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# О НЕКОТОРЫХ НОВЫХ ОЦЕНКАХ ГРАДИЕНТА ФУНКЦИИ В ДЕКАРТОВОМ ПРОИЗВЕДЕНИИ ОБЛАСТЕЙ И СВЯЗАННЫЕ РЕЗУЛЬТАТЫ

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# ON SOME NEW ESTIMATES FOR A GRADIENT OF A FUNCTION IN PRODUCT DOMAINS AND RELATED RESULTS

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Работа содержит некоторые новые оценки связанные с градиентом аналитической и гармонической функции в функциональных пространствах типа Бергмана аналитических и гармонических функций в односвязных областях на комплексной плоскости. Приведены ряд утверждений о $L^p$ -нормах $n$-ых производных или градиентах аналитических или гармонических функций в пространствах типа Бергмана как для одной функции и одной области, так и в технически более трудном случае то есть для случая когда в неравенстве участвуют несколько различных аналитических или гармонических функций (так называемые многофункциональные неравенства). Приведены также новые неравенства подобного типа, в которых участвуют не только простые односвязные области, а их декартовое произведение. Доказательства неравенств более сложного типа выводятся или непосредственно из более простых неравенств такого же типа, или же полностью базируются на некоторых интересных оценках полученных в ходе их доказательств. Такие неравенства привлекали внимание различных авторов в последние годы. Теоремы приведенные в статье могут иметь различные интересные приложения в теории функции как одного так и нескольких комплексных переменных.

***Ключевые слова***: *градиент*, *производная аналитической функции*, *односвязная область*, *декартово произведение областей*, *мультифункциональные пространства*, *декомпозиция Уитни.*

Some new estimates related to the gradient and derivative of an analytic or harmonic function in Bergman type spaces of analytic and harmonic functions in selfconnected domains are presented. Some new results related with $L^p$ -norm estimates of the $n$-th order derivative or with the gradient of an analytic or harmonic function in Bergman type spaces of analytic and harmonic functions or such type multifunctional spaces in the case of simply connected domain in the complex plane $\mathbb{C}$ are considered. New inequalities of a similar type are also presented, in which not only simple simply-connected domains, but their Cartesian product, are involved. Proofs of inequalities of a more complex type are derived either directly from simpler inequalities of the same type, or are completely based on some interesting estimates obtained in the course of their proofs. Such inequalities attracted the attention of various authors in recent years. The theorems given in the article can have various interesting applications in the theory of function of both one and several complex variables.

***Keywords***: *gradient*, *derivative of analytic function*, *simply connected domains*, *product of domains*, *multi functional spaces*, *Whitney decomposition*.

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## Introduction

The gradient of a function is an important tool in the study of various properties of functional classes in various types of domains. This function has close ties with various types of capacities (Besov, Sobolev), fractional integrals, measure densities, quasi-continuous functions and Hausdorff measures (see for example [1]–[4] and various references there). It has various applications in geometric measure theory and potential theory also.

In this note we collect some new estimates for this function in multifunctional functional spaces and in functional spaces on product domains, using as model case simpler one functional estimates on a domain. Some related results for derivatives of a function will be also provided.

The Whitney decomposition of simply connected domain on a complex plane is a base of this note, it has also many other applications in function theory (see, for example, [10] and various references there).

In this note we provide some new estimates in simply connected domains and in products of such domains. For other general types of domains and various other similar estimates for analytic and harmonic functions on them we refer the reader to [5]–[8] and various references there.

## 1 On some estimates for a derivative of an analytic function and the gradient of harmonic function in case of a simply connected domain

In this section we look at some results related with the estimates of $L^p$ -norms of derivatives of n-order of analytic function via norms of the same function and gradient function in the corresponding weighted spaces of functions in the case of arbitrary simply connected domain in the complex plane $\mathbb{C}$.

Let $S=\{z\in\mathbb{C}:|z|<1\}$ be the unit disc in the complex space $\mathbb{C}$; let $G$ be some domain in $\mathbb{C}$; let further $H(G)$, $h(G)$ be the sets of analytic and harmonic functions in $G$; let further $d(w,\partial G)$ be a distance from the point $w$ to $\partial G$.

