УДК 539.12 DOI: https://doi.org/10.54341/20778708_2022_1_50_19

БЕЗМАССОВОЕ ПОЛЕ СО СПИНОМ 3/2: РЕШЕНИЯ С ЦИЛИНДРИЧЕСКОЙ СИММЕТРИЕЙ, УСТРАНЕНИЕ КАЛИБРОВОЧНЫХ СТЕПЕНЕЙ СВОБОДЫ

А.В. Ивашкевич

Институт физики им. Б.И. Степанова Национальной академии наук Беларуси

SPIN 3/2 MASSLESS FIELD, SOLUTIONS WITH CYLINDRICAL SYMMETRY, ELIMINATING THE GAUGE DEGREES OF FREEDOM

A.V. Ivashkevich

B.I. Stepanov Institute of Physics, *National Academy of Sciences of Belarus*

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

Аннотация. Система уравнений, описывающая безмассовую частицу со спином 3/2, решена в цилиндрических координатах пространства Минковского. Используется общековариантный тетрадный формализм. На решениях диагонализируются операторы $i\partial_t, J_3, i\partial_z$. После разделения переменных получена система уравнений для 16 функций, зависящих от переменой *r*. Показано, что построенные согласно теории Паули – Фирца калибровочные решения с цилиндрической симметрией, обращают эти 16 уравнений в тождества. Найдены 6 линейно независимых решений системы уравнений, 4 из них совпадают с калибровочными, два не содержат калибровочных степеней свободы и описывают физически наблюдаемые состояния, они выражаются через функции Бесселя.

Ключевые слова: *спин* 3/2, *цилиндрическая симметрия*, *безмассовое поле*, *точные решения*, *калибровочные степени свободы*.

Для цитирования: *Ивашкевич*, *А.В*. Безмассовое поле со спином 3/2: решения с цилиндрической симметрией, устранение калибровочных степеней свободы / А.В. Ивашкевич // Проблемы физики, математики и техники. – 2022. – № 1 (50). – С. 19–27. – DOI: https://doi.org/10.54341/20778708_2022_1_50_19

Abstract. The system of equations for the vector-bispinor describing a spin 3/2 massless particle, is solved in cylindric coordinates of the Minkowski space. The covariant tetrad formalism is applied. Three operators, $i\partial_t$, J_3, $i\partial_z$ are diagonalized on the constructed solutions. After separating the variables, the system of 16 functions in the variable *r* is derived. It is shown that the gauge solutions with cylindric symmetry, which are specified according to the Pauli – Fierz theory, satisfy these 16 equations identically. There are constructed 6 linearly independent solutions. Four of them coincide with the gauge ones, and two remain solutions do not contain the gauge degrees of freedom. These two solutions are expressed in terms of Bessel functions.

Keywords: *spin* 3/2, *cylindric symmetry*, *massless field*, *exact solutions*, *gauge degrees of freedom*.

For citation: *Ivashkevich*, *A.V.* Spin 3/2 massless field, solutions with cylindrical symmetry, eliminating the gauge degrees of freedom / A.V. Ivashkevich // Problems of Physics, Mathematics and Technics. – 2022. – № 1 (50). – P. 19–27. – DOI: https://doi.org/10.54341/20778708_2022_1_50_19

Introduction

After the primary papers by Pauli and Fierz [1], [2], and Rarita and Schwinger [3], the theory of the particle with spin 3/2 [4]–[25] always has attracted attention. The description of that particle requires 16-component wave function with transformation properties of vector-bispinor with respect to the Lorentz group. Special interest represents the case of the massless particle. As was shown by Pauli and Fierz, there exists specific gauge symmetry, according to which the 4-gradient over arbitrary bispinor gives solutions of the massless wave equation. The gauge states do not contribute to observable physical quantities, like energy-momentum tensor.

In the present paper, we follow the problem of the degrees of freedom of the massless particle for solutions with cylindric symmetry. In explicit form there are constructed two solutions which do not contain the gauge components. Also we find 4 solutions of the wave equation, which may be identified with the gauge ones. The study is based on the use of covariant tetrad formalism according to general method by Tetrode – Weyl – Fock – Ivanenko [19].

1 Separating the variables

The following form of the wave equation for the spin 3/2 particle (for generality, we start with the case of the massive particle; see notations in [19]) is known

$$\gamma^5 \varepsilon_\rho{}^{\sigma\alpha\beta}(x)\gamma_\sigma\left(iD_\alpha - M\gamma_\alpha\right)\Psi_\beta = 0, \qquad D_\alpha = \nabla_\alpha + \Gamma_\alpha + ieA_\alpha; \tag{1.1}$$

$M = mc/2\hbar c$ stands for the massive parameter. It is readily proved that in massless case, if the Riemann curvature tensor for a space-time model equals to zero, then eq. (1.1) has the class of gradient type solutions

$$\Psi^G_\beta(x) = D_\beta\Psi(x), \quad D_\beta = (\nabla_\beta + \Gamma_\beta),$$

© *Ivashkevich A.V.*, 2022

where Φ stands for an arbitrary bispinor field. Indeed, due to the commutator [19]:

$$(D_\alpha D_\beta - D_\beta D_\alpha)\Phi = \frac{1}{2}\sigma^{\mu\nu}(x)R_{\mu\nu\alpha\beta}(x) \equiv 0, \quad (1.2)$$

eq. (1.1) is satisfied identically by this gradient solution in all space-time models with $R_{\mu\nu\alpha\beta}(x) = 0$.

After transition in the field function to the tetrad presentation in vector index $\Psi_\beta = e_\beta^{(b)}\Psi_b$ eq. (1.1) takes on the form

$$\gamma^5 \varepsilon_k^{can}\gamma_c \left[i(D_a)_n^l - M\gamma_a\delta_n^l \right]\Psi_l = 0, \quad (1.3)$$

where we use the extended derivative operator

$$D_a = e_{(a)}^\alpha(\partial_\alpha + ieA_\alpha) + \frac{1}{2}\left(\sigma^{ps}\otimes I + I \otimes j^{ps}\right)\gamma_{[ps]a}. \quad (1.4)$$

Introducing 6 matrices $\varepsilon_k^{can} = (\mu^{[ca]})_k^n$, we can rewrite eq. (1.3) as follows

$$\gamma^5(\mu^{[ca]})_k{}^n\gamma_c\left[i(D_a)_n{}^l - M\gamma_a\delta_n{}^l \right]\Psi_l = 0. \quad (1.5)$$

