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СВОБОДНЫЕ n-ДИНИЛЬПОТЕНТНЫЕ ДИМОНОИДЫ

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FREE n-DINILPOTENT DIMONOIDS

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Построен свободный n-динильпотентный димоноид и охарактеризована наименьшая n-динильпотентная конгруэнция на свободном димоноиде.

***Ключевые слова**: n-динильпотентный димоноид, свободный n-динильпотентный димоноид, димоноид, полугруппа, конгруэнция.*

We construct a free n-dinilpotent dimonoid and characterize the least n-dinilpotent congruence on a free dimonoid.

***Keywords**: n-dinilpotent dimonoid, free n-dinilpotent dimonoid, dimonoid, semigroup, congruence.*

Introduction and preliminaries

In the author's preceding papers [1]–[4] the problem of the construction of some relatively free dimonoids was solved. The structure of the corresponding algebras was described and some least congruences on a free dimonoid were characterized.

The present paper continues this trend of research by considering the so-called n-dinilpotent dimonoids, that is dimonoids with two n-nilpotent semigroups. It turns out that the class of such dimonoids is a subvariety of the variety of all dimonoids. For the indicated variety a free object is constructed and the least n-dinilpotent congruence on a free dimonoid is characterized.

Let us recall that a nonempty set D equipped with two binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \qquad \text{(D1)}$$

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \qquad \text{(D2)}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \qquad \text{(D3)}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \qquad \text{(D4)}$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z) \qquad \text{(D5)}$$

for all $x, y, z \in D,$ is called a dimonoid.

An element 0 of a dimonoid $(D, \dashv, \vdash)$ will be called zero, if

$$x * 0 = 0 = 0 * x$$

for all $x \in D$ and $* \in \{\dashv, \vdash\}$.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

We call a semigroup S nilpotent, if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n we shall call the nilpotency index of S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\le k$ is said to be k-nilpotent.

A dimonoid $(D, \dashv, \vdash)$ with zero will be called dinilpotent, if $(D, \dashv)$ and $(D, \vdash)$ are nilpotent semigroups.

A dinilpotent dimonoid $(D, \dashv, \vdash)$ will be called n-dinilpotent, if $(D, \dashv)$ and $(D, \vdash)$ are n-nilpotent semigroups. If ρ is a congruence on a dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an n-dinilpotent dimonoid, then we say that ρ is an n-dinilpotent congruence.

Note that operations of any 1-dinilpotent dimonoid coincide and it is a zero semigroup.

***Lemma* 0.1.** *The class of all n-dinilpotent dimonoids is a subvariety of the variety of all dimonoids.*

Proof. Indeed, the class of all n-dinilpotent dimonoids is a subclass of the variety of all dimonoids which is closed under taking of homomorphic images, subdimonoids and Cartesian products, and consequently, it is a variety. □

A dimonoid which is free in the variety of n-dinilpotent dimonoids will be called a free n-dinilpotent dimonoid.

The necessary information about varieties of dimonoids can be found in [1].

J.-L. Loday described a free dimonoid [5]. We constructed the dimonoid isomorphic to the free dimonoid in [6]. Recall this construction.

Let $F[X]$ be the free semigroup in the alphabet X. We denote the length of a word $w \in F[X]$ by l_w. Define operations $\dashv$ and $\vdash$ on

$$F = \{(w, m) \in F[X] \times \mathbb{N} \,|\, l_w \ge m\}$$

by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, m_1),$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, l_{w_1} + m_2)$$

for all $(w_1, m_1), (w_2, m_2) \in F$. Denote the algebra $(F, \dashv, \vdash)$ by $\breve{F}[X]$. By Lemma 3 from [6] $\breve{F}[X]$ is the free dimonoid over X.

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f.

1 Free objects

In this section we construct a free n-dinilpotent dimonoid of an arbitrary rank and consider separately free n-dinilpotent dimonoids of rank 1.

