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ORDINARY DIFFERENTIAL EQUATIONS

Calculating the Poincaré Map for Two-Dimensional Periodic Systems and Riccati Equations

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Abstract— Formulas for calculating the Poincaré period map for a two-dimensional linear periodic system of differential equations and for the Riccati differential equation are obtained.

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INTRODUCTION

The present paper is a continuation of [1]. Here we restrict ourselves to two-dimensional time-periodic linear differential systems. To draw the reader's attention to the fact that the basic ideas of the paper can be applied to nonlinear systems, we also consider the Riccati equation.

In what follows, we use the notion of Poincaré period map [2, p. 209; 3, p. 216].

Let us first recall some information from the theory of the reflecting function [4–11], which is necessary for understanding this paper.

Let $n \in \mathbb{N}$ be fixed, and let D be an open domain in $\mathbb{R}^n$. Consider the differential system

$$\frac{dx}{dt} = X(t,x), \quad t \in \mathbb{R}, \quad x \in D, \tag{1}$$

for which the Cauchy problem has a unique solution for any point $(t_0, x_0) \in \mathbb{R} \times D$. Let $x = \varphi(t; t_0, x_0)$ be the general solution of this system in the Cauchy form. The *reflecting function* $F(t,x)$ of system (1) is determined by the formula $F(t,x) = \varphi(-t; t, x)$. It is obvious that $F(0,x) \equiv x$, $x \in D$. For $x \in D$, by $\mathcal{I}_x = (-\alpha_x, \alpha_x)$ we denote the maximum existence interval symmetric about zero for the solution $\varphi(t; 0, x)$. The graphs of the solutions $\varphi(t; 0, x)$, $t \in \mathcal{I}_x$, fill some open domain, which is the domain of the reflecting function $F(t,x)$.

The main property of the reflecting function is that the identity $F(t, x(t)) \equiv x(-t)$, $t \in \mathcal{I}_{x(0)}$, holds for each solution $x(t)$ of system (1). In other words, the reflecting function takes each future state $x(t)$ of the real system modeled by the differential system (1) to its previous state $x(-t)$ at the time symmetric about the current time $t = 0$. This implies the following assertions (see [5, 6]):

1. If system (1) is 2ω-periodic in t and $F(t,x)$ is its reflecting function, then the Poincaré map on the interval $[-\omega, \omega]$ for this system is defined by the rule $x \mapsto F(-\omega, x)$.
2. If $F(t,x)$ is the reflecting function of a 2ω-periodic system (1), then a solution $x(t) = \varphi(t; -\omega, x_0)$ extendible to $[-\omega, \omega]$ is 2ω-periodic if and only if $F(-\omega, x_0) = x_0$. The kind of stability of this solution coincides with the kind of stability of the fixed point x_0 of the Poincaré map $x \mapsto F(-\omega, x)$.
3. A differentiable function $F(t,x)$ defined in some neighborhood of the hyperplane $t = 0$ of the space (t,x) is the reflecting function of system (1) if and only if it is the solution of the Cauchy problem
$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} X(t,x) + X(-t, F) = 0, \quad F(0,x) = x.$$
4. For the linear system
$$\frac{dx}{dt} = P(t)x, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{2}$$
the reflecting function is linear and has the form $\bar{x} = F(t)x$, where the matrix $F(t)$ can be expressed via the Cauchy matrix $\Phi(t)$ of system (2) by the formula $F(t) = \Phi(-t)\Phi^{-1}(t)$. This

matrix is called the *reflecting matrix* and is the solution of the following Cauchy problem for a linear matrix differential equation:

$$\frac{dF}{dt} + FP(t) + P(-t)F = 0, \quad F(0) = E, \tag{3}$$

where E is the identity matrix.

5. If system (2) has been obtained by a change of variables $x = S(t)y$ from a system $dy/dt = Q(t)y$ that has a reflecting function $\bar{y} = F(t)y$, then the reflecting function of system (2) is given by the formula $\bar{x} = S(-t)F(t)S^{-1}(t)x$.