Allowing for the summation formula

$$\mu^{ca}\gamma_c D_a = \mu^{[01]}[\gamma_0 D_1\Psi - \gamma_1 D_0\Psi] +$$
$$+ \mu^{[02]}[\gamma_0 D_2\Psi - \gamma_2 D_0] + \mu^{[03]}[\gamma_0 D_3\Psi - \gamma_3 D_0] +$$
$$+ \mu^{[23]}[\gamma_2 D_3\Psi - \gamma_3 D_2\Psi +$$
$$+ \mu^{[31]}[\gamma_3 D_1\Psi - \gamma_1 D_3] + \mu^{[12]}[\gamma_1 D_2\Psi - \gamma_2 D_1],$$

or (taking in mind that the matrices $\mu^{[ca]}$ act on the vector index)

$$\mu^{[ca]}\gamma_c D_a =$$
$$= \left(\gamma^1\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[03]}\right)D_0\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[12]} - \gamma^3\otimes\mu^{[31]}\right)D_1\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[23]} - \gamma^1\otimes\mu^{[12]}\right)D_2\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[03]} + \gamma^1\otimes\mu^{[31]} - \gamma^2\otimes\mu^{[23]}\right)D_3\Psi,$$

and the other summation formula

$$(\mu^{[ca]})_k{}^n\gamma_c\gamma_a\Psi_n \quad \Rightarrow \quad \mu^{[ca]}\gamma_c\gamma_a\Psi =$$
$$= \{(\gamma_0\gamma_1 - \gamma_1\gamma_0)\otimes\mu^{[01]} + (\gamma_0\gamma_2 - \gamma_2\gamma_0)\otimes\mu^{[02]} +$$
$$+ (\gamma_0\gamma_3 - \gamma_3\gamma_0)\otimes\mu^{[03]} + (\gamma_2\gamma_3 - \gamma_3\gamma_2)\otimes\mu^{[23]} +$$
$$+ (\gamma_3\gamma_1 - \gamma_1\gamma_3)\otimes\mu^{[31]} + (\gamma_1\gamma_2 - \gamma_2\gamma_1)\otimes\mu^{[12]}\}\Psi =$$
$$= \{s_{01}\otimes\mu^{[01]} + s_{02}\otimes\mu^{[02]} + s_{03}\otimes\mu^{[03]} +$$
$$+ s_{23}\otimes\mu^{[23]} + s_{31}\otimes\mu^{[31]} + s_{12}\otimes\mu^{[12]}\}\Psi,$$

we arrive at the detailed form of the basic equation (1.5):

$$\left(\gamma^1\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[03]}\right)D_0\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[12]} - \gamma^3\otimes\mu^{[31]}\right)D_1\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[23]} - \gamma^1\otimes\mu^{[12]}\right)D_2\Psi +$$
$$+ \left(\gamma^0\otimes\mu^{[03]} + \gamma^1\otimes\mu^{[31]} - \gamma^2\otimes\mu^{[23]}\right)D_3\Psi +$$
$$+ iM\{s_{01}\otimes\mu^{[01]} + s_{02}\otimes\mu^{[02]} + s_{03}\otimes\mu^{[03]} +$$
$$+ s_{23}\otimes\mu^{[23]} + s_{31}\otimes\mu^{[31]} + s_{12}\otimes\mu^{[12]}\}\Psi = 0. \quad (1.6)$$

Let us search solutions with cylindrical symmetry. For cylindrical coordinates and tetrad

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$
$$x^\alpha = (t, r, \phi, z), \quad A_\phi = \frac{Br^2}{2}, \quad (1.7)$$

$$e_{(a)}^\beta(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad e_{(a)\beta}(y) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix};$$

the Ricci rotation coefficients are

$$\gamma_{ab0} = 0, \quad \gamma_{ab1} = 0, \quad \gamma_{122} = -\gamma_{212} = +\frac{1}{r}, \quad \gamma_{ab3} = 0.$$

The components of the derivative operator D_c are determined by the formulas

$$D_0 = \partial_t, \quad D_1 = \partial_r, \quad (1.8)$$
$$D_2 = \frac{1}{r}\left(\partial_\phi + \frac{ieBr^2}{2}\right) + \frac{1}{r}\left(\sigma^{12}\otimes + I\otimes j^{12}\right), \quad D_3 = \partial_z.$$

Thus, eq. (1.6) takes on the form

$$\{(\gamma^1\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[03]})\partial_t\Psi +$$
$$+ (\gamma^0\otimes\mu^{[01]} + \gamma^2\otimes\mu^{[12]} - \gamma^3\otimes\mu^{[31]})\partial_r\Psi +$$
$$+ (\gamma^0\otimes\mu^{[02]} + \gamma^3\otimes\mu^{[23]} - \gamma^1\otimes\mu^{[12]})D_2\Psi +$$
$$+ (\gamma^0\otimes\mu^{[03]} + \gamma^1\otimes\mu^{[31]} - \gamma^2\otimes\mu^{[23]})\partial_z\Psi\}_k +$$
$$+ iM\{s_{01}\otimes\mu^{[01]} + s_{02}\otimes\mu^{[02]} + s_{03}\otimes\mu^{[03]} +$$
$$+ s_{23}\otimes\mu^{[23]} + s_{31}\otimes\mu^{[31]} + s_{12}\otimes\mu^{[12]}\}\Psi = 0, \quad (1.9)$$

where $s_{01} = (\gamma^0\gamma^1 - \gamma^1\gamma^0)/4$ and so on. We need expressions for 6 matrices $\mu^{[ca]}$:

$$\mu^{[01]} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \mu^{[02]} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

$$\mu^{[03]} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \mu^{[23]} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\mu^{[31]} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \mu^{[12]} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

and expressions for Dirac matrices:

$$\gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \gamma^1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma^2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \gamma^3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix};$$

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

$$\sigma^{12} = \frac{1}{2}\begin{vmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{vmatrix}, \quad j^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$s_{01} = -\begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{vmatrix}, \quad s_{02} = -\begin{vmatrix} 0 & -2i & 0 & 0 \\ 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & 0 & -2i & 0 \end{vmatrix},$$

$$s_{03} = -\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}, \quad s_{23} = \begin{vmatrix} 0 & -2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & -2i & 0 \end{vmatrix},$$

$$s_{31} = \begin{vmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{vmatrix}, \quad s_{12} = \begin{vmatrix} -2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 2i \end{vmatrix}.$$

We apply the substitution for cylindrically symmetric wave function (assuming $\varepsilon > 0$):

$$\Psi_{A(n)} = e^{-i\varepsilon t} e^{im\phi} e^{ikz} \Phi_{A(n)}(r),$$

$$[\Phi_{A(n)}] = \Phi(r) = \begin{vmatrix} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \\ h_0 & h_1 & h_2 & h_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix}. \qquad (1.10)$$

It should be noted that in the basis of cylindrical tetrad, parameter m stands for eigenvalues of the third projection of the total angular momentum, so m takes on the half-integer values $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. Allowing for the above substitution, we reduce the equation to the form

$$-i\varepsilon\left(\gamma^1 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[03]}\right)\Phi +$$
$$+\left(\gamma^0 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[12]} - \gamma^3 \otimes \mu^{[31]}\right)\frac{d}{dr}\Phi +$$
$$+\left(\gamma^0 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[23]} - \gamma^1 \otimes \mu^{[12]}\right)D_2\Phi +$$
$$+ ik\left(\gamma^0 \otimes \mu^{[03]} + \gamma^1 \otimes \mu^{[31]} - \gamma^2 \otimes \mu^{[23]}\right)\Phi +$$
$$+iM\left\{s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]} +\right.$$
$$\left.+ s_{23} \otimes \mu^{[23]} + s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]}\right\}\Phi = 0, \quad (1.11)$$

where

$$D_2 = \frac{1}{r}\left(im + \sigma^{12} \otimes + I \otimes j^{12}\right). \qquad (1.12)$$

