Southeast Asian Bulletin of Mathematics (2021) 45: 813–824

Southeast Asian
Bulletin of
Mathematics

On σ-Subnormal Subgroups of Finite Groups*

Wenbin Guo
School of Science, Hainan University, Haikou, Hainan, 570228, China
Email: wbguo@ustc.edu.cn

Inna N. Safonova
Department of Applied Mathematics and Computer Science, Belarusian State University, Minsk 220030, Belarus
Email: safonova@bsu.by

Alexander N. Skiba
Department of Mathematics and Technologies of Programming, Francisk Skorina Gomel State University, Gomel 246019, Belarus
Email: alexander.skiba49@gmail.com

Received 5 June 2021
Accepted 26 September 2021

Communicated by Xiuyun Guo

Dedicated to the memory of Professor Yuqi Guo (1940–2019)

AMS Mathematics Subject Classification(2000): 20D10, 20D15, 20D30, 20D35, 20E15

Abstract. Throughout this paper, all groups are finite and σ is some partition of the set of all primes $\mathbb{P}$ (that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$). A subgroup A of a group G is said to be *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a σ_j-group for some $j = j(i)$ for all $i = 1, \ldots, n$.

In this review, we discuss some known results of the theory of σ-subnormal subgroups and also some open questions in this line research.

Keywords: Finite group; σ-soluble group; σ-nilpotent group; $P\sigma T$-group; σ-subnormal subgroup.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group;

*Research was supported by the NNSF of China (No. 11771409.)

$\mathcal{L}(G)$ is the lattice of all subgroups of G. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

Following Shemetkov [46], we use σ to denote some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \cup_{i\in I}\sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

The *σ-property* of a group G [49, 50] is any its property which does not depend on the the choice of the partition of σ of $\mathbb{P}$.

Before continuing, we recall some concepts of the papers [49]–[52] which play a fundamental role in the theory of σ-properties of groups.

First of all, recall that a set $\mathcal{H}$ of subgroups of G is a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$ (here $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \emptyset\}$); a complete Hall σ-set $\mathcal{H}$ of G is a *σ-basis* of G if every its two members A and B are permutable, that is, $AB = BA$. The group G is called *σ-full* if G possesses a complete Hall σ-set.

The group G is said to be: *σ-primary* if G is a σ_i-group for some $i = i(G)$; *σ-soluble* if every chief factor of G σ-primary; *σ-nilpotent* if every chief H/K of G is *σ-central* in G, that is, $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary.

Definition 1.1. [49, 50] *A subgroup A of G is said to be* σ-subnormal *in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.*

Remark 1.2.

(i) The group G is σ-nilpotent if and only if every subgroup of G is σ-subnormal [50].

(ii) Let $\mathfrak{F}$ be a class of groups. Then a subgroup A of G is said to be $\mathfrak{F}$-*subnormal in G in the sense of Kegel* [37] or *K-$\mathfrak{F}$-subnormal* in G [12, 6.1.4] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{F}$ for all $i = 1, \ldots, n$. It is not difficult to show that A is σ-subnormal in G if and only if it is $\mathfrak{N}_\sigma$-subnormal in G in the sense of Kegel, where $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups.

A subgroup A of G is *σ-permutable* in G [49, 50] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$.

Now we give a classical interpretation for the introduced concepts.

Example 1.3.

(i) In the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$: G is σ^1-soluble (respectively σ^1-nilpotent) if and only if G is soluble (respectively nilpotent); a subgroup A of G is subnormal in G if and only if it is σ^1-subnormal in G. The σ^1-permutable subgroups are also called *S-permutable* [11, 21] or *Sylow permutable.*

(ii) In the other classical case when $\sigma = \sigma^{\pi} = \{\pi, \pi'\}$: G is σ^{π}-soluble (respectively σ^{π}-nilpotent) if and only if G is π-separable (respectively π-decomposable, that is, $G = O_{\pi}(G) \times O_{\pi'}(G)$); a subgroup A of G is σ^{π}-subnormal in G if and only if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$, or $A_i/(A_{i-1})_{A_i}$ is a π-group, or $A_i/(A_{i-1})_{A_i}$ is a π'-group for all $i = 1, \ldots, n$. A subgroup A of G is σ^{π}-permutable in G if and only if G has a Hall π-subgroup V and a Hall π'-subgroup W such that $AV^x = V^xA$ and $AW^x = W^xA$ for all $x \in G$.