1. REFLECTING FUNCTION AND PERIODIC SOLUTIONS OF LINEAR TWO-DIMENSIONAL SYSTEMS

Each linear system (2) with continuous coefficient matrix can be written in the form

$$\frac{dx}{dt} = P_{\text{ev}}(t)x + P_{\text{od}}(t)x, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{4}$$

where $P_{\text{ev}}(t) = (P(t) + P(-t))/2$ and $P_{\text{od}}(t) = (P(t) - P(-t))/2$ are, respectively, even and odd continuous matrices. Making the change of variables

$$y = x \exp\left(-\frac{1}{n}\int\limits_0^t \operatorname{tr} P_{\text{ev}}(\tau)\, d\tau\right)$$

in system (4), where tr is the trace of a matrix, we obtain a system of the form (4) in which the matrix $P_{\text{ev}}(t)$ of the even part has identically zero trace. Therefore, in what follows, considering the two-dimensional system (4), without loss of generality, we assume that it has the form

$$\frac{dx}{dt} = \begin{pmatrix} p_1(t) & p_2(t) \\ p_3(t) & -p_1(t) \end{pmatrix} x + \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} x, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2; \tag{5}$$

here

$$P_{\text{ev}}(t) := \begin{pmatrix} p_1(t) & p_2(t) \\ p_3(t) & -p_1(t) \end{pmatrix} \quad \text{and} \quad P_{\text{od}}(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$$

are even and odd continuous matrices.

It is this system that we will consider in what follows unless stated otherwise.

Theorem 1. *Let the matrix $P_{\text{ev}}(t)$ be continuously differentiable in system (5) for all $t \in \mathbb{R}$, and let the matrix $P_{\text{od}}(t)$ be continuous. Assume also that the following conditions are satisfied:*

1. *The inequality $-\det P_{\text{ev}}(t) \equiv p_1^2(t) + p_2(t)p_3(t) > 0$ holds for all $t \in \mathbb{R}$.*
2. *The functions*

$$m(t) := \frac{-p_1(t)}{\sqrt{p_1^2(t) + p_2(t)p_3(t)}}, \quad n(t) := \frac{-p_2(t)}{\sqrt{p_1^2(t) + p_2(t)p_3(t)}}, \quad r(t) := \frac{-p_3(t)}{\sqrt{p_1^2(t) + p_2(t)p_3(t)}}$$

satisfy the relations

$$\frac{dm}{dt} + n\gamma - \beta r = 0, \quad \frac{dn}{dt} + n(\delta - \alpha) + 2m\beta = 0, \quad \frac{dr}{dt} + r(\alpha - \delta) - 2m\gamma = 0. \tag{6}$$

Then the reflecting matrix of system (5) is given by the formula

$$F(t) = E\cosh\varphi(t) + M(t)\sinh\varphi(t),$$

where

$$M(t)=\begin{pmatrix} m(t) & n(t) \\ r(t) & -m(t)\end{pmatrix} \quad \text{and} \quad \varphi(t)=2\int_0^t \sqrt{p_1^2(\tau)+p_2(\tau)p_3(\tau)}\,d\tau.$$

Proof. To prove the theorem, it suffices to verify the main relation (3) for the reflecting matrix, whose differential equation in our case acquires the form