The main equation may be re-written in a shorter form

$$-i\varepsilon B_0\Phi + B_1\frac{d}{dr}\Phi + B_2D_2\Psi + ikB_3\Phi +$$
$$+ iM\left\{s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]} +\right.$$
$$\left.+ s_{23} \otimes \mu^{[23]} + s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]}\right\}\Phi = 0. \quad (1.13)$$

First, we calculate the terms:

$$-i\varepsilon B_0\Phi =$$
$$= \begin{vmatrix} 0 & +\varepsilon(d_3 - ih_2) & +i\varepsilon(h_1 - d_3) & +\varepsilon(-d_1 + id_2) \\ 0 & +\varepsilon(-h_3 + id_2) & +i\varepsilon(-d_1 - h_3) & +\varepsilon(h_1 + ih_2) \\ 0 & -\varepsilon(g_3 - if_2) & -i\varepsilon(f_1 - g_3) & -\varepsilon(-g_1 + ig_2) \\ 0 & -\varepsilon(-f_3 + ig_2) & -i\varepsilon(-g_1 - f_3) & -\varepsilon(f_1 + if_2) \end{vmatrix},$$

$$\frac{d}{dr}B_1\Phi = \begin{vmatrix} +h_2' + id_3' & 0 & h_0' - h_3' & h_2' + id_0' \\ -d_2' - ih_3' & 0 & -d_0' - d_3' & d_2' - ih_0' \\ -f_2' - ig_3' & 0 & -f_0' - f_3' & f_2' - ig_0' \\ g_2' + if_3' & 0 & g_0' - g_3' & g_2' + if_0' \end{vmatrix},$$

$$B_2D_2\Phi =$$
$$= \frac{i}{2r}\begin{vmatrix} \begin{matrix} -(2\mu-1)h_1 + \\ +(2\mu+1)d_3 - \\ -2ih_2 \end{matrix} & \begin{matrix} -(2\mu-1)\times \\ \times(h_0 - h_3) \end{matrix} & 0 & \begin{matrix} -(2\mu-1)h_1 + \\ +(2\mu+1)d_0 - \\ -2ih_2 \end{matrix} \\ \begin{matrix} +(2\mu+1)d_1 + \\ +(2\mu-1)h_3 + \\ +2id_2 \end{matrix} & \begin{matrix} +(2\mu+1)\times \\ \times(d_0 + d_3) \end{matrix} & 0 & \begin{matrix} -(2\mu+1)d_1 + \\ +(2\mu-1)h_0 - \\ -2id_2 \end{matrix} \\ \begin{matrix} (2\mu-1)f_1 - \\ -(2\mu+1)g_3 + \\ +2if_2 \end{matrix} & \begin{matrix} (2\mu-1)\times \\ \times(f_0 + f_3) \end{matrix} & 0 & \begin{matrix} -(2\mu-1)f_1 - \\ -(2\mu+1)g_0 - \\ -2if_2 \end{matrix} \\ \begin{matrix} -(2\mu+1)g_1 - \\ -(2\mu-1)f_3 - \\ -2ig_2 \end{matrix} & \begin{matrix} -(2\mu+1)\times \\ \times(g_0 - g_3) \end{matrix} & 0 & \begin{matrix} -(2\mu+1)g_1 - \\ -(2\mu-1)f_0 - \\ -2ig_2 \end{matrix} \end{vmatrix},$$

$$ikB_3\Phi = \qquad (1.14)$$
$$= \begin{vmatrix} k(d_1 - id_2) & k(d_0 - ih_2) & ik(-d_0 + h_1) & 0 \\ k(-h_1 - ih_2) & k(-h_0 - id_2) & ik(-h_0 + d_1) & 0 \\ -k(g_1 - ig_2) & -k(g_0 + if_2) & -ik(-g_0 - f_1) & 0 \\ -k(-f_1 - if_2) & -k(-f_0 + ig_2) & -ik(-f_0 - g_1) & 0 \end{vmatrix}.$$

For massive term we get

$$+iM\left\{s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]} +\right. \qquad (1.15)$$
$$\left.+ s_{23} \otimes \mu^{[23]} + s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]}\right\}\Phi =$$

$$= 2iM\begin{vmatrix} -i(f_3 + g_1) - g_2 & -f_2 - i(g_0 + g_3) & f_1 - g_0 - g_3 & -if_0 + ig_1 + g_2 \\ -if_1 + f_2 + ig_3 & -if_0 + if_3 + g_2 & f_0 - f_3 - g_1 & -if_1 + f_2 + ig_0 \\ -i(d_1 - id_2 + h_3) & -id_0 + id_3 + h_2 & -d_0 + d_3 - h_1 & -i(d_1 - id_2 + h_0) \\ i(d_3 - h_1) + h_2 & -d_2 - i(h_0 + h_3) & d_1 + h_0 + h_3 & i(d_0 + h_1 + ih_2) \end{vmatrix}.$$

Collect all summands in the equation together:

$$\begin{vmatrix} 0 & +\varepsilon(d_3 - ih_2) & +i\varepsilon(h_1 - d_3) & +\varepsilon(-d_1 + id_2) \\ 0 & +\varepsilon(-h_3 + id_2) & +i\varepsilon(-d_1 - h_3) & +\varepsilon(h_1 + ih_2) \\ 0 & -\varepsilon(g_3 - if_2) & -i\varepsilon(f_1 - g_3) & -\varepsilon(-g_1 + ig_2) \\ 0 & -\varepsilon(-f_3 + ig_2) & -i\varepsilon(-g_1 - f_3) & -\varepsilon(f_1 + if_2) \end{vmatrix} +$$

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

$$
+\begin{vmatrix}
+h'_2+id'_3 & 0 & +h'_0-h'_3 & +h'_2+id'_0 \\
-d'_2-ih'_3 & 0 & -d'_0-d'_3 & +d'_2-ih'_0 \\
-f'_2-ig'_3 & 0 & -f'_0-f'_3 & +f'_2-ig'_0 \\
+g'_2+if'_3 & 0 & +g'_0-g'_3 & +g'_2+if'_0
\end{vmatrix}+
$$

$$
+\frac{i}{2r}\begin{vmatrix}
-(2\mu-1)h_1+(2\mu+1)d_3-2ih_2 & -(2\mu-1)(h_0-h_3) & 0 & -(2\mu-1)h_1+(2\mu+1)d_0-2ih_2 \\
+(2\mu+1)d_1+(2\mu-1)h_3+2id_2 & +(2\mu+1)(d_0+d_3) & 0 & -(2\mu+1)d_1+(2\mu-1)h_0-2id_2 \\
+(2\mu-1)f_1-(2\mu+1)g_3+2if_2 & +(2\mu-1)(f_0+f_3) & 0 & -(2\mu-1)f_1-(2\mu+1)g_0-2if_2 \\
-(2\mu+1)g_1-(2\mu-1)f_3-2ig_2 & -(2\mu+1)(g_0-g_3) & 0 & -(2\mu+1)g_1-(2\mu-1)f_0-2ig_2
\end{vmatrix}+
$$