(iii) In fact, in the theory of π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. Hence G is $\sigma^{1\pi}$-soluble (respectively $\sigma^{1\pi}$-nilpotent) if and only if G is π-soluble (respectively π-*special* [54, 24], that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$). A subgroup A of G is: $\sigma^{1\pi}$-subnormal in G if and only if it is $\mathfrak{F}$-subnormal in G in the sense of Kegel, where $\mathfrak{F}$ is the class of all π'-groups, and a subgroup A is $\sigma^{1\pi}$-permutable in G if and only if A permutes with all Sylow p-subgroups of G for all $p \in \pi$ and G has a Hall π'-subgroup W such that $AW^x = W^xA$ for all $x \in G$.

It is necessary to mention that the σ-subnormality plays a key role in the analysis of many questions and, in particular, in the study of σ-permutable subgroups. Many important properties of such subgroups have already been described in the papers [49, 50] and this made it possible to find new interesting applications of the theory of σ-properties of a group (see, in particular, [3, 4], [13, 16, 23], [25]–[31], [33]–[35], [39, 40], [47, 48, 53]).

In this review, we discuss some applications of the theories of σ-subnormal and σ-permutable subgroups and also some open questions in this line research.

2. σ-Subnormal and σ-Permutable Subgroups

The theory of Sylow permutable subgroups had been mainly developed in the papers by Kegel [36] and Deskins [18] and one of the main results of the theory states that H^G/H_G is nilpotent for every Sylow permutable subgroup H of G and hence every Sylow permutable subgroup of G is subnormal. One of the first primary applications of the theory of σ-subnormal subgroups was found in [50], where the following generalization of this classical result was proved.

Theorem 2.1. [49, 50] *Suppose that G is σ-full. Then H^G/H_G is σ-nilpotent for every σ-permutable subgroup H of G. Hence every σ-permutable subgroup of G is σ-subnormal in G.*

Among other corollaries of Theorem 2.1 we mention also the following two its special cases.

Corollary 2.2. *Suppose that G possesses a Hall π-subgroup and a Hall π'-subgroup (this condition holds, for example, in every π-separable group). If a subgroup H*

of G permutes with all Hall π-subgroups and all Hall π'-subgroups of G, then H^G/H_G is π-decomposable. (See Example 1.3(ii)).

Corollary 2.3. *Suppose that G possesses a Hall π'-subgroup. If a subgroup H of G permutes with all Hall π'-subgroups and all Sylow p-subgroups of G for all $p \in \pi$, then H^G/H_G is a π-special group. (See Example 1.3(iii)).*

A subgroup M of G is said to be: *quasinormal* in G if M permutes with all subgroups of G; *modular* in G if M is a modular element (in the sense of Kurosh [45, p. 43]) of the lattice $\mathcal{L}(G)$, that is,

(i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and
(ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

Schmidt proved [45, Theorem 5.1.1] that a subgroup A of G is quasinormal in G if and only if A is modular and subnormal in G. This elegant observation is a motivation for the following

Definition 2.4. [29, Definition 1.1] *We say that a subgroup A of G is σ-quasinormal in G if A is modular and σ-subnormal in G.*

The following theorem describes the most important properties of σ-quasinormal subgroups.

Theorem 2.5. [29, Theorem C] *Let A be a σ-quasinormal subgroup of G. Then the following statements hold:*

(i) *If G possesses a Hall σ_i-subgroup, then A permutes with each Hall σ_i-subgroup of G.*
(ii) *The quotients A^G/A_G and $G/C_G(A^G/A_G)$ are σ-nilpotent.*
(iii) *Every chief factor of G between A^G and A_G is σ-central in G.*

In the case when $\sigma = \sigma^1$ we get from Theorem 2.5 the following well-known results.

