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О ПОЛУПОКРЫВАЮЩИХ-ИЗОЛИРУЮЩИХ ПОДГРУППАХ ИЛИ *S*-КВАЗИНОРМАЛЬНО ВЛОЖЕННЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

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ON SEMI COVER-AVOIDING OR *S*-QUASINORMALLY EMBEDDED SUBGROUPS OF FINITE GROUPS

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В данной работе мы изучаем нильпотентность и сверхразрешимость конечных групп *G*, некоторые примарные подгруппы которых являются либо полупокрывающими-изолирующими, либо *S*-квазинормально вложенными в *G*. Получено обобщение некоторых известных результатов.

Ключевые слова: *полупокрывающая-изолирующая подгруппа; S-квазинормально вложенная подгруппа; p-нильпотентная группа; сверхразрешимая группа.*

In this paper, we characterize the nilpotency and supersolvability of a finite group *G* by assuming some subgroups of prime power order are either semi cover-avoiding or *S*-quasinormally embeded in *G*. Some known results are generalized.

Keywords: *semi cover-avoiding subgroup; S-quasinormally embeded subgroup; p-nilpotent group; supersolvable group.*

Introduction

All groups considered in this paper are finite and *G* always denotes a finite group. The following notations are used in the paper: $O_p(G)$ is the maximal normal *p*-subgroup of *G*, $\Phi(G)$ is the Frattini subgroup of G and $\mathcal{U}$ is the class of all supersolvable groups. A class of groups $\mathcal{F}$ is called a formation if $\mathcal{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is said to be saturated if $G\in\mathcal{F}$ whenever $G/\Phi(G)\in\mathcal{F}$. All unexplained terminology and notations are standard, as in [13], [9].

If M and N are normal subgroups of G with $N<M$, then we call M/N a normal factor of G. A subgroup H of G is said to cover the normal factor M/N of G provided that $HM=HN$, and H is said to avoid M/N provided that $H\cap M=H\cap N$. If H either covers or avoids each chief factor of *G*, then H is said to possess the cover-avoiding property in G. This concept was introduced by Gaschütz [6] in 1962 and studied by many authors (see, for example, [7], [12], [22], [18]). In 2006, Fan, Guo and Shum [5] introduced the semi cover-avoiding property: a subgroup *H* is said to be semi cover-avoiding in *G* if there is a chief series $1=G_0<G_1<\dots<G_l=G$ of *G* such that *H* either covers or avoids G_j/G_{j-1} for every $j=1,\dots,l$. Many authors have investigated the structures of the group *G* under the assumption that some subgroups of *G* is semi cover-avoiding in *G* and obtained some interesting results (see [10], [11], [25]).

Recall that a subgroup *H* of *G* is called *S*-quasinormal [14] in *G* provided that *H* permutes with all Sylow subgroups of *G*. A subgroup *H* of a group *G* is said to be *S*-quasinormally embedded [3] in *G* if for each prime *p* dividing the order of *H*, a Sylow *p*-subgroup of *H* is also a Sylow *p*-subgroup of some *S*-quasinormal subgroup of *G*. By using *S*-quasinormally embedded subgroups, some authors have obtained many interesting results (see, for example, [1], [2], [15], [17]).

The following examples show that semi cover-avoiding subgroups and *S*-quasinormally embedded subgroups are two independent concepts.

***Example* 0.1.** Let $G=A_4\times Z_2$, where A_4 is an alternating group and $Z_2=\langle c\rangle$ is a cyclic group of order 2. Let $K_4=\langle a,b\rangle$ be the Sylow 2-group of A_4 generated by two elements a and b of order 2 and let $H=\langle a,bc\rangle$. Then $1\trianglelefteq Z_2\trianglelefteq K_4Z_2\trianglelefteq G$ is a chief series of *G*. It is easy to prove that *H* covers K_4Z_2/Z_2 and avoids the factors G/K_4Z_2 and $Z_2/1$, but *H* is not *S*-quasinormally embedded in *G*.

***Example* 0.2.** Let $G=A_5$ be the alternative group of degree 5. Since A_5 is simple, there is no nontrivial semi cover-avoiding subgroup in A_5.

However, if H is any Sylow subgroup of G, then clearly H is S-quasinormally embedded in G.

In this paper, we investigate the structure of a group G under the assumption that all maximal subgroups of a Sylow subgroup is either semi cover-avoiding or S-quasinormally embedded subgroups in G. Some new characterizations on the structure of finite groups are obtained and some known results are generalized.

