МАТЕМАТИКА

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О ЧАСТИЧНО СОПРЯЖЁННО-ПРЕРЕСТАНОВОЧНЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

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ON PARTIALLY CONJUGATE-PERMUTABLE SUBGROUPS OF FINITE GROUPS

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Пусть R – подгруппа группы G. Подгруппу H группы G назовём R-сопряжённо-перестановочной, если $HH^r = H^rH$ для любого $r \in R$. В работе изучаются свойства и влияние R-сопряжённо-перестановочных подгрупп (максимальных, силовских, циклических примарных) на строение конечных групп. В качестве R рассматриваются подгруппа Фиттинга $F(G)$, квазинильпотентный радикал $F^*(G)$ и обобщённая подгруппа Фиттинга $\tilde{F}(G)$, введенная П. Шмидом. В частности, было показано, что группа G нильпотентна тогда и только тогда, когда все её максимальные подгруппы $\tilde{F}(G)$-сопряжённо-перестановочны.

Ключевые слова*: конечная группа, нильпотентная группа, сопряжённо-перестановочная подгруппа, R-сопряжённо-перестановочная подгруппа, подгруппа Фиттинга.*

Let R be a subgroup of a group G. We shall call a subgroup H of G the R-conjugate-permutable subgroup if $HH^r = H^rH$ for all $r \in R$. In this work the properties and the influence of R-conjugate-permutable subgroups (maximal, Sylow, cyclic primary) on the structure of finite groups are studied. As R we consider the Fitting subgroup $F(G)$, quasinilpotent radical $F^*(G)$ and the generalized Fitting subgroup $\tilde{F}(G)$ that was introduced by P. Shmid. In particular, it was shown that group G is nilpotent iff all its maximal subgroups are $\tilde{F}(G)$-conjugate-permutable.

Keywords: *finite group, nilpotent group, R-conjugate-permutable subgroup, conjugate-permutable subgroup, the Fitting subgroup.*

Introduction

All groups considered here are finite. Recall [1] that subgroups H and K of a group G are said to permute if $HK = KH$, which is equivalent to that the set HK is a subgroup of G.

The classic area of group theory is the study of subgroups of a group G which permute with every subgroup of a dedicated system of subgroups of G. This trend goes back to O. Ore [2] who introduced the concept of quasinormal (permutable) subgroup in 1939. Recall that subgroup H of a group G is called quasinormal if it permutes with every subgroup of G. Every normal subgroup is quasinormal. It is known that every quasinormal subgroup is subnormal. There are examples showing that the converse is false.

Another important type of subgroups' permutability was proposed by O. Kegel [3] in 1962. A subgroup H of a group G is called S-permutable (S-quasinormal, π-quasinormal) subgroup of G, if H permutes with every Sylow subgroup of G. Note that every S-permutable subgroup is subnormal. The converse need not hold. Currently, the concept of quasinormal and S-permutable subgroups and their generalizations have been studied intensively by many authors (see monograph [4]).

In 1997 T. Foguel [5] noted in the proof that a quasinormal subgroup is subnormal, one only needs to show that it is permutable with all of its conjugates. This led him to the following concept of subgroups' permutability.

***Definition* 0.1** [5]. *A subgroup H of a group G is called the conjugate-permutable subgroup of G, if $HH^x = H^xH$ for all $x \in G$. Denoted by $H <_{C-P} G$.*

Clearly, every quasinormal subgroup is conjugate-permutable. In [5] there is an example showing that the converse is not true. On the other hand, every 2-subnormal subgroup (i. e. subgroup is a normal subgroup of some normal subgroup of the group) is a conjugate-permutable.

Analyzing the proofs of some results of that works (for example see [5]) we have seen that we can replace conjugate-permutability by permutability with a smaller number of conjugates for proving that results. This observation led us to the following definition.

***Definition* 0.2.** *Let R be a subgroup of a group G. We shall call a subgroup H of G the R-conjugate-permutable subgroup of G, if $HH^x = H^x H$ for all $x \in R$.*

The goal of this paper is to study the influence and the properties of R-conjugate-permutable subgroups on the structure of finite groups.

1 Preliminaries

We use standard notation and terminology, which if necessary can be found in [1], [4], [6] and except through E we denote the unit group.

We recall the following well-known definitions and results (see [1], [6], [7]).

The Fitting subgroup $F(G)$ is the maximal normal nilpotent subgroup of a group G.