Corollary 2.6. *Let A be a quasinormal subgroup of G. Then the following statements hold:*

(i) *A/A_G is nilpotent* (see [32]).
(ii) *Every chief factor H/K of G between A^G and A_G is central in G, that is, $C_G(H/K) = G$* (see [41]).

The classical Wielandt's result [57] states that the set $\mathcal{L}_{sn}(G)$, of all subnormal subgroups of G, forms a sublattice of the lattice $\mathcal{L}(G)$ (that is, $A\cap B, \langle A, B\rangle \in \mathcal{L}_{sn}(G)$ for all $A, B \in \mathcal{L}_{sn}(G)$). The most applications of σ-subnormal subgroups are based on the following generalization of this result.

Theorem 2.7. [49, 50] *For any partition σ of the set of all primes $\mathbb{P}$, the set $\mathcal{L}_\sigma(G)$, of all σ-subnormal subgroups of G, forms a sublattice in $\mathcal{L}(G)$.*

Initially, this theorem was proved in the work [50] on the basis of the methods of the formation theory. Another proof of this result, based on a more detailed study of the σ-subnormal subgroups, was found in [3].

From Theorem 2.7 it follows that the intersection of any set of σ-subnormal subgroups of G is also σ-subnormal in G. In particular, the intersection of all σ-subnormal subgroups of G containing a subgroup H of G is σ-subnormal in G and such the intersection is called the *σ-subnormal closure* of H in G.

Among the most interesting open problems concerning σ-subnormal subgroups, the problem of describing the σ-subnormal closure of a subgroup is still remained open.

Al-Shomrani, Heliel and Ballester-Bolinches [4] provide a solution to this problem in an important for applications case where the group G is σ-soluble.

Recall that G is said to be: a *D_π-group* if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; a *σ-full group of Sylow type* [50] if every subgroup E of G is a D_{σ_i}-group for every i.

It is well known that the set of all qusinormal subgroups in the general case does not form a sublattice in the lattice $\mathcal{L}(G)$ and that the set of all Sylow permutable subgroups of G is a sublattice in $\mathcal{L}(G)$ (see [36]). Theorem 2.7 allows to prove the following generalization of this Kegel's result.

Theorem 2.8. [50, Theorem C] *Let G be a σ-full group of Sylow type. Then the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all σ-subnormal subgroups of G.*

Note, in passing, that Theorem 2.8 not only generalizes the above mentioned Kegel's result on the lattice of the Sylow permutable subgroups but also gives a new proof of it.

Theorems 2.7 and 2.8 make it possible to successfully solve the problem of describing groups with various given systems of σ-subnormal and σ-permutable subgroups. In this regard, we mention first of all the fundamental work of W. Guo and A.N. Skiba [23] devoted to the development of the well-known papers by Mann [42] and Spencer [55].

In the paper [28] it is proved the following result in this line research which gives the answer to Question 7.7 in [52].

Theorem 2.9. [31, Theorem C] *Every subgroup of G is either σ-subnormal or σ-abnormal in G if and only if G is a group of one of the following two types:*

(i) *G is σ-nilpotent;*

(ii) *$G = D \rtimes P$, where*

  (a) *$D = G^{\mathfrak{N}_\sigma} = G'$ is a σ-nilpotent σ-Hall subgroup of G;*

  (b) *$P = N_G(P)$ is a cyclic Sylow subgroup of G;*

(c) *$Z(G)$ is the unique maximal subgroup of P.*

In this theorem, $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual of G*, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Among questions related to this direction, we mention the following two.

Problem 2.10. [50, Question 4.7] Describe groups in which every Schmidt subgroup is σ-subnormal.

In the case when $\sigma = \sigma^1$ the solution to this problem is known (see [56]).