Denote by $L^p_\tau(G)$, $0<p<+\infty$, $\tau>-1$, the space of measurable function $f$ in $G$ such that

$$\int_G |f(w)|^p d^\tau(w,\partial G)dm_2(w)<+\infty.$$

Put also $A^p_\tau(G)=H(G)\cap L^p_\tau(G)$,

$$h^p_\tau(G)=h(G)\cap L^p_\tau(G).$$

Let $f\in H(S): f(z)=\sum_{k=1}^{\infty}a_k z_k$, $z\in S$ then we put $(Df)(z)=\sum_{k=0}^{\infty}(k+1)a_k z^k$ where $f\sum\{a_k\}_1^\infty$. This is a known fractional derivative of an analytic $f$ function.

Let $dm_2$ be the Lebeques measure on $G$ domain. In this note we denote by $c, c_1, c_2, \ldots,$ various positive constants in various estimates below.

To prove our results we first formulate a well-known classical fact (see for example [10]):

***Theorem* A** (Whitney decomposition). *Let $\Omega$ be a connected open set in $\mathbb{C}$. Then there is a set of squares* $\mathrm{P}=\{Q_1,Q_2,\ldots,Q_l,\ldots\}$, $\overset{0}{Q_l}\cap\overset{0}{Q_m}=\varnothing,\ l\neq m$ (*where* $\overset{0}{Q_l}$ *is interior of the* $Q_l$ *cube*) *such that* $\bigcup_l Q_l=\Omega$, *and*

$$c_1 diam(Q_l)\le dist(Q_l,\partial\Omega)\le c_2 diam(Q_l),$$

*where the constant* $c_1$, $c_2$ *do not depend on* $\Omega$.

Next, by $Q_l^*$ we denote the dyadic cube with the same center as $Q_l$, but stretched in $(1+\varepsilon)$ times, $0<\varepsilon<\frac{1}{4}$, $Q_l\subset Q_l^*$ (see for example [1]–[4]).

The following theorem as well as all other results of this and next sections concerning derivatives can be considered as extensions of the well-known classical theorem of Hardy and Littlewood in the unit disk on derivatives (or fractional derivatives). Theorem 1.1 is probably known for experts and it uses standard arguments, but it is a base of the main part of this paper and we provide it with detailed proof.We alert the reader that estimates which appear in proofs of this and next theorem in this section will be used by us in next section.

***Theorem* 1.1.** *Let $G$ be open simply connected domain in $\mathbb{C}$. Let also $f\in H(G)$; $0<p<+\infty$; $\tau>-1$; $n\in\mathbb{Z}_+$. Then for a positive constant $\tilde c_1=\tilde c_1(n,\tau)$ we have the following estimate*

$$\int_G |f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)dm_2(w)\le$$
$$\le \tilde c_1\int_G |f(w)|^p d^\tau(w,\partial G)dm_2(w).$$

*Proof*. Let $G=\bigcup_l Q_l$ be the Whitney decomposition of the set $G$, then

$$\int_G |f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)dm_2(w)=$$
$$=\sum_l\int_{Q_l}|f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)dm_2(w)\le$$
$$\le\sum_l\max_{w\in Q_l}|f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)(diam(Q_l))^2\le$$
$$\le c_1\sum_l\max_{w\in Q_l}|f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)d^2(w,\partial G)=$$
$$=c_1\sum_l\max_{w\in Q_l}|f^{(n)}(w)|^p d^{np+\tau+2}(w,\partial G)\le$$
$$\le c_1\sum_l|f^{(n)}(w_l)|^p d^{np+\tau+2}(w_l,\partial G),$$

where $w_l\in\partial Q_l$.