1 Preliminaries

In this section, we list some known results which will be useful for the proofs of our main results.

***Lemma* 1.1** [20]. *Let H be a p-subgroup of G for some prime p. Then H is S-quasinormal in G if and only if $O^p(G) \le N_G(H)$.*

***Lemma* 1.2.** *If H is an S-quasinormal subgroup of G, then*

(1) *H is subnormal in G* [14];

(2) *H / H_G is nilpotent* [4].

***Lemma* 1.3** [1]. *Let H be a subgroup of G. Then the following two statements are equivalent:*

(1) *H is an S-quasinormal nilpotent subgroup of G.*

(2) *The Sylow subgroups of H are S-quasinormal in G.*

***Lemma* 1.4** [11]. *Let H be a subgroup of G. If H is semi cover-avoiding in G, then H is semi cover-avoiding in K for every subgroup K of G with $H \le K$.*

***Lemma* 1.5** [5]. *Let N be a normal subgroup of G and let H be a subgroup of G which is semi cover-avoiding in G. Then HN / N is semi cover-avoiding in G / N if one of the following holds:*

(1) $N \le H$;

(2) $(|N|,|H|) = 1$.

***Lemma* 1.6** [3]. *Suppose that U is an S-quasinormally embedded subgroup of G and K is a normal subgroup of G. Then*

(1) *U is S-quasinormally embedded in H whenever $U \le H \le G$.*

(2) *UK is S-quasinormally embedded in G and UK / K is S-quasinormally embedded in G / K.*

***Lemma* 1.7** [11]. *Let p be a prime dividing the order of G with $(|G|, p-1) = 1$ and P be a Sylow p-subgroup of G. If there is a maximal subgroup P_1 of P such that P_1 is semi cover-avoiding in G, then G is p-solvable.*

***Lemma* 1.8** [21]. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G has a normal subgroup E such that $G / E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

***Lemma* 1.9** [24]. *Let K be an S-quasinormal subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If either $P \le O_p(G)$ or $K_G = 1$, then P is S-quasinormal in G.*

***Lemma* 1.10** [9]. *Let N be a nontrivial solvable normal subgroup of G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G contained in N.*

2 Main results

***Theorem* 2.1.** *Let p be an odd prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is either semi cover-avoiding or S-quasinormally embedded in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false, and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Suppose that $D = O_{p'}(G) \neq 1$. Obviously, PD / D is a Sylow p-subgroup of G / D. Let T / D be a maximal subgroup of PD / D. Then $T = P_1 D$ for some maximal subgroup P_1 of P. By Lemmas 1.5 and 1.6 (2), $P_1 D / D$ is either semi cover-avoiding or S-quasinormally embedded in G / D. On the other hand, since

$$N_{G/D}(PD / D) = N_G(P)D / D$$

by [9], we see that $N_{G/D}(PD / D)$ is p-nilpotent. This shows that G / D satisfies the hypothesis of the theorem. Thus G / D is p-nilpotent. It follows that G is p-nilpotent, a contradiction.

(2) *If M is a proper subgroup of G with $P \le M$, then M is p-nilpotent.*

Clearly, $N_M(P)$ is p-nilpotent. By Lemmas 1.4 and 1.6 (1), we see that M satisfies the hypothesis. The minimal choice of G implies that M is p-nilpotent.

(3) *$G = PQ$ and $O_p(G) \neq 1$, where Q is a Sylow q-subgroup of G with $q \neq p$.*

Since G is not p-nilpotent, by Thompson's theorem [23], there is a nonidentity characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $H < K \le P$. Since H *char* $P \trianglelefteq N_G(P)$, we have $H \trianglelefteq N_G(P)$, and so $N_G(P) < N_G(H)$. Then by (2), we have $G = N_G(H)$. This shows that $H \le O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $O_p(G) < K \le P$ (if exists). In this case, using Thompson's theorem again, we see that $G / O_p(G)$ is p-nilpotent and so G is p-solvable. Thus for any prime divisor q of $|G|$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G (see [8, Chapter

6, Theorem 3.5]). If $PQ < G$, then PQ is p-nilpotent by (2). It follows from (1) that

$$Q \le C_G(O_p(G)) = O_p(G),$$

a contradiction. Hence $G = PQ$.