***Definition* 1.1.** *A subgroup $F^*(G)$ of a group G is defined by*

$$F^*(G)/F(G) = Soc(C_G(F(G))F(G)/F(G)).$$

***Lemma* 1.2.** *Let G be a group. Then*:

(1) $F^*(G)$ *is the maximal normal quasinilpotent subgroup of* G;

(2) $C_G(F^*(G)) \subseteq F(G)$;

(3) $F(G) \subseteq F^*(G)$.

***Definition* 1.3** (see [6] or [8]). *A subgroup $\tilde{F}(G)$ of a group G is defined by*

1) $\Phi(G) \subseteq \tilde{F}(G)$;

2) $\tilde{F}(G)/\Phi(G) = Soc(G/\Phi(G))$.

***Lemma* 1.4.** *Let G be a group. Then:*

(1) $\tilde{F}(G/\Phi(G)) = \tilde{F}(G)/\Phi(G)$;

(2) $C_G(\tilde{F}(G)) \subseteq F(G)$.

***Lemma* 1.5.** *Let G be a solvable group. Then*:

(1) $\Phi(G) \subset F(G)$;

(2) $C_G(F(G)) \subseteq F(G)$;

(3) $\tilde{F}(G) = F^*(G) = F(G)$.

***Lemma* 1.6.** *Let G be a group. Then $F(G) \subseteq F^*(G) \subseteq \tilde{F}(G)$.*

Proof. $F(G) \subseteq F^*(G)$ is well known. Now we will show that $F^*(G) \subseteq \tilde{F}(G)$ for every group G. The idea of the proof of lemma 1.6 belongs to L. Shemetkov (see also [9]).

Let a group G be the minimal order counterexample for lemma 1.6. If $\Phi(G) \neq E$ then for $G/\Phi(G)$ the statement is true. From

$$F^*(G)/\Phi(G) \subseteq F^*(G/\Phi(G))$$

and

$$\tilde{F}(G/\Phi(G)) = \tilde{F}(G)/\Phi(G)$$

we have that $F^*(G) \subseteq \tilde{F}(G)$. It is a contradiction with the choice of G.

Let $\Phi(G) = E$. Now $\tilde{F}(G) = Soc(G)$. By 13.14.X [7] $F^*(G) = E(G)F(G)$. Note $\Phi(E(G)) = E$. Since 13.7.X [7] $E(G)/Z(E(G))$ is the direct product of simple nonabelian groups, $Z(E(G)) = F(E(G))$. From it and theorem 10.6.A [1] we conclude that $E(G) = HZ(E(G))$ where H is the complement to $Z(E(G))$ in $E(G)$. Now H is the direct product of simple nonabelian groups. Since $H \, char \, E(G) \lhd G$, we have $H \lhd G$. From lemma 14.14.A [1] follows $H \subseteq Soc(G)$. Since $Z(E(G)) \subseteq F(G) \subseteq \tilde{F}(G)$ and $H \subseteq Soc(G)$, it follows that $E(G) \subseteq \tilde{F}(G)$. Now

$$F^*(G) = E(G)F(G) \subseteq \tilde{F}(G).$$

It is a contradiction with the choice of G.

***Example* 1.7** [9]. Let $G \simeq A_5$ be the alternating group of degree 5 and $K = F_3$ be a field composed by three elements. We denoted by $A = A_K(G)$ the Frattini KG-module [10]. In view of [10], A is the faithful irreducible KG-module of the dimension 4. By the known Gaschutz theorem, there exists a Frattini extension $A \rightarrowtail R \twoheadrightarrow G$ such that $A \overset{G}{\simeq} \Phi(R)$ and $R/\Phi(R) \simeq G$. From the properties of module A it follows that $\tilde{F}(G) = R$ and $F^*(G) = \Phi(R)$.

2 Properties of R-conjugate-permutable subgroups

First we begin with showing that if H is R-congugate-permutable subgroup then H need not to be $\langle H, R \rangle$-conjugate permutable.

***Example* 2.1.** Let $B = \langle (1,3,6),(2,4) \rangle$ and $x = (1,2)(3,4)(5,6)$, $B^x = \langle (2,4,5),(1,3) \rangle$. One can check that $B^x B = BB^x$ (the author did it with the help of GAP). Let $R = \langle x \rangle$. It is easy to see that $|R| = 2$. Thus B is R-conjugate-permutable. Consider $G = \langle B, R \rangle$. Let $y = (2,4,5)$. Again one can check that $BB^y \neq B^y B$. Thus B is not G-conjugate-permutable.