Let $D_\rho(w_l)=\{w:|w-w_l|<\rho\}$,

$$0<\rho<\frac{1}{2}dist(Q_l,\partial Q_l^*).$$

Then since $f^{(n)}(w_l)=\frac{n!}{2\pi i}\int_{\partial D_\rho}\frac{f(w)}{(w-w_l)^{n+1}}dw,$ we get

$$|f^{(n)}(w_l)|\le\frac{n!}{2\pi}\frac{1}{\rho^n}\max_{w\in\partial D_\rho}|f(w)|\le\frac{c_2}{d^n(\tilde w_l,\partial G)}|f(\tilde w_l)|,$$

where $\tilde w_l\in\partial D_\rho$. So, $|f^{(n)}(w_l)|^p\le\frac{c_3|f(\tilde w_l)|^p}{d^{np}(\tilde w_l,\partial G)}$.

But from theorem A we have $diam(Q_l)<$ $<diam(Q_l^*)<1\tfrac{1}{4}diam(Q_l)$, hence

$$d(w_l,\partial G)\sim d(\tilde w_l,\partial G).$$

Indeed now we get

$$\sum_l|f^{(n)}(w_l)|^p d^{np+\tau+2}(w_l,\partial G)\le$$
$$\le c_4\sum_l|f(\tilde w_l)|^p d^{\tau+2}(\tilde w_l,\partial G).$$

Let also $0<\rho'<\frac{1}{2}dist(Q_l,\partial Q_l^*)$,

$K_{\rho'}(\tilde w_l)=\{w:|w-\tilde w_l|<\rho'\}$, so, $K_{\rho'}(\tilde w_l)\subset Q_l^*$.

Since $|f|^p$ is a subharmonic function, then

$$|f(\tilde w_l)|^p\le\frac{1}{\pi\rho'^2}\int_{K_{\rho'}(\tilde w_l)}|f(w)|^p dm_2(w)\le$$
$$\le\frac{c_5}{d^2(\tilde w_l,\partial G)}\int_{Q_l^*}|f(w)|^p dm_2(w),\tag{1.1}$$

where $0<p<+\infty$.

Finally, we have

$$\int_G \left|f^{(n)}(w)\right|^p d^{np+\tau}(w,\partial G)dm_2(w) \le$$

$$\le c_6 \sum_l \int_{Q_l^*} \left|f(w)\right|^p d^{\tau}(w,\partial G)dm_2(w) \le$$

$$\le \tilde{c}_1 \int_G \left|f(w)\right|^p d^{\tau}(w,\partial G)dm_2(w).$$

Let $u \in H(S), z = x+iy$, then

$$grad\ u(z) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \qquad \square$$

The same method can be applied for the proof of theorem 1.2 on a gradient of a harmonic function. Indeed similarly to theorem 1, the following theorem can be shown. We provide only the complete sketch of the proof referying interested readers to [9].

***Theorem* 1.2** (see [9]). *Let G be open simply connected domain in* $\mathbb{C}$. *Let also* $f \in h_\tau^p(G)$, $0 < p < +\infty$, $\tau > -1$. *Then for a positive constant* $\tilde{c}_4 = \tilde{c}_4(\tau)$ *we have the following estimate*

$$\int_G \left|grad\ f(w)\right|^p d^{p+\tau}(w,\partial G)dm_2(w) \le$$

$$\le \tilde{c}_4 \int_G \left|f(w)\right|^p d^{\tau}(w,\partial G)dm_2(w).$$

*Sketch of the Proof.* Let $G = \bigcup_l Q_l$ be the Whitney decomposition of the set *G*, then

$$\int_G \left|grad\ f(w)\right|^p d^{p+\tau}(w,\partial G)dm_2(w) =$$

$$= \sum_l \int_{Q_l} \left|grad\ f(w)\right|^p d^{p+\tau}(w,\partial G)dm_2(w) \le$$

$$\le \sum_l \max_{w\in Q_l} \left|grad\ f(w)\right|^p d^{p+\tau}(w,\partial G)(diam(Q_l))^2 \le$$

$$\le c_1 \sum_l \max_{w\in Q_l} \left|grad\ f(w)\right|^p d^{p+\tau+2}(w,\partial G) \le$$

$$\le c_1 \sum_l \left|grad\ f(w_l)\right|^p d^{p+\tau+2}(w_l,\partial G),$$

where $w_l \in \partial Q_l$.