$$
+\begin{vmatrix}
+k(d_1-id_2) & +k(d_0-ih_2) & +ik(-d_0+h_1) & 0 \\
+k(-h_1-ih_2) & +k(-h_0-id_2) & +ik(-h_0+d_1) & 0 \\
-k(g_1-ig_2) & -k(g_0+if_2) & -ik(-g_0-f_1) & 0 \\
-k(-f_1-if_2) & -k(-f_0+ig_2) & -ik(-f_0-g_1) & 0
\end{vmatrix}+
$$

$$
+2M\begin{vmatrix}
+f_3+g_1-ig_2 & -if_2+g_0+g_3 & i(f_1-g_0-g_3) & +f_0-g_1+ig_2 \\
-g_3+f_1+if_2 & +ig_2+f_0-f_3 & i(f_0-f_3-g_1) & -g_0+f_1+if_2 \\
+h_3+d_1-id_2 & +ih_2+d_0-d_3 & i(-d_0+d_3-h_1) & +h_0+d_1-id_2 \\
-d_3+h_1+ih_2 & -id_2+h_0+h_3 & i(d_1+h_0+h_3) & -d_0-h_1-ih_2
\end{vmatrix}=0.
$$

Further we will follow only the massless case.

2 Massless field

We have the system

$$
\begin{gathered}
h'_2+id'_3+\frac{i}{2r}\big[-(2m-1)h_1+(2m+1)d_3-2ih_2\big]+ \\
+k(d_1-id_2)=0, \\
-d'_2-ih'_3+\frac{i}{2r}\big[+(2m+1)d_1+(2m-1)h_3+2id_2\big]+ \\
+k(-h_1-ih_2)=0, \\
-f'_2-ig'_3+\frac{i}{2r}\big[+(2m-1)f_1-(2m+1)g_3+2if_2\big]- \\
-k(g_1-ig_2)=0, \\
g'_2+if'_3+\frac{i}{2r}\big[-(2m+1)g_1-(2m-1)f_3-2ig_2\big]- \\
-k(-f_1-if_2)=0;
\end{gathered} \tag{2.1}
$$

$$
\begin{gathered}
+\varepsilon(d_3-ih_2)+\frac{i}{2r}\big[-(2m-1)(h_0-h_3)\big]+ \\
+k(d_0-ih_2)=0, \\
+\varepsilon(-h_3+id_2)+\frac{i}{2r}\big[(2m+1)(d_0+d_3)\big]+ \\
+k(-h_0-id_2)=0, \\
-\varepsilon(g_3-if_2)+\frac{i}{2r}\big[(2m-1)(f_0+f_3)\big]-k(g_0+if_2)=0, \\
-\varepsilon(-f_3+ig_2)-\frac{i}{2r}\big[(2\mu+1)(g_0-g_3)\big]- \\
-k(-f_0+ig_2)=0;
\end{gathered} \tag{2.2}
$$

$$
\begin{gathered}
+i\varepsilon(h_1-d_3)+h'_0-h'_3+ik(-d_0+h_1)=0, \\
+i\varepsilon(-d_1-h_3)-d'_0-d'_3+ik(-h_0+d_1)=0, \\
-i\varepsilon(f_1-g_3)-f'_0-f'_3-ik(-g_0-f_1)=0, \\
-i\varepsilon(-g_1-f_3)+g'_0-g'_3-ik(-f_0-g_1)=0;
\end{gathered} \tag{2.3}
$$

$$
\begin{gathered}
+\varepsilon(-d_1+id_2)+h'_2+id'_0+ \\
+\frac{i}{2r}\big[-(2m-1)h_1+(2m+1)d_0-2ih_2\big]=0, \\
+\varepsilon(h_1+ih_2)+d'_2-ih'_0+ \\
+\frac{i}{2r}\big[-(2m+1)d_1+(2m-1)h_0-2id_2\big]=0, \\
-\varepsilon(-g_1+ig_2)+f'_2-ig'_0+ \\
+\frac{i}{2r}\big[-(2m-1)f_1-(2m+1)g_0-2if_2\big]=0, \\
-\varepsilon(f_1+if_2)+g'_2+if'_0+ \\
+\frac{i}{2r}\big[-(2m+1)g_1-(2m-1)f_0-2ig_2\big]=0.
\end{gathered} \tag{2.4}
$$

Let us verify correctness of the system by substituting the gauge functions into it. The gauge solutions with cylindrical symmetry are determined by the formulas

$$
\begin{gathered}
\bar{\Psi}_n=D_n\Phi,\ \ D_n=e^{\alpha}_{(n)}\partial_\alpha+\frac{1}{2}\sigma^{kl}\gamma_{kln}, \\
\Phi=e^{-i\varepsilon x^0}e^{im\phi}e^{ikz}\begin{vmatrix}K_1(r)\\K_2(r)\\K_3(r)\\K_4(r)\end{vmatrix}, \\
D_0=\partial_0,\ \ D_1=\frac{\partial}{\partial r}, \\
D_2=\frac{1}{r}(\partial_\phi+\sigma^{12}),\ \ D_3=\partial_z;
\end{gathered} \tag{2.5}
$$

so they are given by the matrix (exponential multipliers are omitted)

$$
\bar{\Psi}_n=(\bar{\Psi}_{An})=\begin{vmatrix}
-i\varepsilon K_1 & K'_1 & \frac{i}{r}\left(m-\frac{1}{2}\right)K_1 & ikK_1 \\
-i\varepsilon K_2 & K'_2 & \frac{i}{r}\left(m+\frac{1}{2}\right)K_2 & ikK_2 \\
-i\varepsilon K_3 & K'_3 & \frac{i}{r}\left(m-\frac{1}{2}\right)K_3 & ikK_3 \\
-i\varepsilon K_4 & K'_4 & \frac{i}{r}\left(m+\frac{1}{2}\right)K_4 & ikK_4
\end{vmatrix}=
$$

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

$$=\begin{vmatrix} \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \\ \bar{g}_0 & \bar{g}_1 & \bar{g}_2 & \bar{g}_3 \\ \bar{h}_0 & \bar{h}_1 & \bar{h}_2 & \bar{h}_3 \\ \bar{d}_0 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 \end{vmatrix}. \qquad (2.6)$$

It is readily verified that all 16 equations are satisfied by these functions. It should be especially noted that any explicit form of separated gauge functions K_1, K_2, K_3, K_4 does not matter.