Problem 2.11. [43, Question 19.85] Suppose that every Schmidt subgroup of G is σ-subnormal in G. Is it true that then there is a normal σ-nilpotent subgroup N such that G/N is cyclic?

On the base of Theorem 2.7, Problem 2.10 is partially solved in the papers [3, 30]. The complete positive answer to this problem was given by S.F. Kamornikov and X. Yi in [35]. But Problem 2.11 is still open.

It is quite natural and important to find characterizations and criteria for σ-subnormality and σ-permutability of subgroups.

First mention that as another application of Theorem 2.1, the following fact was proved in [50].

Theorem 2.12. [50, Theorem 4.1] *Let G be a σ-full group of Sylow type. Then a subgroup A of G is σ-permutable in G if and only if A is σ-subnormal in G and A is σ-permutable in $\langle A, x\rangle$ for all $x \in G$.*

Since a subgroup A of G is subnormal in G if and only if A is subnormal in $\langle A, x\rangle$ for all $x \in G$ [19, A, 14.10], we get from Theorem 2.12 the following known result.

Corollary 2.13. (see [9] or [11, Theorem 1.2.13]) *A subgroup A of G is Sylow permutable in G if and only if A is Sylow permutable in $\langle A, x\rangle$ for all $x \in G$.*

The following problem was posed by Skiba first in "Advances of Group Theory and Applications" (2016, **1**, p. 159]) and some later in [43, Question 19.86].

The generalized Wielandt-Kegel problem:

Problem 2.14. Let A be a subgroup of a σ-full group G. Is it true then that A is σ-subnormal in G if and only if $H \cap A$ is a Hall σ_i-subgroup of A for all i and every Hall σ_i-subgroup H of G?

Note that in the cases when $\sigma = \sigma^1$ (Kleidman [38]) or when $\sigma = \{\{p\}, \{p\}'\}$ (see [33]) the answer to this question is positive.

We also know some other special conditions under which the problem has a positive solution. For example, the problem has a positive solution in the class of all σ-soluble groups [51] and in the class of all $3'$-groups [34].

Wielandt's classical results on subnormal subgroups motivated also the following two questions.

Problem 2.15. [50, Question 4.10] Let for each element $x \in G$ the subgroup H of G σ-subnormal in $\langle H, x\rangle$. Is it true that the subgroup H σ-subnormal in G?

Problem 2.16. (see [43, Question 19.86] or [52, Question 7.5]) Is it true that a subgroup H is σ-subnormal in G if H is σ-subnormal in $\langle H, H^x\rangle$ for any element $x \in G$?

A positive solution to Problem 2.15 was found by A. Ballester-Bolinches, S.F. Kamornikov, M.C. Pedraza-Aguilera and V. Perez-Calabuig in [13]. A counterexample to Problem 2.16 was found in [14].

3. $P\sigma T$-groups and $Q\sigma T$-groups

Many parers are related to the study of T-groups and various their generalizations, in particular, of PT-groups and PST-groups.

Recall that a group G is a *T-group* if normality is a transitive relation in G, that is, if K is a normal subgroup of H and H is a normal subgroup of G, then K is a normal subgroup of G. It is rather clear that the T-groups are exactly the groups G in which every subnormal subgroup is normal in G.

A group G is called a *PT-group* (respectively, a *PST-group*) [11] if permutability (respectively, Sylow permutability) is a transitive relation in G. Since every Sylow permutable subgroup is subnormal in the group, G is a PST-group if and only if every subnormal subgroup is Sylow permutable in G; G is PT-group if and only if every its subnormal subgroup is modular (and so quasinormal) in G.

The group G is called a *$P\sigma T$-group* [50] (respectively, a *$Q\sigma T$-group*) if σ-permutability (respectively, σ-quasinormality) is a transitive relation in G, that is, if K is a σ-permutable (respectively, σ-quasinormal) subgroup of H and H is a σ-permutable (respectively, σ-quasinormal) subgroup of G, then K is a σ-permutable (respectively, σ-quasinormal) subgroup of G.