$$\begin{aligned}\frac{dF}{dt}&+F(P_{\text{ev}}+P_{\text{od}})+(P_{\text{ev}}-P_{\text{od}})F\\ &=\frac{d}{dt}(E\cosh\varphi+M\sinh\varphi)+(E\cosh\varphi+M\sinh\varphi)(P_{\text{ev}}+P_{\text{od}})\\ &\qquad+(P_{\text{ev}}-P_{\text{od}})(E\cosh\varphi+M\sinh\varphi)\\ &=E\sinh\varphi\frac{d\varphi}{dt}+\frac{dM}{dt}\sinh\varphi+M\frac{d\varphi}{dt}\cosh\varphi+2P_{\text{ev}}\cosh\varphi\\ &\qquad+(MP_{\text{ev}}+P_{\text{ev}}M)\sinh\varphi+(MP_{\text{od}}-P_{\text{od}}M)\sinh\varphi\\ &=\left(M\frac{d\varphi}{dt}+2P_{\text{ev}}\right)\cosh\varphi+\left(E\frac{d\varphi}{dt}+MP_{\text{ev}}+P_{\text{ev}}M\right)\sinh\varphi\\ &\qquad+\left(\frac{dM}{dt}+MP_{\text{od}}-P_{\text{od}}M\right)\sinh\varphi=0.\end{aligned}$$

Indeed, here each of the expressions in parentheses vanishes. This can readily be verified if we take into account the fact that condition 2 implies the identity

$$P_{\text{ev}}=-M\sqrt{p_1^2+p_2p_3}=-\frac{1}{2}M\frac{d\varphi}{dt}.$$

The condition $F(0)=E$ is obviously satisfied. The proof of the theorem is complete.

Remark 1. In accordance with the general proposition of the theory of reflecting functions, the Poincaré map of the 2ω-periodic system (5) on the interval $[-\omega,\omega]$ is given by the formula

$$F(-\omega,x)=\begin{pmatrix}\cosh\varphi(\omega)-m(\omega)\sinh\varphi(\omega) & -n(\omega)\sinh\varphi(\omega)\\ -r(\omega)\sinh\varphi(\omega) & \cosh\varphi(\omega)+m(\omega)\sinh\varphi(\omega)\end{pmatrix}x.$$

Theorem 2. *Let the matrix $P_{\text{ev}}(t)$ in system (5) be continuously differentiable for all $t\in\mathbb{R}$, and let the matrix $P_{\text{od}}(t)$ be continuous. Assume also that the following conditions are satisfied:*

1. *The inequality $-\det P_{\text{ev}}(t)\equiv p_1^2+p_2(t)p_3(t)<0$ holds for all $t\in\mathbb{R}$.*
2. *The functions*

$$m(t):=\frac{-p_1(t)}{\sqrt{-p_1^2(t)-p_2(t)p_3(t)}},\ n(t):=\frac{-p_2(t)}{\sqrt{-p_1^2(t)-p_2(t)p_3(t)}},\ r(t):=\frac{-p_3(t)}{\sqrt{-p_1^2(t)-p_2(t)p_3(t)}}$$

satisfy relations (6).

Then the reflecting matrix of system (5) is given by the formula

$$F(t)=E\cos\varphi(t)+M(t)\sin\varphi(t),$$

where

$$M(t)=\begin{pmatrix} m(t) & n(t) \\ r(t) & -m(t)\end{pmatrix} \quad \text{and} \quad \varphi(t)=2\int_0^t \sqrt{-p_1^2(\tau)-p_2(\tau)p_3(\tau)}\,d\tau.$$

Proof. Let us reduce the proof of Theorem 2 to proving Theorem 1 by using the statement of the latter. Here we assume that this formulation uses the notation φ_1, m_1, n_1, and r_1 instead of φ, m, n, and r, respectively. Then

$$\varphi_1(t) = 2\int_0^t \sqrt{p_1^2(\tau) + p_2(\tau)p_3(\tau)}\, d\tau = 2i\int_0^t \sqrt{-p_1^2(\tau) - p_2(\tau)p_3(\tau)}\, d\tau = -i\varphi(t),$$

$$m_1(t) = \frac{m(t)}{i}, \quad n_1(t) = \frac{n(t)}{i}, \quad r_1(t) = \frac{r(t)}{i}, \quad i^2 = -1.$$

Therefore, for the reflecting matrix of the system in question we obtain the relation

$$F(t) = E\cosh(i\varphi(t)) + M(t)\frac{\operatorname{sh}(i\varphi(t))}{i} = E\cos\varphi(t) + M(t)\sin\varphi(t).$$

This implies the assertion in Theorem 2.