Put $D_\rho(w_l) = \{w : \left|w-w_l\right| < \rho\}$,

$$0 < \rho < \frac{1}{2} dist(Q_l, \partial Q_l^*).$$

Then following [9] we have

$$\rho\left|grad\ f(w)\right| \le c_1 \max_{\zeta\in D_\rho(w_l)} \left|f(\zeta)\right| \frac{1}{1-\frac{|w-w_l|}{\rho}}$$

for all $w \in D_\rho(w_l)$.

Then $\rho\left(1-\frac{\left|w-w_l\right|}{\rho}\right)\left|grad\ f(w)\right| \le c_2 \left|f(\tilde{w}_l)\right|$,

where $\tilde{w}_l \in \partial D_\rho$.

Thus we get $\rho\left|grad\ f(w_l)\right| \le c_3 \left|f(\tilde{w}_l)\right|$ and $d(w_l,\partial G)\left|grad\ f(w_l)\right| \le c_3 \left|f(\tilde{w}_l)\right|$.

Finally, repeating the proof of theorem 1.1, we will have the statement of theorem 1.2. Note as a substitution of (1.1) the well known Hardy-Littlewood type theorem on subharmonic behaviour of an harmonic functions (or Fefferman-Stein theorem) should be applied. $\square$

These two onefunctional results and some estimates we had in their proofs serve as a base for proofs of our similar type multifunctional assertions in next section.

## 2 On some new estimates of the derivatives of an analytic function and the gradient of harmonic function in the multi functional and multidimensional cases

The goal of this section to provide new estimates for gradient and some related results for multi functional spaces and spaces on product domains. Our proofs are fully based on proofs of theorems of previous section and we will omit them partially providing short hints for readers. These estimates may have further various applications.

Issues related with multifunctional function spaces and spaces on product domains were under attention in recent years (see, for example, [11], [12], [13] and various references there).

These theorems below extend some results of previous section to the case of several functions and to some functional spaces on product domains.

***Theorem* 2.1.** *Let G be open simply connected domain in* $\mathbb{C}$. *Let also* $f_i \in H(G)$; $0 < p_i < +\infty$, $i = \overline{1,m}$, $m \ge 1$; $n \in \mathbb{Z}_+$; $\tau > -1$. *Then for a positive constant* $c_1 = c_1(\tau,n,m)$ *we have the following estimate*

$$\int_G \left|f_1^{(n)}(w)\right|^{p_1} \ldots \left|f_m^{(n)}(w)\right|^{p_m} d^{(np+\tau+2)m-2}(w,\partial G)dm_2(w) \le$$

$$\le c_1 \int_G \left|f_1(w)\right|^{p_1} d^{\tau}(w,\partial G)dm_2(w) \times \ldots \times$$

$$\times \int_G \left|f_m(w)\right|^{p_m} d^{\tau}(w,\partial G)dm_2(w),$$

where $p = \sum_{j=1}^n p_j$.

***Theorem* 2.2.** *Let G be open simply connected domain in* $\mathbb{C}$. *Let also* $u_i \in h_\tau^{p_i}(G)$, $0 < p_i < +\infty$, $i = \overline{1,m}$, $m \ge 1$, $\tau > -1$. *Then for a positive constant* $c_2 = c_2(\tau,m)$ *we have the following estimate*

$$\int_G \left|grad\ u_1(w)\right|^{p_1} \ldots \left|grad\ u_m(w)\right|^{p_m} \times$$

$$\times d^{(p+\tau)m+2m-2}(w,\partial G)dm_2(w) \le$$

$$\le c_2 \int_G \left|grad\ u_1(w)\right|^{p_1} d^{\tau_1}(w,\partial G)dm_2(w) \times \ldots \times$$

$$\times \int_G \left|grad\ u_m(w)\right|^{p_m} d^{\tau_m}(w,\partial G)dm_2(w).$$

*The proofs of theorem* 2.1 *and* 2.2.

Note the following estimate can be found in the proof of theorem 1.1. Indeed carefull inspection shows that for every analytic *f* function and for all $0 < p < +\infty$ we have

$$\sup_{w\in G}|f(w)|(d(w,\partial G))^{\frac{2}{p}}\le \le c\left(\int_G|f(w)|^p\,dm_2(w)\right)^{\frac{1}{p}} \qquad (2.1)$$

for some constant $c$.