In view of Theorem 2.1, G is a *$P\sigma T$-group* if and only if every σ-subnormal subgroup of G is σ-permutable in G; G is a *$Q\sigma T$-group* if and only if every σ-subnormal subgroup of G is modular in G. Therefore G is a PT-group (respectively, a PST-group) if and only if G is a $Q\sigma T$-group (respectively, a $P\sigma T$-group), where $\sigma = \sigma^1$.

The description of PST-groups was first obtained by Agrawal [1], for the soluble case, and by Robinson in [44], for the general case. In the further publications, authors (see, for example, the recent papers [2], [5]–[8], [10, 15, 17, 58]) have found out and described many other interesting characterizations of soluble PST-groups. A significant place to the theory of PST-groups is given in the nice book [11].

A new approach to the study of PST-groups and PT-groups was also proposed in the recent publication [20].

Let $\mathfrak{F}$ be a class of groups. We call, following Guo, Shum and Skiba [22], a set Σ of subgroups of G a *G-covering subgroup system for the class* $\mathfrak{F}$ if $G \in \mathfrak{F}$ whenever $\Sigma \subseteq \mathfrak{F}$.

In [20], the following two results are proved.

Theorem 3.1. [20, Theorem B] *Suppose that a set of subgroups Σ contains at least one supplement to each maximal subgroup of every Sylow subgroup of G. Then G is a soluble PT-group (respectively, a soluble T-group) if and only if every subgroup in Σ is a soluble PT-group (respectively, a soluble T-group) and at least one of the non-identity Sylow subgroups of G is an Iwasawa (respectively, a Dedekind) group.*

The example of extraspecial 3-group of order p^3 shows that the set Σ in Theorem 3.1 is not a G-covering subgroup system for the classes of all soluble PT-groups and all soluble T-groups.

Now we indicate a system of subgroups that is a G-covering subgroup system simultaneously for classes of all soluble PST-, PT-, and T-groups.

Theorem 3.2. [20, Theorem C] *Let Σ be the set of all two-generated subgroups of G. Then Σ is a G-covering subgroup system for any class $\mathfrak{F}$ in the following list:*

(i) *$\mathfrak{F}$ is the class of all soluble PST-groups.*
(ii) *$\mathfrak{F}$ is the class of all soluble PT-groups.*
(iii) *$\mathfrak{F}$ is the class of all soluble T-groups.*

Theorem 3.2 partially answers the following open question.

Problem 3.3. Let Σ be the set of all two-generated subgroups of G. Is it true then that Σ is a G-covering subgroup system simultaneously for the classes of all PST-groups, all PT-groups, and all T-groups?

In the most general case (i.e., without of any restriction on σ), the following theorem is true, which in fact is the main result of the observations found in [50, 54].

Problem 3.4. [54, Theorem B]) If G is a σ-soluble $P\sigma T$-group and $D = G^{\mathfrak{N}_\sigma}$, then the following conditions hold:

(i) $G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent and every element of G induces a power automorphism in D;
(ii) $O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.

Conversely, if Conditions (i) and (ii) hold for some subgroups D and M of G, then G is a $P\sigma T$-group.

In the case when $\sigma = \sigma^1$, we get from Theorem 3.4 the following

Corollary 3.5. [1, Theorem 2.3] *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of G. If G is a soluble PST-group, then D is an abelian Hall subgroup of G of odd order and every element of G induces a power automorphism in D.*

In the case when $\sigma = \sigma^\pi$ we get from Theorem 3.4 the following corollary.

Corollary 3.6. *G is a π-separable $P\sigma^\pi T$-group if and only if the following conditions hold:*
(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is π-decomposable and every element of G induces a power automorphism in D;*
(ii) *$O_\pi(D)$ has a normal complement in a Hall π-subgroup of G;*
(iii) *$O_{\pi'}(D)$ has a normal complement in a Hall π'-subgroup of G.*

In the case when $\sigma = \sigma^{1\pi}$ we get from Theorem 3.4 the following

Corollary 3.7. *G is a π-soluble $P\sigma^{1\pi}T$-group if and only if the following conditions hold:*
(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, $M = O_{p_1}(M) \times \cdots \times O_{p_n}(M) \times O_{\pi'}(M)$ and every element of G induces a power automorphism in D;*
(ii) *$O_{\pi'}(D)$ has a normal complement in a Hall π'-subgroup of G.*

Theorem 3.4 gives a solution to the following problem in the class of all σ-soluble groups.