The proof can also be conducted directly by verifying relation (3).

Corollary 1. *Let all conditions in Theorem 2 be satisfied for the 2ω-periodic system (5), and let*

$$\int_0^\omega \sqrt{-p_1^2(\tau) - p_2(\tau)p_3(\tau)}\, d\tau = \pi\frac{q}{p},$$

where $p, q \in \mathbb{N}$ and q/p is an irreducible fraction.

Then all solutions of this system are $2\omega p$-periodic.

Proof. Since the function $\varphi(t)$ in the statement of Theorem 2 is, according to its definition, odd, we see that the function $\dot\varphi(t)$ is even and, by virtue of the assumptions in the corollary, 2ω-periodic. It is well known that for each 2ω-periodic even function $\dot\varphi(t)$ there exists a 2ω-periodic function $\psi(t)$ and a constant $c = \frac{1}{2\omega}\int_{-\omega}^{\omega}\dot\varphi(\tau)\,d\tau = \frac{1}{\omega}\int_0^\omega \dot\varphi(\tau)\,d\tau$ such that $\int_0^t \dot\varphi(\tau)\,d\tau = ct + \psi(t)$.

Therefore, the function

$$\cos\varphi(t) = \cos\big(ct + \psi(t)\big) = \cos\left(t\frac{\pi}{\omega}\frac{q}{p} + \psi(t)\right)$$

is $2p\omega$-periodic. It follows that the Poincar'e map $F(-\omega)x = Ex$ is an identity on the interval $[-\omega p, \omega p]$. The proof of Corollary 1 is complete.

Corollary 2. *Let all assumptions in Theorem 2 be satisfied. Then all solutions of the 2ω-periodic system (5) are bounded on $\mathbb{R}$, and it is stable nonasymptotically.*

The **proof** readily follows from Theorem 1 in [11, p. 81].

Remark 2. We will consider the three differential relations which must be satisfied by the functions $m(t)$, $n(t)$, and $r(t)$ according to the second condition in Theorem 2 as a system of differential equations for these functions. It can readily be verified that such a system has the first integral $m^2 + rn = \text{const}$, which is in agreement with the formulas defining these functions. It follows from these formulas that $m^2 + rn = 1$. Thus, instead of the three differential relations (6), it suffices to require the validity of only the following two differential relations

$$\frac{dn}{dt} + n(\delta - \alpha) + 2\beta\sqrt{1 - rn} = 0, \quad \frac{dr}{dt} + r(\alpha - \delta) - 2\gamma\sqrt{1 - rn} = 0.$$

By way of example, consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & k(t) \\ p^2(t)k(t) & 0 \end{pmatrix} x + \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} x, \tag{7}$$

where $k(t)$ is an even continuous function, $p(t)$ is an even continuously differentiable function assuming only positive values, and the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are continuous and odd.

For the system under consideration, we find $d\varphi(t)/dt = 2\sqrt{p_1^2 + p_3 p_4} = 2p(t)k(t)$ and

$$m(t) \equiv 0, \quad n(t) \equiv \frac{-p_2}{\dot\varphi} \equiv \frac{-k}{2pk} \equiv \frac{-1}{2p(t)}, \quad r(t) \equiv \frac{-p_3}{\dot\varphi} \equiv -\frac{p(t)}{2}; \quad M(t) = \frac{1}{2}\begin{pmatrix} 0 & 1/p(t) \\ p(t) & 0 \end{pmatrix}.$$

As calculations show, conditions (6) are reduced to the following two:

$$\gamma(t) = \beta(t)p^2(t) \quad \text{and} \quad \frac{dp(t)}{dt} = p(t)\big(\delta(t) - \alpha(t)\big). \tag{8}$$

Therefore, Theorem 1 implies the following corollary.