Fix $n \in \mathbb{N}$ and define operations $\dashv$ and $\vdash$ on

$$FD_n = \{(w, m) \in F[X] \times \mathbb{N} \mid m \le l_w, m \le n, l_w - m + 1 \le n\} \cup \{0\}$$

by

$$(w_1, m_1) \dashv (w_2, m_2) = \begin{cases} (w_1 w_2, m_1), \text{ if } l_{w_1 w_2} - m_1 + 1 \le n, \\ 0, \text{ if } l_{w_1 w_2} - m_1 + 1 > n, \end{cases}$$

$$(w_1, m_1) \vdash (w_2, m_2) = \begin{cases} (w_1 w_2, l_{w_1} + m_2), \text{ if } l_{w_1} + m_2 \le n, \\ 0, \text{ if } l_{w_1} + m_2 > n, \end{cases}$$

$$(w_1, m_1) * 0 = 0 * (w_1, m_1) = 0 * 0 = 0$$

for all $(w_1, m_1), (w_2, m_2) \in FD_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(FD_n, \dashv, \vdash)$ will be denoted by $FD_n(X)$.

***Theorem* 1.1.** *$FD_n(X)$ is the free n-dinilpotent dimonoid.*

Proof. First prove that $FD_n(X)$ is a dimonoid. Let (w_1, m_1), (w_2, m_2), $(w_3, m_3) \in FD_n \setminus \{0\}$ and let

$$l_{w_1} + l_{w_2} + l_{w_3} - m_1 + 1 \le n. \quad (1.1)$$

From (1.1) it follows

$$l_{w_1} + l_{w_2} - m_1 + 1 < n, \quad (1.2)$$

$$l_{w_2} + l_{w_3} - m_2 + 1 < n, \quad (1.3)$$

$$l_{w_2} + m_3 < n. \quad (1.4)$$

Using (1.1)–(1.4), we get

$$((w_1, m_1) \dashv (w_2, m_2)) \dashv (w_3, m_3) =$$
$$= (w_1 w_2, m_1) \dashv (w_3, m_3) = (w_1 w_2 w_3, m_1) =$$
$$= (w_1, m_1) \dashv (w_2 w_3, m_2) =$$
$$= (w_1, m_1) \dashv ((w_2, m_2) \dashv (w_3, m_3)),$$
$$(w_1, m_1) \dashv ((w_2, m_2) \vdash (w_3, m_3)) =$$
$$= (w_1, m_1) \dashv (w_2 w_3, l_{w_2} + m_3) = (w_1 w_2 w_3, m_1)$$

and so, the axioms $(D1)$ and $(D2)$ of a dimonoid hold. If

$$l_{w_1} + l_{w_2} + l_{w_3} - m_1 + 1 > n,$$

then, obviously, the axioms $(D1)$ and $(D2)$ hold too.

Let

$$l_{w_1} + m_2 \le n, \quad (1.5)$$

$$l_{w_2} + l_{w_3} - m_2 + 1 \le n. \quad (1.6)$$

Using (1.5), (1.6), we obtain

$$((w_1, m_1) \vdash (w_2, m_2)) \dashv (w_3, m_3) =$$
$$= (w_1 w_2, l_{w_1} + m_2) \dashv (w_3, m_3) =$$
$$= (w_1 w_2 w_3, l_{w_1} + m_2) =$$
$$= (w_1, m_1) \vdash (w_2 w_3, m_2) =$$
$$= (w_1, m_1) \vdash ((w_2, m_2) \dashv (w_3, m_3)).$$

If

$$l_{w_2} + m_2 > n \quad \text{or}$$
$$l_{w_2} + l_{w_3} - m_2 + 1 > n,$$

then, obviously,

$$((w_1, m_1) \vdash (w_2, m_2)) \dashv (w_3, m_3) = 0 =$$
$$= (w_1, m_1) \vdash ((w_2, m_2) \dashv (w_3, m_3)).$$

Thus, the axiom $(D3)$ holds.

Let

$$l_{w_1} + l_{w_2} + m_3 \le n. \quad (1.7)$$

From (1.7) it follows

$$l_{w_1} + m_2 < n, \quad (1.8)$$

$$l_{w_2} + m_3 < n, \quad (1.9)$$

$$l_{w_1} + l_{w_2} - m_1 + 1 < n. \quad (1.10)$$

According to (1.7)–(1.10), we have

$$((w_1, m_1) \vdash (w_2, m_2)) \vdash (w_3, m_3) =$$
$$= (w_1 w_2, l_{w_1} + m_2) \vdash (w_3, m_3) =$$
$$= (w_1 w_2 w_3, l_{w_1 w_2} + m_3) =$$
$$= (w_1 w_2 w_3, l_{w_1} + l_{w_2} + m_3) =$$
$$= (w_1, m_1) \vdash (w_2 w_3, l_{w_2} + m_3) =$$
$$= (w_1, m_1) \vdash ((w_2, m_2) \vdash (w_3, m_3)),$$
$$((w_1, m_1) \dashv (w_2, m_2)) \vdash (w_3, m_3) =$$
$$= (w_1 w_2, m_1) \vdash (w_3, m_3) =$$
$$= (w_1 w_2 w_3, l_{w_1 w_2} + m_3)$$

and so, the axioms $(D5)$ and $(D4)$ of a dimonoid hold.