To proceed with the system, let us transform it to the new variables

$$f_1 + if_2 = F_1, \;\; f_1 - if_2 = F_2,$$
$$f_1 = \frac{1}{2}(F_1 + F_2), \;\; f_2 = \frac{1}{2i}(F_1 - F_2);$$
$$g_1 + ig_2 = G_1, \;\; g_1 - ig_2 = G_2,$$
$$g_1 = \frac{1}{2}(G_1 + G_2), \;\; g_2 = \frac{1}{2i}(G_1 - G_2);$$
$$h_1 + ih_2 = H_1, \;\; h_1 - ih_2 = H_2,$$
$$h_1 = \frac{1}{2}(H_1 + H_2), \;\; h_2 = \frac{1}{2i}(H_1 - H_2);$$
$$d_1 + id_2 = D_1, \;\; d_1 - id_2 = D_2,$$
$$d_1 = \frac{1}{2}(D_1 + D_2), \;\; d_2 = \frac{1}{2i}(D_1 - D_2);$$
$$f_0 + f_3 = F_0, \;\; f_0 - f_3 = F_3,$$
$$f_0 = \frac{1}{2}(F_0 + F_3), \;\; f_3 = \frac{1}{2}(F_0 - F_3);$$
$$g_0 + g_3 = G_0, \;\; g_0 - g_3 = G_3,$$
$$g_0 = \frac{1}{2}(G_0 + G_3), \;\; g_3 = \frac{1}{2}(G_0 - G_3);$$
$$h_0 + h_3 = H_0, \;\; h_0 - h_3 = H_3,$$
$$h_0 = \frac{1}{2}(H_0 + H_3), \;\; h_3 = \frac{1}{2}(H_0 - H_3);$$
$$d_0 + d_3 = D_0, \;\; d_0 - d_3 = D_3,$$
$$d_0 = \frac{1}{2}(D_0 + D_3), \;\; d_3 = \frac{1}{2}(D_0 - D_3); \qquad (2.7)$$

in this way we arrive at new equations, which may be divided into two unlinked groups, each of 8 equations. It is convenient to apply special notations for two operators:

$$a = \frac{d}{dr} + \frac{1+2m}{2r}, \;\; b = \frac{d}{dr} + \frac{1-2m}{2r}. \qquad (2.8)$$

First, let us consider the first subsystem:

$$iaF_1 + iaG_3 - i\left(b + \frac{1}{r}\right)F_2 = (\varepsilon + k)G_2,$$
$$ibG_2 + ibF_0 - i\left(a + \frac{1}{r}\right)G_1 = (\varepsilon - k)F_1,$$
$$iaG_0 = (\varepsilon - k)G_2, \quad ibF_3 = (\varepsilon + k)F_1,$$
$$i\left(a - \frac{1}{r}\right)F_0 = F_2(\varepsilon - k), \;\; i\left(b - \frac{1}{r}\right)G_3 = G_1(\varepsilon + k),$$
$$ibF_0 = F_1(\varepsilon - k) - G_0(\varepsilon + k) + G_3(\varepsilon - k),$$
$$iaG_3 = G_2(\varepsilon + k) - F_3(\varepsilon - k) + F_0(\varepsilon + k). \qquad (2.9)$$

With the help of equations 7 and 8 we can exclude the variables F_0, G_3 in equations 1 and 2:

$$iaF_1 - F_3(\varepsilon - k) + F_0(\varepsilon + k) - i(b + 1/r)F_2 = 0,$$
$$ibG_2 - G_0(\varepsilon + k) + G_3(\varepsilon - k) - i(a + 1/r)G_1 = 0.$$

Further, with the help of equations 3, 4, 5, 6 we exclude the variables F_1, F_2 and G_1, G_2 :

$$-\frac{1}{(\varepsilon + k)}abF_3 - F_3(\varepsilon - k) + F_0(\varepsilon + k) +$$
$$+\frac{1}{(\varepsilon - k)}\left(b + \frac{1}{r}\right)\left(a - \frac{1}{r}\right)F_0 = 0,$$
$$-b\frac{1}{(\varepsilon - k)}aG_0 - G_0(\varepsilon + k) + G_3(\varepsilon - k) +$$
$$+\left(a + \frac{1}{r}\right)\frac{1}{(\varepsilon + k)}\left(b - \frac{1}{r}\right)G_3 = 0,$$

or

$$\left[ab + \varepsilon^2 - k^2\right]\frac{F_3}{\varepsilon + k} =$$
$$= \left[\left(b + \frac{1}{r}\right)\left(a - \frac{1}{r}\right) + \varepsilon^2 - k^2\right]\frac{F_0}{\varepsilon - k},$$
$$\left[ba + \varepsilon^2 - k^2\right]\frac{G_0}{\varepsilon - k} =$$
$$= \left[\left(a + \frac{1}{r}\right)\left(b - \frac{1}{r}\right) + \varepsilon^2 - k^2\right]\frac{G_3}{\varepsilon + k}.$$

Taking in mind the following identities

$$ab = \left(b + \frac{1}{r}\right)\left(a - \frac{1}{r}\right) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m - 1/2)^2}{r^2} = \Delta_-,$$
$$ba = \left(a + \frac{1}{r}\right)\left(b - \frac{1}{r}\right) =$$
$$= \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + 1/2)^2}{r^2} = \Delta_+, \qquad (2.10)$$

we get two 2-order equations:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m - 1/2)^2}{r^2}\right]\left(\frac{F_0}{\varepsilon - k} - \frac{F_3}{\varepsilon + k}\right) = 0,$$
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + 1/2)^2}{r^2}\right]\left(\frac{G_0}{\varepsilon - k} - \frac{G_3}{\varepsilon + k}\right) = 0. \qquad (2.11)$$

We will impose constraints of two types:

$$A, \;\; \frac{1}{\varepsilon + k}F_3(r) = +\frac{1}{\varepsilon - k}F_0(r) = f(r),$$
$$\frac{1}{\varepsilon + k}G_3(r) = +\frac{1}{\varepsilon - k}G_0(r) = g(r); \qquad (2.12)$$
$$B, \;\; \frac{1}{\varepsilon + k}F_3(r) = -\frac{1}{\varepsilon - k}F_0(r) = f(r),$$
$$\frac{1}{\varepsilon + k}G_3(r) = -\frac{1}{\varepsilon - k}G_0(r) = g(r). \qquad (2.13)$$

In the case A, eqs. (2.11) are satisfied identically, and the functions $f(r), g(r)$ may be arbitrary. Below we will see that the case A corresponds to the gauge solutions.

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

In the case B, the functions $f(r), g(r)$ obey the following equations

$$F_0, F_3, \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \varepsilon^2 - k^2 - \frac{(m-1/2)^2}{r^2}\right] f(r) = 0;$$

$$G_0, G_3,$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \varepsilon^2 - k^2 - \frac{(m+1/2)^2}{r^2}\right] g(r) = 0; \quad (2.14)$$

and all remaining functions can be found by means of the formulas

$$F_1 = \frac{ib}{(\varepsilon + k)} F_3 = ibf,$$

$$F_2 = \frac{i(a - 1/r)}{\varepsilon - k} F_0 = i(a - 1/r) f,$$

$$G_1 = \frac{i(b-1/r)}{\varepsilon + k} G_3 = i(b - 1/r)\, g,$$

$$G_2 = \frac{ia}{\varepsilon - k} G_0 = ia\, g. \quad (2.15)$$

Let us prove consistency of restrictions B with the complete system of equations. To this end, in all equations we should take into account the constraints

$$B, \ \frac{1}{\varepsilon + k} F_3(r) = -\frac{1}{\varepsilon - k} F_0(r),$$

$$\frac{1}{\varepsilon + k} G_3(r) = -\frac{1}{\varepsilon - k} G_0(r).$$

In this way, eqs. 3, 4, 5, 6 from (2.9) lead to

$$F_1 = -\frac{ib}{\varepsilon - k} F_0, \ F_2 = \frac{i(a-1/r)}{\varepsilon - k} F_0,$$

$$G_1 = -\frac{i(b-1/r)}{(\varepsilon - k)} G_0, \ G_2 = \frac{ia}{(\varepsilon - k)} G_0.$$