(4) *G has a unique minimal normal subgroup N such that $G = N \rtimes M$, where M is a maximal subgroup of G, $N = O_p(G) = C_G(N)$.*

Let N be a minimal normal subgroup of G. Then by (1) and (3), N is an elementary abelian p-group, and $N \subseteq O_p(G) < P$. It is easy to see that G/N satisfies the hypothesis. Hence G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $N \not\leq \Phi(G)$. Consequently, $G = N \rtimes M$ for some maximal subgroup M of G. Clearly, $N = O_p(G)$.

(5) *Final contradiction.*

Since $P \ntrianglelefteq G$ and $P = NM_p$, we see that $N \not\leq \Phi(G)$. Hence there exists a maximal subgroup P_1 of P such that $N \not\leq P_1$. If $P_1 = 1$, then P is a cyclic subgroup of order p. It follows that $N_G(P) = C_G(P)$ since $N_G(P)$ is p-nilpotent. Hence G is p-nilpotent by Burnside Theorem, a contradiction. Hence we assume that $P_1 \ne 1$.

Assume that P_1 is semi cover-avoiding in G. Then there is a chief factor series

$$1 = G_0 < G_1 < \cdots < G_l = G$$

such that for every $j = 1, ..., l$, P_1 either covers or avoids G_j / G_{j-1}. In particular, P_1 covers or avoids $G_1/1$, which means that $G_1P_1 = P_1$ or $G_1 \cap P_1 = 1$. By (3), $G_1 = N$. If $NP_1 = P_1$, then $N \le P_1$, a contradiction. Hence $N \cap P_1 = 1$ and so $|N| = p$. It follows that $P = N(P \cap M) = N \times (P \cap M)$, which contradicts (4).

Now assume that G has an S-quasinormal subgroup K such that P_1 is a Sylow p-subgroup of K. If $K_G \ne 1$, then $N \le K_G \le K$, and thereby $N \le P_1$, a contradiction. Therefore $K_G = 1$. Then by Lemmas 1.2 (2) and 1.3, P_1 is S-quasinormal in G. Thus P_1 is subnormal in G by Lemma 1.2(1). By [9], we have that $P_1 \le O_p(G) = N \le P$. Since P_1 is a maximal subgroup of P, $P_1 = N$, a contradiction also. The final contradiction completes the proof of the theorem.

***Corollary* 2.2.** *Let H be a normal subgroup of G such that G/H is p-nilpotent, where p is a prime dividing the order of G. If there exists a Sylow p-subgroup P of H such that $N_G(P)$ is either semi cover-avoiding or S-quasinormally embedded in G, then G is p-nilpotent.*

Proof. By Lemmas 1.4 and 1.6 (1) and Theorem 2.1, H is p-nilpotent. Let $H_{p'}$ be a normal Hall p'-subgroup of H. Assume that $H_{p'} \ne 1$. Then clearly, $(G/H_{p'})/(H/H_{p'}) \cong G/H$ is p-nilpotent. Applying Lemmas 1.5 and 1.6 (2) and [9], we see that $G/H_{p'}$ satisfies the hypothesis. Hence by induction on $|G|$, $G/H_{p'}$ is p-nilpotent. It follows that G is p-nilpotent. We may, therefore, assume $H_{p'} = 1$. Then $H = P$ is a p-group. In this case, $G = N_G(P)$ is p-nilpotent.

***Theorem* 2.3.** *Let p be the smallest prime dividing $|G|$ and P be a Sylow p-subgroup of G. If every maximal subgroup of P is either semi cover-avoiding or S-quasinormally embedded in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We prove it via the following steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \ne 1$, then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Suppose that $M/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then there exists a maximal subgroup P_1 of P such that $M = P_1O_{p'}(G)$. By the hypothesis, P_1 is either semi cover-avoiding or S-quasinormally embedded in G. Then $M/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ is either semi cover-avoiding or S-quasinormally embedded in $G/O_{p'}(G)$ by Lemmas 1.5 and 1.6(2). The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Therefore, we have $O_{p'}(G) = 1$.

(2) $O_p(G) \ne 1$.

If all maximal subgroups of P are S-quasinormally embedded in G, then G is p-nilpotent by [1]. Hence there exists at least a maximal subgroup P_1 of P which is semi cover-avoiding in G. By Lemma 1.7, G is p-solvable. It follows from (1) that $O_p(G) \ne 1$.