Based on this we have

$$\int_G|f_1^{(n)}(w)|^{p_1}\ldots|f_m^{(n)}(w)|^{p_m}\times \times d^{(np+\tau+2)m-2}(w,\partial G)dm_2(w)\le$$
$$\le c\sup_{w\in G}\left(|f_1^{(n)}(w)|^{p_1}\,d^{v_1}(w,\partial G)\right)\times\ldots\times$$
$$\times\sup_{w\in G}\left(|f_{m-1}^{(n)}(w)|^{p_{m-1}}\,d^{v_{m-1}}(w,\partial G)\right)\times$$
$$\times\int_G|f_m^{(n)}(w)|^{p_m}\,d^{v_m}(w,\partial G)dm_2(w)$$

for some $v_1,\ldots,v_m$ so that $\sum_{j=1}^m v_j=(np+\tau+2)m-2.$

It remains to use (2.1) $(m-1)$ times and then theorem 1 and separately theorem 1 for $f_m$ function to get the estimate we need.

The proof of theorem 2.2 is also based on (2.1), but for $(grad\,u)$ which can be obtained from the proof of theorem 1.2.

As we see proofs mainly are based on elementary estimates like

$$\max_{z\in X}\{u_1(z)\ldots u_n(z)\}\le\max_{z\in X}u_1(z)\ldots\max_{z\in X}u_n(z),$$

for some positive functions $u_i$, $i=\overline{1,n}$, $n\ge1$, on sets $X$, and on some similar type estimates for sets of $n$-functions or for only one single function and on arguments of proof of theorem 1.1.

Let now $G_i, i=\overline{1,n}$, be an open simply connected domain in $\mathbb{C}$. Denote by $H(G_1\times\ldots\times G_n)$ a space of all analytic functions in $G_1\times\ldots\times G_n$; let $h^p_{\tau_i}(G_1\times\ldots\times G_n)$, $0<p<+\infty$; $\tau_i>-1$; $i=\overline{1,n}$, be a space of all harmonic functions in $G_1\times\ldots\times G_n$ such that

$$\underbrace{\int_{G_1}\ldots\int_{G_n}}_{n}|f(\vec{w})|^p\left(\prod_{i=1}^n d^{\tau_i}(w_i,\partial G_i)dm_2(w_i)\right)<+\infty,$$

where $\vec{w}=(w_1,\ldots,w_n), w_i\in G_i$, $i=\overline{1,n}$.

We now formulate some versions of theorems of previous section in case of product domains namely for so called polydomains.

***Theorem* 2.3.** *Let $G_i, i=\overline{1,n}$, be an open simply connected domain in $\mathbb{C}$. Let also $f\in H(G_1\times\ldots\times G_n)$; $0<p<+\infty$; $m_j\ge0, j=\overline{1,n}$; and $\tau=(\tau_1,\ldots,\tau_n)$, $\tau_i>-1$, $i=\overline{1,n}$; $n\in Z_+$. Then there is a positive constant $c_3=c_3(\tau,n,m)$ so that*

$$\underbrace{\int_{G_1}\ldots\int_{G_n}}_{n}\left|\frac{\partial^{|m|}}{\partial w_1^{m_1}\ldots\partial w_n^{m_n}}f(w_1,\ldots,w_n)\right|^p d^{pm_1+\tau_1}(w_1,\partial G_1)\ldots$$
$$\ldots d^{pm_n+\tau_n}(w_n,\partial G_n)dm_2(w_1)\ldots dm_2(w_n)\le$$
$$\le c_3\underbrace{\int_{G_1}\ldots\int_{G_n}}_{n}|f(w_1,\ldots,w_n)|^p d^{\tau_1}(w_1,\partial G_1)\ldots$$
$$\ldots d^{\tau_n}(w_n,\partial G_n)dm_2(w_1)\ldots dm_2(w_n),$$

where $|m|=\sum_{j=1}^n m_j$.