Problem 3.8. (see Question in [50]) Let G be a σ-full group. What is the structure of G provided that every σ-subnormal subgroup of G is σ-permutable ?

In [28], Theorem 3.4 was used to obtain the description of σ-soluble $Q\sigma T$-groups.

Theorem 3.9. [28, Theorem C] *If G is a σ-soluble $Q\sigma T$-group and $D = G^{\mathfrak{N}_\sigma}$, then the following conditions hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order and M is a σ-nilpotent group with modular lattice $\mathcal{L}(M)$;*
(ii) *every element of G induces a power automorphism in D,*
(iii) *$O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.*

Conversely, if Conditions (i), (ii) and (iii) hold for some subgroups D and M of G, then G is a $Q\sigma T$-group.

Note that, in view of [45, 2.3.2, 2.4.4], if G is a nilpotent group with modular lattice $\mathcal{L}(G)$, then G is an *Iwasawa group* [11, 1.4.2], that is, every subgroup of G is quasinormal in G. Therefore in the case where $\sigma = \sigma^1$, we get from Theorem 3.9 the following well-known result.

Corollary 3.10. [59] *A group G is a soluble PT-group if and only if the following conditions hold:*

(i) *the nilpotent residual $D = G^{\mathfrak{N}}$ of G is an abelian Hall subgroup of odd order;*
(ii) *every element of G induces a power automorphism in D,*
(iii) *G/D is an Iwasawa group.*

References

[1] R.K. Agrawal, Finite groups whose subnormal subgroups permute with all Sylow subgroups, *Proc. Amer. Math. Soc.* **47** (1975) 77–83.

[2] K.A. Al-Sharo, J.C. Beidleman, H. Heineken, M.F. Ragland, Some characterizations of finite groups in which semipermutability is a transitive relation, *Forum Math.* **22** (2010) 855–862.

[3] K.A. Al-Sharo, A.N. Skiba, On finite groups with σ-subnormal Schmidt subgroups, *Comm. Algebra* **45** (2017) 4158–4165.

[4] M.M. Al-Shomrani, A.A. Heliel, A. Ballester-Bolinches, On σ-subnormal closure, *Comm. in Algebra* **48** (8) (2020) 3624–3627. doi: 10.1080/00927872.2020.1742348

[5] M. Asaad, Finite groups in which normality or quasinormality is transitive, *Arch. Math.* **83** (4) (2004) 289–296.

[6] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken, A local approach to certain classes of finite groups, *Comm. Algebra* **31** (2003) 5931–5942. doi:10.1081/AGB-120024860.

[7] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken, Groups in which Sylow subgroups and subnormal subgroups permute, *Illinois J. Math.* **47** (1-2) (2003) 63–69.

[8] A. Ballester-Bolinches, J. Cossey, Totally permutable products of finite groups satisfying SC or PST, *Monatsh. Math.* **145** (2005) 89–93.

[9] A. Ballester-Bolinches, R. Esteban-Romero, On finite soluble groups in which Sylow permutability is a transitive relation, *Acta Math. Hungar.* **101** (2003) 193–202.

[10] A. Ballester-Bolinches, R. Esteban-Romero, Sylow permutable subnormal subgroups, *J. Algebra* **251** (2002) 727–738.

[11] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, *Products of Finite Groups*, Walter de Gruyter, Berlin-New York, 2010.

[12] A. Ballester-Bolinches, L.M. Ezquerro, *Classes of Finite Groups*, Springer, Dordrecht, 2006.