(3) *G is solvable.*

If G is not solvable, then $p = 2$ by Feit-Thompson's theorem. Suppose that $M/O_2(G)$ is a maximal subgroup of $P/O_2(G)$. Then M is a maximal subgroup of P. By Lemmas 1.5 and 1.6(2), $M/O_2(G)$ is either semi cover-avoiding or S-quasinormally embedded in $G/O_2(G)$. Therefore $G/O_2(G)$ satisfies the hypothesis. The minimal choice of G implies that $G/O_2(G)$ is 2-nilpotent, and so $G/O_2(G)$ is solvable. It follows that G is solvable, a contradiction. Thus (3) holds.

(4) *G has a unique minimal normal subgroup* $N = O_p(G)$, $G = NM$, *where* M *is p-nilpotent and* $|N| > p$.

Let N be a minimal normal subgroup of G. By (3), N is an elementary abelian subgroup. Since $O_{p'}(G) = 1$, $N \le O_p(G)$. It is easy to see that G/N satisfies the hypothesis. The choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \not\le \Phi(G)$. This implies that $G = N \rtimes M$, $N = O_p(G)$ and M is p-nilpotent. If $|N| = p$, then $G/C_G(N)$ is an abelian group exponent $p-1$. It follows that $N \le Z(G)$ and so G is p-nilpotent, a contradiction.

(5) *Final contradiction.*

Clearly, $P = N(P \cap M)$ and $P \cap M < P$. Thus, there exists a maximal subgroup P_1 of P such that P_1 containing $P \cap M$. Then $P = NP_1$ and $P_1 \ne 1$. By the hypothesis, P_1 is either semi cover-avoiding or S-quasinormally embedded in G. Suppose that P_1 is semi cover-avoiding in G. Then P_1 covers or avoids $N/1$. If $P_1N = P_1$, then $N \le P_1$, a contradiction. Hence $P_1 \cap N = 1$. Consequently $|N| = p$, a contradiction. Now assume that P_1 is S-quasinormally embedded in G. Then there exists an S-quasinormal subgroup K such that P_1 is a Sylow p-subgroup of K. If $K_G \ne 1$, then $N \le K_G \le K$ by (4) and so $N \le P_1$. This contradiction shows that $K_G = 1$. Then by Lemmas 1.2 (2) and 1.3, P_1 is S-quasinormal in G. It follows from Lemma 1.2 (1) that P_1 is subnormal in G. Now by [9], we have that $P_1 \le O_p(G) = N$. The final contradiction completes the proof.

***Corollary* 2.4.** *Let p be the smallest prime dividing* $|G|$ *and H a normal subgroup of G such that* G/H *is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either semi cover-avoiding or S-quasinormally embedded in* G, *then* G *is p-nilpotent.*

Proof. By Lemmas 1.4 and 1.6 (1), every maximal subgroup of P is either semi cover-avoiding or S-quasinormally embedded in H. Applying Theorem 2.3, H is p-nilpotent. Let $H_{p'}$ be the normal p-complement of H. Then $H_{p'}$ is normal in G. By using the same argument as in the proof of Corollary 2.2, we may assume $H_{p'} = 1$ and so $H = P$ is a p-group. Since G/H is p-nilpotent, we may let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 2.3 again, we see that K is p-nilpotent. Hence $K = H \times K_{p'}$. In this case, $K_{p'}$ is a normal p-complement of G, thus G is p-nilpotent.

***Corollary* 2.5.** *Suppose that every maximal subgroup of any Sylow subgroup of G is either semi cover-avoiding or S-quasinormally embedded in* G. *Then G is a Sylow tower group of supersolvable type.*

***Corollary* 2.6** [11, Theorem 3.2]. *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of* G. *If* P *is cyclic or every maximal subgroup of P is semi cover-avoiding in* G, *then* G *is p-nilpotent.*

Proof. If P is a cyclic group, then by [19], G is p-nilpotent. Hence we assume that every maximal subgroup of P is semi cover-avoiding in G. By Corollary 2.3, G is p-nilpotent.

***Theorem* 2.7.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Then* $G \in \mathcal{F}$ *if and only if there is a normal subgroup* H *of* G *such that* $G/H \in F$ *and every maximal subgroup of the Sylow subgroup of H is either semi cover-avoiding or S-quasinormally embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Assume that it is false and let G be a counterexample of minimal order. Then:

(1) *There is a normal Sylow subgroup P of G contained in H.*

By Corollary 2.5, H has a Sylow tower of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in H. Since $P \; char \; H \trianglelefteq G$, we have that $P \trianglelefteq G$.