To get the estimate in this theorem we have to apply our first theorem by each variable separately and then use induction by amount of variables.This type procedure is well-known for the unit polydisk in $C^n$.

***Remark* 2.1.** These results (theorem 2.1, 2.3) are also valid when $G$ is in $\mathbb{C}^n$ for which some analogue of theorem A is valid. As an example we mention bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ (see for example [13]).

***Remark* 2.2.** Note we used some technique in proofs of these assertions which is close to the one we presented previously for analytic functions in the unit ball (see [11]). Note, we also modify already known proofs one functional case for gradient function $(n=1)$ provided in previous section which has various applications also (see [14]–[15]).

We finally present another two versions of such type results.

***Theorem* 2.4.** *Let $G$ be open simply connected domain in $\mathbb{C}$, let $G=\bigcup_l Q_l$, be the Whitney decomposition of the set $G$. Let also $0<p<+\infty$, $\tau>-1$, $\sum_{i=1}^n\frac{1}{q_i}=1$, $n\in\mathbb{Z}_+$.*

*1) If $f_i\in H(G)$, $i=\overline{1,n}$, then for a positive constant $c_6=c_6(\tau,n)$ we have the following estimate*

$$\int_G\left|(f_1\ldots f_n)^{(n)}(w)\right|^p d^{np+\tau}(w,\partial G)dm_2(w)\le$$
$$\le c_6\sum_{l\ge0}\left[\left(\int_{Q_l^*}\left|f_1(w)\right|^{pq_1}d^{\tau}(w,\partial G)dm_2(w)\right)^{\frac{1}{q_1}}\times\ldots\right.$$
$$\left.\ldots\times\left(\int_{Q_l^*}\left|f_n(w)\right|^{pq_n}d^{\tau}(w,\partial G)dm_2(w)\right)^{\frac{1}{q_n}}\right];$$

*2) If $u_i\in h^{pq_i}_\tau(G)$, $i=\overline{1,n}$, then for a positive constant $c_7=c_7(\tau,n)$ we have the following estimate*

$$\int_G\left|grad(u_1\ldots u_n)(w)\right|^p d^{p+\tau}(w,\partial G)dm_2(w)\le$$
$$\le c_7\sum_{l\ge0}\prod_{i=1}^n\left(\int_{Q_l^*}\left|u_i(w)\right|^{pq_i}d^{\tau}(w,\partial G)dm_2(w)\right)^{\frac{1}{q_i}}.$$

***Theorem* 2.5.** *Let $G_i, i=\overline{1,n}$, be open simply connected domains in $\mathbb{C}$, $G_i=\bigcup_{l_i}Q_{l_i}, i=\overline{1,n}$, be the*

*Whitney decomposition of the set G. Let also* $0<p<+\infty,\ q\le 1,$ *and* $m_j\ge 0,\ j=\overline{1,n},\ \tau=(\tau_1,...,\tau_n),$ $\tau_i>-1,\ z=(z_1,...z_n), z_i\in G_i,\ i=\overline{1,n},\ n\in\mathbb{Z}_+.$

1) *If* $f_i\in H(G_i),\ i=\overline{1,n},$ *then there is a positive constant* $c_8=c_8(\vec{\tau},n)$ *so that*

$$\underbrace{\int_{G_1}...\int_{G_n}}_{n}\left|\frac{\partial^{|m|}(f_1...f_n)(\vec{z})}{\partial z_1^{m_1}...\partial z_n^{m_n}}\right|^p\prod_{j=1}^{n}(d^{pm_j+\tau_j}(z_j,\partial G_j)dm_2(z_j))\le$$

$$\le c_8\prod_{i=1}^{n}\left(\sum_{l_i\ge 0}\left(\int_{Q_{l_i}^*}|f_i(w)|^p d^{\tau_i}(w,\partial G)dm_2(w)\right)^q\right)^{\frac{1}{q}}.$$