***Lemma* 2.2 (Properties of R-conjugate-permutable subgroups).** *Let H and R be a subgroups of a group G and N be a normal subgroup of G. Then*:

(1) *If H is R-conjugate-permutable and $HR = RH$ then H is conjugate-permutable in RH, in particular subnormal in RH*;

(2) *If H is R-conjugate-permutable then H^x is R^x-conjugate-permutable for all $x \in G$*;

(3) *If H is R-conjugate-permutable then NH is NR-conjugate-permutable*;

(4) *If* H *is* R*-conjugate-permutable then* HN/N *is* RN/N*-conjugate-permutable*;

(5) *If* H/N *is* RN/N*-conjugate-permutable then* H *is* R*-conjugate-permutable*;

(6) *If* H *is a maximal* R*-conjugate-permutable subgroup then* $R \subseteq N_G(H)$;

(7) *If* H *is* R*-conjugate-permutable and* $r_1,\dots,r_n \in R$ *then* $H^{r_1}\dots H^{r_n}$ *is* R*-conjugate-permutable*;

(8) *If* H *is* R*-conjugate-permutable then* $H^R = H^{r_1}\dots H^{r_n}$ *for some* $r_i \in R$;

(9) *If* H_i *is* R_i*-conjugate-permutable subgroup of a group* G_i *, where* $i=1,2,$ *then* $H_1 \times H_2$ *is* $R_1 \times R_2$*-conjugate-permutable subgroup in the group* $G_1 \times G_2$.

Proof. It is straightforward to check.

Recall that a subgroup H of a group G is the pronormal subgroup if H and H^x are conjugate in $\langle H, H^x \rangle$. Moreover if $x \in \langle H, H^x \rangle$ for all $x \in G$ then H is called abnormal. Note that every abnormal subgroup is pronormal.

***Lemma* 2.3.** *Let* R *be a subgroup of a group* G. *If a pronormal subgroup* H *of* G *is* R*-conjugate-permutable then* $R \subseteq N_G(H)$. *In particular, if* H *is also abnormal then* $R \subseteq H$.

Proof. Let $r \in R$. Then $H^r H = HH^r$. Since H is pronormal in G, H and H^r are conjugate in $H^r H$. Hence there is $y \in HH^r$ such that $HH^r = HH^y$. But then there is $h_1, h_2 \in H$ such that $y = h_1 h_2^y$. Hence $y = h_2 h_1$, ie $H = H^y = H^r$. Thus $R \subseteq N_G(H)$. It is easy to see that if H is abnormal in G then $H = N_G(H)$. Hence, if H is abnormal and R-conjugate-permutable in G then $R \subseteq H$.

3 Applications of R-conjugate-permutable subgroups

The following example shows that $F(G)$-conjugate-permutable subgroup need not to be conjugate-permutable, even subnormal.

***Example* 3.1.** Let $G \simeq S_4$ be the symmetric group of degree 4. Let H be Sylow 2-subgroup of G. Then H is a maximal subgroup of G which is not normal in G. Note that

$$\tilde{F}(G) = F^*(G) = F(G) \subseteq H.$$

Hence H is $F(G)$-conjugate-permutable in G. But H is abnormal maximal subgroup of G, hence H is neither conjugate-permutable nor subnormal subgroup of G. Now consider $K = G \times S_4$. Since $S_4 \subseteq C_K(G)$, H is $F(K)$-conjugate-permutable in K. But $F(K)$ is not subgroup of H. Also H is not self-normalizing in K and not conjugate-permutable in K.

***Theorem* 3.2.** *A group* G *is nilpotent if and only if every maximal subgroup of* G *is* $\tilde{F}(G)$*-conjugate-permutable.*

Proof. Let G be a nilpotent group. Then $\tilde{F}(G) = G$. Since G is nilpotent, every maximal subgroup of G is normal in G, and hence, $\tilde{F}(G)$-conjugate-permutable.

Conversely. Assume the result is false. Let a group G be a counterexample of minimal order. Then all maximal subgroups of G are $\tilde{F}(G)$-conjugate-permutable but G is not nilpotent group.

Suppose that $\Phi(G) \neq E$. Consider the quotient $G/\Phi(G)$. We have $\tilde{F}(G/\Phi(G)) = \tilde{F}(G)/\Phi(G)$. By lemma 2.2 it is easy to see that all maximal subgroups of $G/\Phi(G)$ are $\tilde{F}(G/\Phi(G))$-conjugate-permutable. Since $|G| > |G/\Phi(G)|$, we have $G/\Phi(G)$ is nilpotent. From Theorem 9.3(b) [1, p. 30] it follows that G is nilpotent, a contradiction.