(2) *Let N be a minimal normal subgroup of G contained in* P. *Then* $G/N \in \mathcal{F}$ *and* $N = P$.

It is easy to see that

$$(G/N)/(H/N) \cong G/H \in \mathcal{F}.$$

Let P_1/N be a maximal subgroup of P/N. By Lemmas 1.5 and 1.6, P_1/N is either semi cover-avoiding property or S-quasinormally embedded in G/N. Let Q be a Sylow q-subgroup of H, where $q \ne p$, and M_1/N be a maximal subgroup of the Sylow q-subgroup QN/N of H/N. It is clear that $M_1 = Q_1N$ for some maximal subgroup Q_1 of Q. By the hypothesis, Q_1 is either semi cover-avoiding or S-quasinormally embedded in G. Hence M_1/N is either semi cover-avoiding or S-quasinormally embedded in G/N by Lemmas 1.5 and 1.6. Thus G/N satisfies the hypothesis of the theorem. The choice of G implies that $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P, $\Phi(P) = 1$ and $N \not\le \Phi(G)$. It follows from Lemma 1.10 that $P = F(P) = N$.

(3) *Final contradiction.*

Suppose that every maximal subgroup of P is S-quasinormally embedded in G. Then by [1], $G\in\mathcal{F}$, a contradiction. So we may assume that there is some maximal subgroup P_1 of P such that P_1 is semi cover-avoiding in G. Then there exists a chief series of G

$$1=G_0<G_1<\cdots<G_l=G$$

such that P_1 covers or avoids every factor G_j/G_{j-1}, $j=1,...,l$. Since N=P is a minimal normal in G, there exists j such that $G_j\cap N=N$ and $G_{j-1}\cap N=1$. If P_1 covers G_j/G_{j-1}, then $P_1G_j=P_1G_{j-1}$. It follows that $P_1(G_j\cap N)=P_1(G_{j-1}\cap N)$, that is, $P_1N=P_1$, a contradiction. If P_1 avoids G_j/G_{j-1}, then $P_1\cap G_j=P_1\cap G_{j-1}$ and so

$$P_1\cap G_j\cap N=P_1\cap G_{j-1}\cap N.$$

This means that $P_1=1$ and so $|N|=p$. Then by (2) and Lemma 1.8, $G\in\mathcal{F}$. This contradiction completes the proof.

***Corollary* 2.8** [16, Theorem 3.6]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal Hall subgroup H of G such that $G/H\in\mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H has the semi cover-avoiding property in G, then $G\in\mathcal{F}$.*

***Theorem* 2.9.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and H be a solvable normal subgroup of G such that $G/H\in\mathcal{F}$. If every maximal subgroup of any Sylow subgroup of $F(H)$ is either semi cover-avoiding or S-quasinormally embedded in G, then $G\in\mathcal{F}$.*

Proof. Assume that the theorem is false and let (G,H) be a counterexample with $|G|+|H|$ is minimal.

Firstly, assume that $H\cap\Phi(G)\neq 1$. Let Q be a Sylow q-subgroup of H, where q is a prime divisor of $|H|$. Since $Q\ char\ H\trianglelefteq G$, we have that $Q\trianglelefteq G$ and so $(G/Q)/(H/Q)\cong G/H\in\mathcal{F}$. By [13, Chapter 3, Theorem 3.5], $F(H/Q)=F(H)/Q$. It is easy to see that $(G/Q,H/Q)$ satisfies the hypothesis of the theorem. Hence $G/Q\in\mathcal{F}$ by minimal choice of G. Since $Q\leq\Phi(G)$ and $\mathcal{F}$ is a saturated formation, we have that $G\in\mathcal{F}$. This contradiction shows that $H\cap\Phi(G)=1$. By Lemma 1.10, $F(H)$ is the direct product of minimal normal subgroups of G contained in H. Let P be the Sylow p-subgroup of $F(H)$ and assume that $P=N_1\times N_2\times\cdots\times N_t$, where $N_1,...,N_t$ are minimal normal subgroups of G. We now prove that $|N_i|=p$ for each $i\in\{1,...,t\}$. If P is cyclic, then it is clear. Assume that P is not cyclic and there exists some N_i such that $|N_i|>p$. Without loss of generality, we may assume that $i=1$. Clearly, there exists a maximal subgroup M of G such that $G=N_1M$ and $N_1\cap M=1$. Let M_p be a Sylow p-subgroup of M. Then $G_p=N_1M_p=PM_p$ is a Sylow p-subgroup of G. Take a maximal subgroup G_p^* of G_p containing M_p and let $P_1=G_p^*\cap P$. Then