[13] A. Ballester-Bolinches, S.F. Kamornikov, M.C. Pedraza-Aguilera, V. Perez-Calabuig, On σ-subnormality criteria in finite σ-soluble groups, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas* **114** (2) (2020), doi.org/10.1007/s13398-020-00824-4.

[14] A. Ballester-Bolinches, S.F. Kamornikov, X. Yi, On σ-subnormality criteria in finite groups, *J. Pure Appl. Algebra* (to appear)

[15] J.C. Beidleman, M.F. Ragland, Subnormal, permutable, and embedded subgroups in finite groups, *Central Eur. J. Math.* **9** (4) (2011) 915–921.

[16] J.C. Beidleman, A.N. Skiba, On τ_σ-quasinormal subgroups of finite groups, *J. Group Theory* **20** (5) (2017) 955–964.

[17] Z. Chi, A.N. Skiba, On a lattice characterisation of finite soluble PST-group, *Bull. Austral. Math. Soc.* **101** (2020) 113–120.

[18] W.E. Deskins, On quasinormal subgroups of finite groups, *Math. Z.* **82** (1963) 125–132.

[19] K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

[20] J. Guo, W. Guo, I.N. Safonova, A.N. Skiba, G-covering subgroup systems for the classes of finite soluble PST-groups, *Comm. in Algebra* **49** (9) (2021) 1–12. doi: 10.1080/00927872.2021.1909057.

[21] W. Guo, *Structure Theory for Canonical Classes of Finite Groups*, Springer, Heidelberg-New York-Dordrecht-London, 2015.

[22] W. Guo, K.P. Shum, A.N. Skiba, G-covering subgroup systems for the classes of supersoluble and nilpotent groups, *Israel J. Math.* **138** (2003) 125–138.

[23] W. Guo, A.N. Skiba, Finite groups whose n-maximal subgroups are σ-subnormal, *Science China. Math.* **62** (7) (2019) 1355–1372.

[24] W. Guo, A.N. Skiba, On σ-supersoluble groups and one generalization of CLT-groups, *J. Algebra* **512** (2018) 92–108.

[25] W. Guo, A.N. Skiba, On the lattice of $\Pi_\mathfrak{I}$-subnormal subgroups of a finite group, *Bull. Austral. Math. Soc.* **96** (2) (2017) 233–244.

[26] W. Guo, C. Zhang, A.N. Skiba, D.A. Sinitsa, On H_σ-permutable embedded subgroups of finite groups, *Rend. Sem. Math. Univ. Padova* **102** (2018) 143–158.

[27] A.A. Heliel, M. Al-Shomrani, A. Ballester-Bolinches, On the σ-length of maximal subgroups of finite σ-soluble groups *Mathematics* **8** (12) (2020) 2165–2169. https://doi.org/10.3390/math8122165.

[28] B. Hu, J. Huang, A.N. Skiba, Groups with only σ-semipermutable and σ-abnormal subgroups, *Acta Math. Hung.* **153** (1) (2017) 236–248.

[29] B. Hu, J. Huang, A.N. Skiba, On σ-quasinormal subgroups of finite groups, *Bull. Austral. Math. Soc.* **99** (3) (2019) 413–420.

[30] B. Hu, J. Huang, A.N. Skiba, On finite groups with generalized σ-subnormal Schmidt subgroups, *Comm. Algebra* **46** (2) (2017) 1–8. doi:org/10.1080/00927872.2017.1404091.

[31] J. Huang, B. Hu, X. Wu, Finite groups all of whose subgroups are σ-subnormal or σ-abnormal, *Comm. Algebra* **45** (1) (2017) 4542–4549. doi:10.1080/00927872.2016.1270956

[32] N. Ito, J. Szép, Uber die Quasinormalteiler von endlichen Gruppen, *Act. Sci. Math.* **23** (1962) 168–170.

[33] S.F. Kamornikov, V.N. Tyutyanov, On σ-subnormal subgroups of finite groups, *Siberian Math. Z.* **61** (2) (2020) 337–343.