Corollary 3. *Let the following conditions be satisfied for system* (7):

1. $p(t)$ *is an even differentiable function on* $\mathbb{R}$ *assuming only positive values.*
2. $k(t)$ *is an even function continuous on* $\mathbb{R}$.
3. *Identities* (8) *hold.*

Then the reflecting matrix of the system in question is given by the formula

$$F(t) = E\cosh\varphi(t) + M(t)\sinh\varphi(t),$$

where

$$\frac{d\varphi(t)}{dt} = 2\int_0^t p(\tau)k(\tau)\,d\tau, \quad M(t) = \frac{1}{2}\begin{pmatrix} 0 & 1/p(t) \\ p(t) & 0 \end{pmatrix}.$$

In conclusion, let us make the following remark, which is important for the practical use of the results obtained.

Remark 3. Consider the two-dimensional differential system

$$\frac{dx}{dt} = a(t)x + b(t)y + P(t,x,y), \quad \frac{dy}{dt} = c(t)x + d(t)y + Q(t,x,y), \tag{9}$$

where $P(t,x,y)$ and $Q(t,x,y)$ are series (or polynomials) in powers of x and y with coefficients depending on time t.

If

$$\begin{pmatrix} F_1(t,x,y) \\ F_2(t,x,y) \end{pmatrix} = \begin{pmatrix} f_1(t) & f_2(t) \\ f_3(t) & f_4(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is the reflecting function of the linear approximation system

$$\frac{dx}{dt} = a(t)x + b(t)y, \quad \frac{dy}{dt} = c(t)x + d(t)y,$$

then this reflecting function quite often is also the reflecting function of the entire system (9). This occurs if and only if the relations

$$f_1(t)P(t,x,y) + f_2(t)Q(t,x,y) + P(-t, f_1(t)x + f_2(t)y, f_3(t)x + f_4(t)y) \equiv 0,$$
$$f_3(t)P(t,x,y) + f_4(t)Q(t,x,y) + Q(-t, f_1(t)x + f_2(t)y, f_3(t)x + f_4(t)y) \equiv 0$$

hold true. These relations can be verified for known $f_k(t)$, $k = 1,\dots,4$, by simple albeit (for large powers of P and Q) cumbersome calculations. Here the calculations for polynomials $P_s(t,x,y)$ and $Q_s(t,x,y)$ homogeneous of degree s with respect to x and y for which

$$P(t,x,y) = \sum_{s=2}^n P_s(t,x,y), \quad Q(t,x,y) = \sum_{s=2}^n Q_s(t,x,y)$$

are made for each s separately.

2. REFLECTING FUNCTION OF THE RICCATI EQUATION

Consider the differential Riccati equation

$$\frac{dx}{dt} = a(t) + b(t)x + c(t)x^2 \tag{10}$$

with differentiable coefficients $a(t)$, $b(t)$, and $c(t)$ in the form

$$\frac{dx}{dt} = \big(a_{\rm ev}(t) + b_{\rm ev}(t)x + c_{\rm ev}(t)x^2\big) + \big(a_{\rm od}(t) + b_{\rm od}(t)x + c_{\rm od}(t)x^2\big),$$

where the expression in the first parentheses is the sum of t-even terms and the expression in the second parentheses is the sum of t-odd terms.

Theorem 3. *Let $b_{\rm ev}^2(t) - 4a_{\rm ev}(t)c_{\rm ev}(t) > 0$ for all $t \in \mathbb{R}$, and let the functions*

$$r(t) := \frac{-2a_{\rm ev}}{\sqrt{b_{\rm ev}^2 - 4a_{\rm ev}c_{\rm ev}}}, \quad s(t) := \frac{2c_{\rm ev}}{\sqrt{b_{\rm ev}^2 - 4a_{\rm ev}c_{\rm ev}}}, \quad n(t) := \frac{-b_{\rm ev}}{\sqrt{b_{\rm ev}^2 - 4a_{\rm ev}c_{\rm ev}}}$$

satisfy the system of relations

$$\frac{dr}{dt} = b_{\rm od}(t)r - 2a_{\rm od}(t)n, \quad \frac{ds}{dt} = -b_{\rm od}(t)s - 2c_{\rm od}(t)n, \quad \frac{dn}{dt} = c_{\rm od}(t)r + a_{\rm od}(t)s. \tag{11}$$