Let us follow the remaining four equations from (2.9):

$$iaF_1 + iaG_3 - i\left(b + \frac{1}{r}\right) F_2 = (\varepsilon + k) G_2 \Rightarrow$$

$$-ia \frac{ib}{\varepsilon - k} F_0 - ia \frac{\varepsilon + k}{\varepsilon - k} G_0 -$$

$$-i\left(b + \frac{1}{r}\right) \frac{i(a-1/r)}{\varepsilon - k} F_0 = \frac{\varepsilon + k}{(\varepsilon - k)} iaG_0 \Rightarrow$$

$$[ab + (b + 1/r)(a - 1/r)] F_0 = 2i(\varepsilon + k) aG_0 \Rightarrow$$

$$abF_0 = i(\varepsilon + k) aG_0,$$

next

$$bF_0 = i(\varepsilon + k) G_0;$$

$$ibG_2 + ibF_0 - i\left(a + \frac{1}{r}\right) G_1 = +(\varepsilon - k) F_1 \Rightarrow$$

$$ib \frac{ia}{(\varepsilon - k)} G_0 + ibF_0 + i\left(a + \frac{1}{r}\right) \frac{i(b-1/r)}{(\varepsilon - k)} G_0 =$$

$$= -(\varepsilon - k) \frac{ib}{\varepsilon - k} F_0,$$

$$[ba + (a + 1/r)(b - 1/r)] G_0 = 2i(\varepsilon - k) bF_0 \Rightarrow$$

$$baG_0 = i(\varepsilon - k) bF_0,$$

next

$$aG_0 = i(\varepsilon - k) F_0;$$

$$ibF_0 = F_1(\varepsilon - k) - G_0(\varepsilon + k) + G_3(\varepsilon - k) \Rightarrow$$

$$ibF_0 = -ibF_0 - (\varepsilon + k) G_0 - (\varepsilon + k) G_0,$$

that is

$$ibF_0 = -(\varepsilon + k) G_0 \ ;$$

next

$$iaG_3 = G_2(\varepsilon + k) - F_3(\varepsilon - k) + F_0(\varepsilon + k) \Rightarrow$$

$$-ia \frac{\varepsilon + k}{\varepsilon - k} G_0 =$$

$$= (\varepsilon + k) \frac{ia}{(\varepsilon - k)} G_0 + (\varepsilon + k) F_0 + F_0(\varepsilon + k),$$

that is

$$iaG_0 = -(\varepsilon - k) F_0.$$

Thus, restrictions B are consistent with the system (2.9), only when the following four equations hold

$$abF_0 = i(\varepsilon + k) aG_0, \ baG_0 = i(\varepsilon - k) bF_0,$$

$$bF_0 = i(\varepsilon + k) G_0, \ aG_0 = i(\varepsilon - k) F_0; \quad (2.16)$$

we can see that only two equations are independent

$$bF_0 = i(\varepsilon + k) G_0, \ aG_0 = i(\varepsilon - k) F_0. \quad (2.17)$$

In the formulas $F_0 = -(\varepsilon - k) f, G_0 = -(\varepsilon - k) g$, the multipliers before *f* and *g* may be hidden in new notations:

$$F_0 = -(\varepsilon - k) f = F, \ G_0 = -(\varepsilon - k) g = G; \quad (2.18)$$

correspondingly, eqs. (2.17) are written as follows

$$bF = i(\varepsilon + k) G, \ aG = i(\varepsilon - k) F. \quad (2.19)$$

This 1-st order equations assume the above equations (2.14):

$$bF = i(\varepsilon + k) G, \quad aG = i(\varepsilon - k) F, \Rightarrow$$

$$abF = i(\varepsilon + k) aG = -(\varepsilon + k)(\varepsilon - k) F,$$

$$baG = i(\varepsilon - k) bF = -(\varepsilon - k)(\varepsilon + k) G. \quad (2.20)$$

Existence of the constrains (2.19) means that the system of 8 equations with restriction B describes only one solution, because the constrains (2.19) permit us to fix the relative coefficient between the variables $F(r)$ and $G(r)$.

To follow the consequences from the other 8 equations, let us write down both subsystems and compare them:

the first is

$$iaF_1 + iaG_3 - i(b + 1/r) F_2 = (\varepsilon + k) G_2,$$

$$ibG_2 + ibF_0 - i(a + 1/r) G_1 = (\varepsilon - k) F_1,$$

$$iaG_0 = (\varepsilon - k) G_2, \ ibF_3 = (\varepsilon + k) F_1,$$

$$i(a - 1/r) F_0 = F_2(\varepsilon - k), \ i(b - 1/r) G_3 = G_1(\varepsilon + k),$$

$$ibF_0 = F_1(\varepsilon - k) - G_0(\varepsilon + k) + G_3(\varepsilon - k),$$

$$iaG_3 = G_2(\varepsilon + k) - F_3(\varepsilon - k) + F_0(\varepsilon + k); \quad (2.21)$$

the second is

$$iaH_1 - iaD_0 - i(b + 1/r) H_2 = -(\varepsilon - k) D_2,$$

$$ibD_2 - ibH_3 - i(a + 1/r) D_1 = -(\varepsilon + k) H_1,$$

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

$$-iaD_3 = -(\varepsilon+k)D_2,\ -ibH_0 = -(\varepsilon-k)H_1,$$
$$-i(a-1/r)H_3 = -(\varepsilon+k)H_2,$$
$$-i(b-1/r)D_0 = -(\varepsilon-k)D_1,$$
$$-ibH_3 = -H_1(\varepsilon+k) - D_3(\varepsilon-k) + D_0(\varepsilon+k),$$
$$-iaD_0 = -D_2(\varepsilon-k) - H_0(\varepsilon+k) + H_3(\varepsilon-k). \quad (2.22)$$

We can immediately notice that eqs. (2.22) follow from (2.21) at the changes:

$$F_1 \Rightarrow H_1,\ F_2 \Rightarrow H_2,\ F_0 \Rightarrow -H_3,\ F_3 \Rightarrow -H_{03},$$
$$(\varepsilon+k) \Rightarrow -(\varepsilon-k),$$
$$G_1 \Rightarrow D_1,\ G_2 \Rightarrow D_2,\ G_0 \Rightarrow -D_3,\ G_3 \Rightarrow -D_0,$$
$$(\varepsilon-k) \Rightarrow +(\varepsilon+k); \quad (2.23)$$

further we derive the rules

$$F \Rightarrow \frac{-H_3}{-(\varepsilon+k)} + \frac{H_0}{-(\varepsilon-k)} = -H,$$
$$G \Rightarrow \frac{-D_3}{-(\varepsilon+k)} + \frac{D_0}{-(\varepsilon-k)} = -D. \quad (2.24)$$