[34] S.F. Kamornikov, V.N. Tyutyanov, On σ-subnormal subgroups of finite $3'$-groups, *Ukranian Math. J.* **72** (6) (2020) 806–811.

[35] S.F. Kamornikov, X. Yi, Finite groups with σ-subnormal Schmidt subgroups, *Journal of Algebra* **560**(15) (2020) 181–191.

[36] O.H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962) 205–221.

[37] O.H. Kegel, Untergruppenverbande endlicher Gruppen, die den subnormalteilerverband each enthalten, *Arch. Math.* **30** (3) (1978) 225–228.

[38] P.B. Kleidman, A proof of the Kegel–Wielandt conjecture on subnormal subgroups, *Ann. Math.* **133** (1977) 369–428.

[39] V.A. Kovaleva, A criterion for a finite group to be σ-soluble, *Comm. Algebra* **46** (12) (2019) 5410–5415. doi:10.1080/00927872.2018.1468907

[40] A.M. Liu, W. Guo, I.N. Safonova, A.N. Skiba, G-covering subgroup systems for some classes of σ-soluble groups, *J. Algebra* **585** (2021) 280–293. doi: 10.1016/j.jalgebra.2021.06.010

[41] R. Maier, P. Schmid, The embedding of permutable subgroups in finite groups, *Math. Z.* **131** (1973) 269–272.

[42] A. Mann, Finite groups whose n-maximal subgroups are subnormal, *Trans Amer. Math. Soc.* **132** (1968) 395–409.

[43] V.D. Mazurov, E.I. Khukhro, *The Kourovka Notebook: Unsolved Problems in Group Theory*, Russ. Acad. of Sciences, Siberian Div., Sobolev Institute of Mathematics **19**, 2018.

[44] D.J.S. Robinson, The structure of finite groups in which permutability is a transitive relation, *J. Austral. Math. Soc.* **70** (2001) 143–159.

[45] R. Schmidt, *Subgroup Lattices of Groups*, Walter de Gruyter, Berlin, 1994.

[46] L.A. Shemetkov, *Formations of Finite Groups*, Moscow, Nauka, Main Editorial Board for Physical and Mathematical Literature, 1978.

[47] D.A. Sinitsa, Finite groups with H_σ-permutable embedded subgroups, *Advances in Group Theory and Applications* **4** (2017) 29–40.

[48] A.N. Skiba, On some classes of sublattices of the subgroup lattice, *J. Belarusian State Univ. Math. Informatics* **3** (2019) 35–47.

[49] A.N. Skiba, On σ-properties of finite groups I, *Problems of Physics, Mathematics and Technics* **4** (2014) 89–96.

[50] A.N. Skiba, On σ-subnormal and σ-permutable subgroups of finite groups, *J. Algebra* **436** (2015) 1–16.

[51] A.N. Skiba, A generalization of a Hall theorem, *J. Algebra Appl.* **15** (4) (2015) 21–36.

[52] A.N. Skiba, On some results in the theory of finite partially soluble groups, *Commun. Math. Stat.* **4** (2016) 281–309.

[53] A.N. Skiba, On sublattices of the subgroup lattice defined by formation fitting sets, *J. Algebra* **550** (2020) 69–85.

[54] A.N. Skiba, Some characterizations of finite σ-soluble $P\sigma T$-groups, *J. Algebra* **495** (2018) 114–129.

[55] A.E. Spencer, Maximal nonnormal chains in finite groups, *Pacific J Math.* **27** (1968) 167–173.

[56] V.A. Vedernikov, Finite groups with subnormal Schmidt subgroups, *Algebra and Logica* **46** (6) (2007) 669–687.

[57] H. Wielandt, Eine Verallgemenerung der invarianten Untergruppen, *Math. Z.* **45** (1939) 200–244.

[58] X. Yi, A.N. Skiba, Some new characterizations of PST-groups, *J. Algebra* **399** (2014) 39–54.

[59] G. Zacher, I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normaly, *Atti Accad, Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur.* **37** (8) (1964) 150–154.