If $l_{w_1} + l_{w_2} + m_3 > n$, then, obviously, the axioms $(D4)$ and $(D5)$ hold too.

The proofs of the remaining cases are obvious.

Consequently, $FD_n(X)$ is a dimonoid.

Take arbitrary elements $(w_i, m_i) \in FD_n \setminus \{0\}$, $1 \le i \le n+1$. It is clear that $l_{w_1 w_2 \ldots w_{n+1}} - m_1 + 1 > n$. From here

$$(w_1, m_1) \dashv (w_2, m_2) \dashv \ldots \dashv (w_{n+1}, m_{n+1}) = 0.$$

Besides, $(x_1, 1) \dashv (x_2, 1) \dashv \ldots \dashv (x_n, 1) \ne 0$ for any $(x_i, 1) \in FD_n \setminus \{0\}$, where $x_i \in X$, $1 \le i \le n$. From the last arguments we conclude that $(FD_n, \dashv)$ is a nilpotent semigroup of nilpotency index n.

Further note that $l_{w_1w_2\ldots w_n}+m_{n+1}>n.$ From the above it follows that

$$(w_1,m_1)\vdash(w_2,m_2)\vdash\ldots\vdash(w_{n+1},m_{n+1})=0.$$

Moreover,

$$(x_1,1)\vdash(x_2,1)\vdash\ldots\vdash(x_n,1)\neq 0$$

for any $(x_i,1)\in FD_n\setminus\{0\},$ where $x_i\in X,$ $1\le i\le n.$ According to the above remarks we deduce that $(FD_n,\vdash)$ is a nilpotent semigroup of nilpotency index $n.$

Thus, by the definition, $FD_n(X)$ is an n-dinilpotent dimonoid.

Let us show that $FD_n(X)$ is free.

Let $(T,\dashv',\vdash')$ be an arbitrary n-dinilpotent dimonoid and $\gamma:X\to T$ be an arbitrary map. Define a map

$$\lambda: FD_n(X)\to(T,\dashv',\vdash'):u\mapsto u\lambda,$$

assuming

$$u\lambda=\begin{cases}x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma, & \text{if } u=(x_1\ldots x_s,l),\ x_i\in X, 1\le i\le s,\\ 0, & \text{if } u=0.\end{cases}$$

Show that λ is a homomorphism. We will use the axioms of a dimonoid.

For arbitrary elements

$$(x_1\ldots x_i\ldots x_s,l),\ (y_1\ldots y_j\ldots y_r,t)\in FD_n\setminus\{0\},$$

where $x_i,y_j\in X,$ $1\le i\le s,$ $1\le j\le r,$ we obtain

$$\big((x_1\ldots x_i\ldots x_s,l)\dashv(y_1\ldots y_j\ldots y_r,t)\big)\lambda=$$
$$=\begin{cases}(x_1\ldots x_s y_1\ldots y_r,l)\lambda, & \text{if } s+r-l+1\le n,\\ 0\lambda, & \text{if } s+r-l+1>n,\end{cases}$$

$$\big((x_1\ldots x_i\ldots x_s,l)\vdash(y_1\ldots y_j\ldots y_r,t)\big)\lambda=$$
$$=\begin{cases}(x_1\ldots x_s y_1\ldots y_r,s+t)\lambda, & \text{if } s+t\le n,\\ 0\lambda, & \text{if } s+t>n.\end{cases}$$

If $s+r-l+1\le n,$ then we have

$$(x_1\ldots x_s y_1\ldots y_r,l)\lambda=$$
$$=x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma\dashv' y_1\gamma\dashv'\ldots\dashv' y_r\gamma=$$
$$=(x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma)\dashv'$$
$$\dashv'(y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma)=$$
$$=(x_1\ldots x_i\ldots x_s,l)\lambda\dashv'(y_1\ldots y_j\ldots y_r,t)\lambda.$$