Assume that $\Phi(G) = E$. Then $\tilde{F}(G) = Soc(G)$. Now assume that $\tilde{F}(G)$ is not nilpotent. So there is a subgroup $S \in Syl_p(\tilde{F}(G))$ such that S is not normal in $\tilde{F}(G)$. Let $P \in Syl_p(G)$ and $P \cap \tilde{F}(G) = S$. Note that $S^x = P^x \cap \tilde{F}(G)^x = P \cap \tilde{F}(G) = S$ for every $x \in N_G(P)$. It means that $N_G(P) \subseteq N_G(S)$. Since $N_G(S) \neq G$, we have $N_G(P) \neq G$. Let M be a maximal subgroup of G such that $N_G(S) \subseteq M$. By lemma 6.20 [1, p. 247] M is the abnormal subgroup of G. By Frattini's argument $N_G(S)\tilde{F}(G) = M\tilde{F}(G) = G$. Since M is the maximal and $\tilde{F}(G)$-conjugate-permutable subgroup, M is normal in G by lemmas 2.2 and 2.3, a contradiction.

Therefore we see that $\tilde{F}(G)$ is nilpotent. Then $\tilde{F}(G) = F(G) = Soc(G) = N_1 \times \dots \times N_t$ where N_i runs over all minimal normal subgroups of G. From $\Phi(G) = E$ and nilpotency of $\tilde{F}(G)$ it follows that N_i is an abelian subgroup for all $i = 1,\dots,t$. Then there is a maximal subgroup M_i such that $N_i M_i = G$ for all $i = 1,\dots,t$. Note that $M_i \tilde{F}(G) = G$. Since M_i is $\tilde{F}(G)$-conjugate-permutable, M_i is normal in G for all $i = 1,\dots,t$ by lemmas 2.2 and 2.3. Since N_i is abelian subgroup, we have $N_i \subseteq C_G(N_i)$ and $N_i \cap M_i = E$. Then $M_i \lhd G$ implies $M_i \subseteq C_G(N_i)$ for all $i = 1,\dots,t$. We show that $G = M_i N_i \subseteq C_G(N_i)$ for every $i = 1,\dots,t$. Therefore

$N_i \subseteq Z(G)$ for all $i = 1,...,t$. Then $\tilde{F}(G) \subseteq Z(G)$. Hence $G \subseteq C_G(\tilde{F}(G)) \subseteq F(G)$. Thus G is nilpotent, a contradiction.

***Corollary* 3.3**. *If G is not a nilpotent group then there is an abnormal maximal subgroup M of G such that $\tilde{F}(G) \nsubseteq M$.*

Proof. Assume the contrary. Then if $\tilde{F}(G) \subseteq M$ for all abnormal maximal subgroups M of G then all of them are $\tilde{F}(G)$-conjugate-permutable. It means that all maximal subgroups of G are $\tilde{F}(G)$-conjugate-permutable. Thus G is nilpotent, a contradiction.

***Corollary* 3.4** (Foguel, [5]). *If every maximal subgroup a group G is conjugate-permutable then G is nilpotent.*

From the example 1.7 it is follows that we can not use $F^*(G)$ in place of $\tilde{F}(G)$ in theorem 3.2.

***Theorem* 3.5.** *The following statements for a group G are equivalent:*

(1) *G is nilpotent*;

(2) *Every abnormal subgroup of G is $F^*(G)$-conjugate-permutable subgroup of G*;

(3) *Normalizers of all Sylow subgroups of G are $F^*(G)$-conjugate-permutable subgroups of G*;

(4) *Sylow subgroups of G are $F^*(G)$-conjugate-permutable subgroups of G.*

Proof. Prove that (1) implies (2). Since G is nilpotent, $F^*(G) = G$ and any subgroup of G is subnormal. It means that the subgroup G is the only one abnormal subgroup in G. It is clear that G is the $F^*(G)$-conjugate-permutable. Thus (1) implies (2).

It is well known that normalizers of all Sylow subgroups of G are abnormal in G. Therefore (2) implies (3).

Prove that (3) implies (4). By lemma 2.3 we see that $F^*(G) \subseteq N_G(P)$ for every Sylow subgroup P of G. Hence every Sylow subgroup of G is the $F^*(G)$-conjugate-permutable subgroup of G. Thus (3) implies (4).

Finally we show that (4) implies (1). Assume that (1) is not true and G is a counterexample of least order.