Then the reflecting function of Eq. (10) *is given by the formula*

$$F(t,x) = \frac{\big(\cosh\varphi(t) + n(t)\sinh\varphi(t)\big)x + r(t)\sinh\varphi(t)}{xs(t)\sinh\varphi(t) + \cosh\varphi(t) - n(t)\sinh\varphi(t)},$$

where $\varphi(t) := \int_0^t \sqrt{b_{\rm ev}^2 - 4a_{\rm ev}(\tau)c_{\rm ev}(\tau)}\, d\tau$.

The **proof** of this theorem can be conducted by verifying the main relation for the reflecting function in property 3.

Corollary 4. *Under the assumptions of Theorem* 3, *a solution $x(t;\omega,x(\omega))$ of Eq.* (10) *extendible to the interval $[-\omega,\omega]$ is a solution of the boundary value problem*

$$G(x(\omega), x(-\omega)) = 0$$

if and only if

$$G\Big(x(\omega), F\big(\omega, x(\omega)\big)\Big) = 0.$$

This assertion follows from the main property $F(t,x(t)) \equiv x(-t)$ of the reflecting function.

In the case of $b_{\rm ev}^2(t) - 4a_{\rm ev}(t)c_{\rm ev}(t) < 0$, for all $t \in \mathbb{R}$ we have

$$\varphi(t) = i\int_0^t \sqrt{4a_{\rm ev}c_{\rm ev} - b_{\rm ev}^2} =: \psi(t)i,$$

and the reflecting function of Eq. (10) acquires the form

$$F(t,x) = \frac{\big(\cos\varphi(t) + n(t)\sin\varphi(t)\big)x + r(t)\sin\varphi(t)}{xs(t)\sin\varphi(t) + \cos\varphi(t) - n(t)\sin\varphi(t)},$$

where

$$r(t) := \frac{-2a_{\rm ev}}{\sqrt{4a_{\rm ev}c_{\rm ev} - b_{\rm ev}^2}}, \quad s(t) := \frac{-2c_{\rm ev}}{\sqrt{4a_{\rm ev}c_{\rm ev} - b_{\rm ev}^2}}, \quad n(t) := \frac{-b_{\rm ev}}{\sqrt{4a_{\rm ev}c_{\rm ev} - b_{\rm ev}^2}}. \tag{12}$$

In this case, if the coefficients of Eq. (10) are 2ω-periodic, we have

$$\int\limits_0^t \sqrt{4a_{\mathrm{ev}}c_{\mathrm{ev}} - b_{\mathrm{ev}}^2}\, d\tau = c_0 t + \theta(t),$$

where the constant $c_0 = \omega^{-1}\int\limits_0^\omega \sqrt{4a_{\mathrm{ev}}c_{\mathrm{ev}} - b_{\mathrm{ev}}^2}\, d\tau$ and $\theta(t)$ is a 2ω-periodic function.

Hence we arrive at the following assertion.

Theorem 4. *Let all continuously differentiable coefficients of the Riccati equation* (10) *be* 2ω*-periodic, and let* $4a_{\mathrm{ev}}(t)c_{\mathrm{ev}}(t) - b_{\mathrm{ev}}^2(t) < 0$ *for all* $t \in \mathbb{R}$. *Further, let the functions* (12) *satisfy relations* (11).

Then all solutions of Eq. (10) *extendible to the interval* $[-\omega, \omega]$ *are* $2\omega p$*-periodic if*

$$\int\limits_0^\omega \sqrt{4a_{\mathrm{ev}}c_{\mathrm{ev}} - b_{\mathrm{ev}}^2}\, d\tau = \pi\frac{q}{p},$$

where $p, q \in \mathbb{N}$ *and* q/p *is an irreducible fraction.*

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