If $s+r-l+1>n,$ then

$$0\lambda=0=$$
$$=x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma\dashv' y_1\gamma\dashv'\ldots\dashv' y_r\gamma=$$
$$=(x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma)\dashv'$$
$$\dashv'(y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma)=$$
$$=(x_1\ldots x_i\ldots x_s,l)\lambda\dashv'(y_1\ldots y_j\ldots y_r,t)\lambda.$$

In the case $s+t\le n,$

$$(x_1\ldots x_s y_1\ldots y_r,s+t)\lambda=$$
$$=x_1\gamma\vdash'\ldots\vdash' x_s\gamma\vdash' y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma=$$
$$=(x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma)\vdash'$$
$$\vdash'(y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma)=$$
$$=(x_1\ldots x_i\ldots x_s,l)\lambda\vdash'(y_1\ldots y_j\ldots y_r,t)\lambda.$$

If $s+t>n,$ then

$$0\lambda=0=$$
$$=x_1\gamma\vdash'\ldots\vdash' x_s\gamma\vdash' y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma=$$
$$=(x_1\gamma\vdash'\ldots\vdash' x_l\gamma\dashv'\ldots\dashv' x_s\gamma)\vdash'$$
$$\vdash'(y_1\gamma\vdash'\ldots\vdash' y_t\gamma\dashv'\ldots\dashv' y_r\gamma)=$$
$$=(x_1\ldots x_i\ldots x_s,l)\lambda\vdash'(y_1\ldots y_j\ldots y_r,t)\lambda.$$

The proofs of the remaining cases are obvious. Thus, λ is a homomorphism. This completes the proof of Theorem 1.1. □

Now we construct a dimonoid which is isomorphic to the free n-dinilpotent dimonoid of rank 1.

Fix $n\in\mathbb{N}$ and define operations $\dashv$ and $\vdash$ on

$$\overline{\mathbb{N}}_n=\{(m,t)\in\mathbb{N}\times\mathbb{N}\mid$$
$$t\le m,\ t\le n,\ m-t+1\le n\}\cup\{0\}$$

by

$$(m_1,t_1)\dashv(m_2,t_2)=$$
$$=\begin{cases}(m_1+m_2,t_1), & \text{if } m_1+m_2-t_1+1\le n,\\ 0, & \text{if } m_1+m_2-t_1+1>n,\end{cases}$$

$$(m_1,t_1)\vdash(m_2,t_2)=$$
$$=\begin{cases}(m_1+m_2,m_1+t_2), & \text{if } m_1+t_2\le n,\\ 0, & \text{if } m_1+t_2>n,\end{cases}$$

$$(m_1,t_1)*0=0*(m_1,t_1)=0*0=0$$

for all $(m_1,t_1),(m_2,t_2)\in\overline{\mathbb{N}}_n\setminus\{0\}$ and $*\in\{\dashv,\vdash\}.$ An immediate verification shows that axioms of a dimonoid hold concerning operations $\dashv$ and $\vdash$. So, $(\overline{\mathbb{N}}_n,\dashv,\vdash)$ is a dimonoid. Denote it by $\mathbb{N}_{(n)}.$

***Lemma* 1.1.** *If* $|X|=1,$ *then* $FD_n(X)\cong\mathbb{N}_{(n)}.$

Proof. Assume $X=\{a\}$ and define a map

$$\eta: FD_n(X)\to\mathbb{N}_{(n)}$$

by the rule

$$u\eta=\begin{cases}(k,l), & u=(a^k,l),\\ 0, & u=0.\end{cases}$$

An easy verification shows that η is an isomorphism. □

2 The least n-dinilpotent congruence on a free dimonoid

In this section we present the least n-dinilpotent congruence on a free dimonoid.