By lemma 2.3 we have that $F^*(G) \subseteq N_G(P)$ for every Sylow subgroup P of G. By Baers's result (see [11]), $F^*(G) \subseteq Z_\infty(G)$ the hypercenter of G. Note that $F(G) = F^*(G) = Z_\infty(G)$.

Assume that $\Phi(G) \neq E$.

Let $H / \Phi(G) = F^*(G / \Phi(G))$.

Show that $H / \Phi(G) = F^*(G) / \Phi(G)$. It is clear that $F^*(G) / \Phi(G) \subseteq H / \Phi(G)$. Suppose that $H / \Phi(G) / F^*(G) / \Phi(G) \simeq H / F^*(G) \neq E$. Note that $H / \Phi(G)$ and $F^*(G) / \Phi(G)$ are quasinilpotent. It follows that $H / F^*(G)$ is quasinilpotent. Now $H / Z_\infty(G) / Z_\infty(H) / Z_\infty(G) \simeq H / Z_\infty(H)$ is quasinilpotent. By theorem 13.6 [7, p. 125] H is the normal quasinilpotent subgroup of G. Hence $H \subseteq F^*(G)$. We have the contradiction with $H / F^*(G) \neq E$. Thus $F^*(G / \Phi(G)) = F^*(G) / \Phi(G)$.

Let $S / \Phi(G)$ be a Sylow subgroup of $G / \Phi(G)$. There is a Sylow subgroup P of G such that $P\Phi(G) / \Phi(G) = S / \Phi(G)$.

From $F^*(G / \Phi(G)) = F^*(G) / \Phi(G)$ it follows that $S / \Phi(G)$ is the $F^*(G / \Phi(G))$-conjugate-permutable subgroup of $G / \Phi(G)$. By minimality of G we have that $G / \Phi(G)$ is nilpotent. Hence, G is nilpotent by theorem 9.3(b) [1, p. 30], a contradiction.

Suppose now that $\Phi(G) = E$. By theorems 8.6, 8.8 [6, p. 96–97] we have $Z_\infty(G) = Z(G)$. Therefore $F^*(G) = Z(G)$. Now we have

$$G = C_G(F^*(G)) \subseteq F(G).$$

Thus G is nilpotent. This is the final contradiction.

***Corollary* 3.6.** *A group G is nilpotent if and only if the normalizers of all Sylow subgroups of G contains $F^*(G)$.*

Proof. If G is nilpotent then for every Sylow subgroup P of G we have $F^*(G) = N_G(P) = G$. Thus the normalizers of all Sylow subgroups of G contains $F^*(G)$. If the normalizers of all Sylow subgroups of G contains $F^*(G)$ then they are $F^*(G)$-conjugate-permutable. Thus G is nilpotent.

***Corollary* 3.7** (Foguel, [5]). *If every Sylow subgroup a group G is conjugate-permutable then G is nilpotent.*

***Theorem* 3.8.** *If all cyclic primary subgroups of a group G are $F^*(G)$-conjugate-permutable then G is nilpotenet.*

Proof. Let $P \in Syl_p(G)$ and $x \in P$. Then the subgroup $\langle x \rangle$ is the $F^*(G)$-conjugate-permutable subgroup. So $\langle x \rangle \lhd\lhd \langle x \rangle F^*(G)$ by (1) of lemma 2.2. Note that $\langle x \rangle \lhd\lhd P$. Since $\langle x \rangle \leq P \cap \langle x \rangle F^*(G)$, by theorem 1.1.7 [4, p. 3] $\langle x \rangle$ is the subnormal subgroup in the product $P(\langle x \rangle F^*(G))$. Since P is generated by its cyclic subnormal in $PF^*(G)$ subgroups, by theorem 7.5 [6, p. 70] we have that $P \lhd\lhd PF^*(G)$. Thus every Sylow subgroup of G is $F^*(G)$-conjugate-permutable. Now theorem 3.8 immediately follows from theorem 3.5.

***Corollary* 3.9** (Foguel, [5]). *If every cyclic primary subgroup a group G is conjugate-permutable then G is nilpotent.*

As follows from example 1.2 [5] the converse of the theorem 3.8 are false.

Remark. In theorems 3.5 and 3.8 we can not use $F(G)$ in place of $F^*(G)$. Let $G \simeq A_5$ be the alternating group of degree 5. Then $F(G) = E$ and every subgroup of G is $F(G)$-conjugate-permutable. But G is not nilpotent.

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