathcal{H}_{\Phi,\mu}f$ depends on $s_n \in [-1, 1]$ only.

To apply Theorem 1 first we shaw that $k(u) = 1$ for $u \in O(n-1)$. Indeed, $k(u) = \|d(A(u^{-1}))_{1_n}\|$ (here 1_n stands for the unit $n \times n$ matrix). It is easy to verify that for every $X \in \mathfrak{so}(n)$, the Lie algebra of $SO(n)$

$$d(A(u^{-1}))_{1_n}X = \widetilde{u}X\widetilde{u}^{-1}.$$

On the other hand, $\widetilde{u} \in O(n)$ for $u \in O(n-1)$. Thus,

$$\|d(A(u^{-1}))_{1_n}\| = \max_{\|X\|=1, \|Y\|=1} |\langle d(A(u^{-1}))_{1_n}X, d(A(u^{-1}))_{1_n}Y\rangle| =$$

$$\max_{\|X\|=1, \|Y\|=1} |\langle \widetilde{u}X\widetilde{u}^{-1}, \widetilde{u}Y\widetilde{u}^{-1}\rangle| = \max_{\|X\|=1, \|Y\|=1} |\langle X, Y\rangle| = 1$$

(here $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product). Since $SO(n)$ is compact, it is doubling [11]. Next, since $SO(n)$ is unimodular, we get that $\mathrm{mod}A = 1$ for all $A \in \mathrm{Aut}(SO(n))$. So if $\Phi \in L^1(O(n-1), \mu)$ [8, Theorem 1] yields that the operator (4) is bounded on $L^p(\mathbb{S}^{n-1})$ and $\|\mathcal{H}_{\Phi,\mu}\|_{\mathcal{L}(L^p(\mathbb{S}^{n-1})} \leq \|\Phi\|_{L^1(\mu)}$. Moreover, Theorem 1 yields that

$$\|\mathcal{H}_{\Phi,\mu}\|_{\mathcal{L}(H^{1,q}(\mathbb{S}^{n-1}))} \leq C_\nu^{1-\frac{1}{q}}\|\Phi\|_{L^1(\mu)}$$

10

where C_ν is the doubling constant for $SO(n)$.

In closing let us consider the following special case. Let $\Phi = 1$ and m be a Haar measure of the (compact) group $O(n-1)$. Then for every $f \in H^1(\mathbb{S}^{n-1})$ the function

$$(\mathcal{H}_{1,m}f)(s) = \int_{O(n-1)} f(u^{-1}s', s_n)dm(u)$$

belongs to $H^1(\mathbb{S}^{n-1})$. On the other hand, this function depends on s_n only. Indeed, $(s', s_n) \in \mathbb{S}^{n-1}$ if and only if s' belongs to the sphere $\mathbb{S}_r^{n-2}$ centered at $0 \in \mathbb{R}^{n-1}$ of radius $r = \sqrt{1 - s_n^2}$. Fix $s_0' \in \mathbb{S}_r^{n-2}$. Since $SO(n-1)$ acts transitively on $\mathbb{S}_r^{n-2}$, for every $s' \in \mathbb{S}_r^{n-2}$ there is such $v \in SO(n-1)$ that $vs_0' = s'$. Taking into account that $O(n-1)$ is unimodular, we get

$$(\mathcal{H}_{1,m}f)(s) = \int_{O(n-1)} f(uvs_0', s_n)dm(u) = \int_{O(n-1)} f(us_0', s_n)dm(u)$$

which completes the proof.

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