Let $\breve{F}[X]$ be the free dimonoid over X (see introduction and preliminaries). Fix $n\in\mathbb{N}$ and let

$$I_{(n)}=\{(w,m)\in\breve{F}[X]\mid m>n\quad\text{or}$$
$$l_w-m+1>n\}.$$

Define a relation $\xi_{(n)}$ on $\breve{F}[X]$ by

$$(w_1,m_1)\xi_{(n)}(w_2,m_2) \text{ if and only if } (w_1,m_1)=(w_2,m_2) \text{ or } (w_1,m_1),(w_2,m_2)\in I_{(n)}.$$

***Theorem* 2.1.** *The relation $\xi_{(n)}$ on the free dimonoid $\breve{F}[X]$ is the least n-dinilpotent congruence.*

Proof. Let $(w_1,m_1),(w_2,m_2)\in \breve{F}[X]$. It is not difficult to see that

$$l_{w_1}+l_{w_2}-m_1+1\le n \qquad (2.1)$$

implies

$$m_2<n, \qquad (2.2)$$

$$l_{w_1}-m_1+1<n, \qquad (2.3)$$

$$l_{w_2}-m_2+1<n \qquad (2.4)$$

and

$$l_{w_1}+m_2\le n \qquad (2.5)$$

implies (2.2), (2.3) and

$$m_1<n. \qquad (2.6)$$

Define a map $\varphi:\breve{F}[X]\to FD_n(X)$ by

$$(w,m)\varphi = \begin{cases}(w,m), \text{ if } m\le n, \quad l_w-m+1\le n,\\ 0, \text{ if } m>n \quad\text{or}\quad l_w-m+1>n\end{cases}$$

$((w,m)\in \breve{F}[X])$. Show that φ is a homomorphism.

Let $m_1>n$. Then

$$((w_1,m_1)\dashv(w_2,m_2))\varphi=(w_1w_2,m_1)\varphi=0=$$
$$=0\dashv(w_2,m_2)\varphi=(w_1,m_1)\varphi\dashv(w_2,m_2)\varphi.$$

In the case $m_1\le n$ we consider the following two cases. If (2.1) holds, then, using (2.2)–(2.4), we have

$$((w_1,m_1)\dashv(w_2,m_2))\varphi=(w_1w_2,m_1)\varphi=$$
$$=(w_1w_2,m_1)=(w_1,m_1)\dashv(w_2,m_2)=$$
$$=(w_1,m_1)\varphi\dashv(w_2,m_2)\varphi.$$

If

$$l_{w_1}+l_{w_2}-m_1+1>n,$$

then

$$((w_1,m_1)\dashv(w_2,m_2))\varphi=(w_1w_2,m_1)\varphi=$$
$$=0=(w_1,m_1)\varphi\dashv(w_2,m_2)\varphi.$$

Thus,

$$((w_1,m_1)\dashv(w_2,m_2))\varphi=$$
$$=(w_1,m_1)\varphi\dashv(w_2,m_2)\varphi$$

for all $(w_1,m_1),(w_2,m_2)\in \breve{F}[X]$.

Let $l_{w_2}-m_2+1>n$. Then

$$((w_1,m_1)\vdash(w_2,m_2))\varphi=(w_1w_2,l_{w_1}+m_2)\varphi=$$
$$=0=(w_1,m_1)\varphi\vdash 0=(w_1,m_1)\varphi\vdash(w_2,m_2)\varphi.$$

In the case $l_{w_2}-m_2+1\le n$ we consider the following two cases. If (2.5) holds, then, using (2.2), (2.3), (2.6), we have

$$((w_1,m_1)\vdash(w_2,m_2))\varphi=(w_1w_2,l_{w_1}+m_2)\varphi=$$
$$=(w_1w_2,l_{w_1}+m_2)=(w_1,m_1)\vdash(w_2,m_2)=$$
$$=(w_1,m_1)\varphi\vdash(w_2,m_2)\varphi.$$

If $l_{w_1}+m_2>n$, then

$$((w_1,m_1)\vdash(w_2,m_2))\varphi=(w_1w_2,l_{w_1}+m_2)\varphi=$$
$$=0=(w_1,m_1)\varphi\vdash(w_2,m_2)\varphi.$$

Thus,

$$((w_1,m_1)\vdash(w_2,m_2))\varphi=$$
$$=(w_1,m_1)\varphi\vdash(w_2,m_2)\varphi$$

for all $(w_1,m_1),(w_2,m_2)\in \breve{F}[X]$. Consequently, φ is a surjective homomorphism.

By Theorem 1.1 $FD_n(X)$ is the free n-dinilpotent dimonoid. Then Δ_φ is the least n-dinilpotent congruence on $\breve{F}[X]$. From the definition of φ it follows that $\Delta_\varphi=\xi_{(n)}$. □

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