Therefore, the final equations from the first subsystem

$$iaG = -(\varepsilon-k)F,\ (\Delta_- + \varepsilon^2 - k^2)F = 0,$$
$$ibF = -(\varepsilon+k)G,\ (\Delta_+ + \varepsilon^2 - k^2)G = 0 \quad (2.25)$$

transform to those for the second subsystem as shown below

$$iaD = (\varepsilon+k)H,\ (\Delta_- + \varepsilon^2 - k^2)H = 0,$$
$$ibH = (\varepsilon-k)D,\ (\Delta_+ + \varepsilon^2 - k^2)H = 0. \quad (2.26)$$

3 Gauge solutions

Let us collect together the results of solving the system of 16 equations for the case A:

$$F_0 = (\varepsilon-k)f,\ F_1 = ibf,$$
$$F_2 = i(a-1/r)f,\ F_3 = (\varepsilon+k)f,$$
$$G_0 = (\varepsilon-k)g,\ G_1 = i(b-1/r)g,$$
$$G_2 = iag,\ G_3 = (\varepsilon+k)g,$$
$$H_0 = (\varepsilon-k)h,\ H_1 = ibh, \quad (3.1)$$
$$H_2 = i(a-1/r)h,\ H_3 = (\varepsilon+k)h,$$
$$D_0 = (\varepsilon-k)d,\ D_1 = i(b-1/r)d,$$
$$D_2 = iad,\ D_3 = (\varepsilon+k)d.$$

In fact, here we have four independent solutions, determined by four arbitrary functions

$$(f,0,0,0),\ (0,g,0,0),\ (0,0,h,0),\ (0,0,0,d).$$

In turn, let us transform the above gauge solutions (2.11) to similar variables, so we obtain

$$\bar{F}_0 = -i(\varepsilon-k)K_1,\ \bar{F}_1 = \left(\frac{d}{dr} - \frac{m-1/2}{r}\right)K_1,$$
$$\bar{F}_2 = \left(\frac{d}{dr} + \frac{m-1/2}{r}\right)K_1,\ \bar{F}_2 = -i(\varepsilon+k)K_1$$
$$\bar{G}_0 = -i(\varepsilon-k)K_1,\ \bar{G}_1 = \left(\frac{d}{dr} - \frac{m+1/2}{r}\right)K_2,$$
$$\bar{G}_2 = \left(\frac{d}{dr} + \frac{m+1/2}{r}\right)K_2,\ \bar{G}_3 = -i(\varepsilon+k)K_2$$
$$\bar{H}_0 = -i(\varepsilon-k)K_1,\ \bar{H}_1 = \left(\frac{d}{dr} - \frac{m-1/2}{r}\right)K_3,$$
$$\bar{H}_2 = \left(\frac{d}{dr} + \frac{m-1/2}{r}\right)K_3,\ \bar{H}_2 = -i(\varepsilon+k)K_3$$
$$\bar{D}_0 = -i(\varepsilon-k)K_4,\ \bar{D}_1 = \left(\frac{d}{dr} - \frac{m+1/2}{r}\right)K_4,$$
$$\bar{D}_2 = \left(\frac{d}{dr} + \frac{m+1/2}{r}\right)K_4,\ \bar{D}_3 = -i(\varepsilon+k)K_4. \quad (3.2)$$

We can readily see that the matrix of gauge solutions coincides with the matrix (3.1) of solutions for the case A (up to the multiplier *i*).

4 Solving the second order equations, taking into account the constraints

Let us find solutions of two equations

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \varepsilon^2 - k^2 - \frac{(m+1/2)^2}{r^2}\right]G(r) = 0,$$
$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \varepsilon^2 - k^2 - \frac{(m-1/2)^2}{r^2}\right]F(r) = 0. \quad (4.1)$$

In the variable $x = \sqrt{\varepsilon^2 - k^2}\, r$, they have the form of Bessel equations:

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{p^2}{x^2}\right]G(x) = 0,$$
$$p = m + \frac{1}{2},\ -p = -m - \frac{1}{2};$$
$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{s^2}{x^2}\right]F(x) = 0, \quad (4.2)$$
$$s = m - \frac{1}{2} = p - 1,\ -s = -m + \frac{1}{2} = -p + 1.$$

Their independent solutions are

$$G_{(1)}(x) \sim J_p(x),\ G_{(2)}(x) \sim J_{-p}(x);$$
$$F_{(1)}(x) \sim J_s(x) = J_{p-1}(x), \quad (4.3)$$
$$F_{(2)}(x) \sim J_{-s}(x) = J_{-p+1}(x).$$

Below we will consider only solutions which are regular in $x = 0$. First, we assume positive values of *m*. Let us transform the constraints

$$aG = i(\varepsilon-k)F,\ \left(\frac{d}{dr} + \frac{m+1/2}{r}\right)G = i(\varepsilon-k)F;$$
$$bF = i(\varepsilon+k)G,\ \left(\frac{d}{dr} - \frac{m-1/2}{r}\right)F = i(\varepsilon+k)G$$

to Bessel form. The first constraint gives

$$\sqrt{\varepsilon+k}\left(\frac{d}{dx} + \frac{m+1/2}{x}\right)G = i\sqrt{\varepsilon-k}F,$$
$$G_{(1)} = \beta J_p,\ F_{(1)} = \alpha J_{p-1},$$

that is

$$\sqrt{\varepsilon+k}\left(\frac{d}{dx} + \frac{p}{x}\right)\beta J_p = i\sqrt{\varepsilon-k}\,\alpha J_{p-1}. \quad (4.4)$$

The second constraint gives

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

$$\sqrt{\varepsilon-k}\left(\frac{d}{dx}-\frac{m-1/2}{x}\right)F=i\sqrt{\varepsilon+k}G,$$

$$F_{(1)}=\alpha J_{p-1},\ G_{(1)}=\beta J_{p},$$

so that

$$\sqrt{\varepsilon-k}\ (\frac{d}{dx}-\frac{p-1}{x})\ \alpha J_{p-1}=i\sqrt{\varepsilon+k}\ \beta J_{p}. \quad (4.5)$$

Further, taking in mind the known properties of the Bessel functions

$$\left(\frac{d}{dx}+\frac{p}{x}\right)J_{p}=J_{p-1},\ \left(\frac{d}{dx}-\frac{p-1}{x}\right)J_{p-1}=-J_{p},$$

we derive two algebraic relations which determine coefficients α and β:

$$\sqrt{\varepsilon+k}\beta=i\sqrt{\varepsilon-k}\alpha,\ \sqrt{\varepsilon-k}\alpha=-i\sqrt{\varepsilon+k}\beta\Rightarrow$$

$$\alpha=\sqrt{\varepsilon+k},\ \beta=i\sqrt{\varepsilon-k}.$$

Therefore, the needed solution is

$$F_{(1)}(x)=\sqrt{\varepsilon+k}J_{p-1}(x),$$
$$G_{(1)}(x)=i\sqrt{\varepsilon-k}J_{p}(x),\ m>0. \quad (4.6)$$

Similar solution at negative m has the form

$$F_{(2)}(x)=\sqrt{\varepsilon+k}J_{-p+1}(x),$$
$$G_{(2)}(x)=i\sqrt{\varepsilon-k}J_{-p}(x),\ m<0. \quad (4.7)$$

In accordance with the above noted symmetry between two subsystems, we get similar results for functions $H(x)$ and $D(x)$:

$$H_{(1)}(x)=\sqrt{\varepsilon-k}J_{p-1}(x),$$
$$D_{(1)}(x)=i\sqrt{\varepsilon+k}J_{p}(x),\ m>0; \quad (4.8)$$

$$H_{(2)}(x)=\sqrt{\varepsilon-k}J_{-p+1}(x),$$
$$D_{(2)}(x)=i\sqrt{\varepsilon+k}J_{-p}(x),\ m<0. \quad (4.9)$$

Conclusion

The wave equation for the massless spin 3/2 has been studied in cylindrical coordinates of Minkowski space. After separating the variables, the system of 16 equations was derived. There are constructed 6 independent solutions. Four of them coincide with the gauge ones, two solutions do not contain the gauge components and describe physically observable states of the particle.