2) *If* $u_i\in h^p_{\tau_i}(G_i),\ i=\overline{1,n},$ *then there is a positive constant* $c_9=c_9(\tau,n)$ *so that*

$$\underbrace{\int_{G_1}...\int_{G_n}}_{n}|grad(u_1...u_n)(\vec{z})|^p\prod_{i=1}^{n}(d^{p+\tau_i}(z_i,\partial G_i)dm_2(z_i))\le$$

$$\le c_9\prod_{i=1}^{n}\left(\sum_{l_i\ge 0}\left(\int_{Q_{l_i}^*}|u_i(w)|^p d^{\tau_i}(w,\partial G)dm_2(w)\right)^q\right)^{\frac{1}{q}}.$$

*The proofs of theorem* 2.4 *and* 2.5. First we provide proofs of estimates with derivatives. During the proof of theorem 1.1 and theorem 1.2 the following estimates were obtained in particular.

$$\int_G|f^{(n)}(w)|^p d^{np+\tau}(w,\partial G)dm_2(w)\le \le c\sum_l\int_{Q_l^*}|f(w)|^p d^{\tau}(w,\partial G)dm_2(w) \quad (2.2)$$

$$\int_G|grad\ f(w)|^p d^{p+\tau}(w,\partial G)dm_2(w)\le \le \tilde{c}_1\sum_l\int_{Q_l^*}|f(w)|^p d^{\tau}(w,\partial G)dm_2(w) \quad (2.3)$$

Using (2.2) we have from theorem 1.1

$$\int_G|(f_1,...,f_n)^{(n)}(w)|^p d^{np+\tau}(w,\partial G)dm_2(w)\le$$

$$\le c\int_G|(f_1,...,f_n)(w)|^p d^{\tau}(w,\partial G)dm_2(w)\le$$

$$\le c\sum_l\int_{Q_l}|f_1(w)|^p...|f_n(w)|^p d^{\tau}(w,\partial G)dm_2(w).$$

It remains to use Holders inequality for *n* functions to get the first estimate in theorem 2.4. The second estimate is based on theorem 1.2 and estimate (2.3) and can be shown similarly. We omit details here leaving them to interested readers.

Estimates of theorem 2.5 are related to product domains $G=G_1\times...\times G_n$. We must use theorem 1.1 to get first estimate in this theorem by each $z_j$ variable separately fixing others to get by induction for $F(z)=f_1(z_1)...f_n(z_n)$ function the following estimate

$$M=\int_{G_1}...\int_{G_n}\left|\frac{\partial^{|m|}(f_1...f_n)(\vec{z})}{\partial z_1^{m_1}...\partial z_n^{m_n}}\right|^p\prod_{j=1}^{n}d^{pm_j+\tau_j}(z_j,\partial G_j)\le$$

$$\le c\int_{G_1}...\int_{G_n}|F(z_1,...,z_n)|^p\times$$

$$\times\prod_{j=1}^{n}d^{\tau_j}(z_j,\partial G_j)dm_2(z_1)...dm_2(z_n),$$

then we repeat arguments provided in the proof of previous theorem using Holder inequality again to get

$$M\le c\prod_{i=1}^{n}\left(\sum_{l_i\ge 0}\left(\int_{Q_{l_i}^*}|f_i(w)|^p d^{\tau_i}(w,\partial G)dm_2(w)\right)^q\right)^{\frac{1}{q}}.$$

The second estimate for gradient in this theorem can be shown similarly based on theorem 1.2 used *n*-times then again by application of Holder's inequality for one sum. We omit easy details. This finishes proofs of theorems 2.4 and 2.5. It is interesting to note that practically all results given by us can be extended even to more general cases when the amount of functions involved is different from the amount of variables, to be more precise where expressions like $\prod_{j=1}^{m}|f_j(w_1,...,w_n)|^{p_j}$ are involved.

***Remark* 2.3.** Under certain additional conditions these type estimates (for appropriately defined derivatives) can be also shown in Siegel domains of the second type, in tubular domains over symmetric cones, in bounded strictly pseudoconvex domain with smooth boundary and bounded symmetric domains. For these type domains the analogue of theorem A is valid. In tube these are so-called Hardy type inequalities (see, i.o., [16]), which also have various interesting applications (see [16]). Results of this note may also have various applications.

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