$$G_p^*=G_p^*\cap PM_p=(G_p^*\cap P)M_p=P_1M_p$$

and

$$P_1=G_p^*\cap(N_1\times N_2\times\cdots\times N_t)=$$
$$=(G_p^*\cap N_1)N_2\cdots N_t=N_1^*N_2\cdots N_t,$$

where $N_1^*=G_p^*\cap N_1$. Since

$$|N_1:G_p^*\cap N_1|=|N_1G_p^*:G_p^*|=|G_p:G_p^*|=p,$$

$N_1^*=G_p^*\cap N_1$ is a maximal subgroup of N_1. This implies that $P_1=N_1^*N_2\cdots N_t$ is a maximal subgroup of P. By the hypothesis, P_1 is semi cover-avoiding or S-quasinormal embedded in G. Let $T=N_2\times\cdots\times N_t$.

Assume that P_1 is semi cover-avoiding in G. Then by Lemma 1.5, P_1/T is semi cover-avoiding in G/T. Let

$$1=\overline{T}\trianglelefteq G_1/T=\overline{G_1}\trianglelefteq\cdots\trianglelefteq G/T=\overline{G_n}$$

be the chief series of G/T such that P_1 either covers or avoids every factor of this series. Let i be the smallest index in $\{1,...,n\}$ such that P_1/T covers $\overline{G_{i+1}}/\overline{G_i}$. Then it is easy to see that $G_i\cap P_1=T$ and

$$G_{i+1}\leq G_iP_1=G_iN_1^*.$$

It follows that $G_{i+1}=G_i(N_1^*\cap G_{i+1})$, and so $N_1^*\cap G_{i+1}>1$. But since N_1 is a minimal normal subgroup of G, we obtain that $N_1\leq G_{i+1}$ and $N_1\cap G_i=1$. Therefore,

$$|N_1|=|G_{i+1}/G_i|=|N_1^*\cap G_{i+1}|=|N_1^*|.$$

This contradiction shows that P_1/T avoids every chief factor $\overline{G_{i+1}}/\overline{G_i}$, for $i=0,1,...,n$. This implies that $P_1/T=1$ and $|N_1|=p$, a contradiction. Now we assume that there is an S-quasinormal subgroup K of G such that P_1 is a Sylow p-subgroup of K. Clearly, $P_1\leq O_p(G)$. Hence by Lemma 1.9, P_1 is S-quasinormal in G. It follows from Lemma 1.1 that $O^p(G)\leq N_G(P_1)$. Since G_p^* and P are both normal in G_p, we have that $P_1=G_p^*\cap P\trianglelefteq G_p$. Hence $G=G_pO^p(G)\leq N_G(P_1)$. Consequently, $P_1\trianglelefteq G$. Since $N_1\not\leq P_1$, we have $P_1\cap N_1=1$. This induces that $|N_1|=p$, a contradiction again.

The above discussion shows that $F(H) = R_1 \times R_2 \times \cdots \times R_n$, where R_i is the minimal normal subgroup of G of prime order for all $i = 1, ..., n$. Since $G / C_G(R_i)$ is isomorphic to some subgroup of $Aut(R_i)$, $G / C_G(R_i)$ is abelian. It follows that $G / C_G(F(G)) = G / \bigcap_{i=1}^{n} C_G(R_i)$ is abelian and hence $G / C_G(F(G)) \in \mathcal{F}$. Then since $G / H \in \mathcal{F}$, we see that

$$G / (H \cap C_G(F(H))) = G / C_H(F(H)) \in \mathcal{F}.$$

Since $F(H)$ is abelian, $F(H) \le C_H(F(H))$. On the other hand, since H is solvable,

$$C_H(F(H)) \le F(H).$$

Thus $F(H) = C_H(F(H))$ and so $G / F(H) \in \mathcal{F}$. Now by Theorem 2.7, we obtain that $G \in \mathcal{F}$, a contradiction. This completes the proof.

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