REFERENCES

1. *Pauli*, *W.* Über relativistische Feldleichungen von Teilchen mit beliebigem Spin im elektromagnetishen Feld / W. Pauli, M. Fierz. – Helvetica Physica Acta. – 1939. – Bd. 12. – S. 297–300.

2. *Fierz*, *M.* On relativistic wave equations for particles of arbitrary spin in an electromagnetic field / M. Fierz, W. Pauli // Proc. Roy. Soc. London. A. – 1939. – Vol. 173. – P. 211–232.

3. *Rarita*, *W.* On a theory of particles with half-integral spin / W. Rarita, J. Schwinger // Phys. Rev. – 1941. – Vol. 60, № 1. – P. 61–64.

4. *Ginzburg*, *V.L.* To the theory of particles of spin 3/2 / V.L. Ginzburg // Journal of Experimental and Theoretical Physics. – 1942. – Vol. 12. – P. 425–442.

5. *Davydov*, *A.S.* Wave equation for a particle with spin 3/2, in absence of external field / A.S. Davydov // Journal of Experimental and Theoretical Physics. – 1943. – Vol. 13. – P. 313–319.

6. *Bhabha*, *H.J.* Relativistic Wave Equations for the Elementary Particles / H.J. Bhabha // Reviews of Modern Physics. – 1945. – Vol. 17, № 2-3. – P. 200–216.

7. *Gelfand*, *I.M.* General relativistically invariant equations and infinite-dimensional representations of the Lorentz group / I.M. Gelfand, A.M. Yaglom // Journal of Experimental and Theoretical Physics. – 1948. – Vol. 18, № 8. – P. 703–733 (in Russian).

8. *Fradkin*, *E.S.* To the theory of particles with higher spins / E.S. Fradkin // Journal of Experimental and Theoretical Physics. – 1950. – Vol. 20, № 1. – P. 27–38 (in Russian).

9. *Fedorov*, *F.I.* Generalized relativistic wave equations / F.I. Fedorov // Proceedings of the Academy of Sciences of the USSR. – 1952. – Vol. 82, № 1. – P. 37–40 (in Russian).

10. *Feinberg*, *V.Ya.* On the theory of interaction of particles with higher spins with electromagnetic and meson fields / V.Ya. Feinberg // Proceedings of the Lebedev Physics Institute of the Academy of Sciences of the USSR. – 1955. – Vol. 6. – P. 269–332 (in Russian).

11. *Petras*, *M.A*. Note to Bhabha's equation for a particle with maximum spin 3/2 / M.A. Petras // Czechoslovak Journal of Physics. – 1955. – Vol. 5, № 3. – P. 418–419.

12. *Bogush*, *A.A*. Equation for a 3/2 particle with anomalous magnetic moment / A.A. Bogush // Russian Physics Journal, 1984. – Vol. 1. – P. 23–27 (in Russian).

13. *Pletyukhov*, *V.A.* To the theory of particles of spin 3/2 / V.A. Pletyukhov, V.I. Strazhev // Russian Physics Journal. – 1985. – Vol. 28, № 1. – P. 91–95 (in Russian).

14. *Pletyukhov*, *V.A.* On the relationship between various formulations of particle theory with spin 3/2 / V.A. Pletyukhov, V.I. Strazhev // Proceedings of the Academy of Sciences of the BSSR. Physics and Mathematics series. – 1985. – Vol. 5. – P. 90–95 (in Russian).

15. *Johnson*, *K.* Inconsistency of the local field theory of charged spin 3/2 particles / K. Johnson, E.C.G. Sudarshan // Ann. Phys. N.Y. – 1961. – Vol. 13. – P. 121–145.

16. *Bender*, *C.M.* Peculiarities of a free massless spin 3/2 field theory / C.M. Bender, M. McCoy Barry // Phys. Rev. – 1966. – Vol. 148. – P. 1375–1380.

17. *Hagen*, *C.R.* Search for consistent interactions of the Rarita-Schwinger field / C.R. Hagen, L.P.S. Singh // Phys. Rev. D. – 1982. – Vol. 26. – P. 393–398.

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ

18. *Loide*, *R.K.* Equations for a vector-bispinor / R.K. Loide // J. Phys. A. – 1984. – Vol. 17. – P. 2535–2550.

19. *Red'kov*, *V.M.* Particle fields in the Riemann space and the Lorentz group / V.M. Red'kov. – Minsk, Belorusian science Publ, 2009. – 486 p.

20. *Pletyukhov*, *V.A.* Relativistic wave equations and internal degrees of freedom / V.A. Pletyukhov, V.M. Red'kov, V.I. Strazhev. – Minsk, Belaruskaya Navuka Publ. – 2015. – 328 p. (in Russian).

21. *Elementary particles with internal structure in external fields.* I. *General theory.* II. *Physical problems.* / V.V. Kisel [et al.]. – New York: Nova Science Publishers Inc. – 2018. – 404 p., 402 p.

22. *Fradkin Equation for a Spin* 3/2 *Particle in Presence of External Electromagnetic and Gravitational Fields* / V.V. Kisel [et al.] // Ukr. J. Phys. – 2019. – Vol. 64, № 12. – P. 1112–1117.

23. *Ivashkevich*, *A.V.* Zero mass field with the spin 3/2: solutions of the wave equation and the helicity operator / A.V. Ivashkevich, E.M. Ovsiyuk, V.M. Red'kov // Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series. – 2019. – Vol. 55, № 3. – P. 338–354 (in Russian).

24. *Spherical solutions of the wave equation for a spin* 3/2 *particle* / A.V. Ivashkevich, E.M. Ovsiyuk, V.V. Kisel, V.M. Red'kov // Doklady of the National Academy of Sciences of Belarus. – 2019. – Vol. 63, № 3. – P. 282–290.

25. *Spin* 3/2 *particle*: *Pauli – Fierz theory*, *non-relativistic approximation* / A.V. Ivashkevich, Ya.A. Voynova, E.M. Ovsiyuk, V.V. Kisel, V.M. Red'kov // Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series. – 2020. – Vol. 56, № 3. – P. 335–349.

Поступила в редакцию 11.08.2021.

Информация об авторах

Ивашкевич Алина Валентиновна – аспирант

РЕПОЗИТОРИЙ ГГУ ИМЕНИ Ф. СКОРИНЫ