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## РЕЛЯТИВИСТСКАЯ ЗАДАЧА О СОСТОЯНИЯХ РАССЕЯНИЯ ДВУХЧАСТИЧНЫХ СИСТЕМ С ПОТЕНЦИАЛАМИ ОДНОБОЗОННОГО ОБМЕНА

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## RELATIVISTIC SCATTERING PROBLEM FOR TWO-PARTICLE SYSTEMS WITH ONE-BOSON EXCHANGE POTENTIALS

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Найдены численные решения уравнений квантовой теории поля, описывающих  $s$ -состояния рассеяния систем двух скалярных частиц в случаях потенциалов однобозонного обмена и потенциала Юкавы. На основании полученных решений вычислены амплитуды рассеяния, длины рассеяния и фазовые сдвиги. Показано, что найденные амплитуды рассеяния удовлетворяют условию унитарности. Проведено сравнение полученных в такой простой модели результатов с экспериментальными данными для нейтрон-протонных систем.

**Ключевые слова:** двухчастичные уравнения, релятивистское конфигурационное представление, потенциал однобозонного обмена, амплитуда рассеяния, длина рассеяния, фазовый сдвиг, составные квадратуры Гаусса, дейтрон.

Numerical solutions of quantum field theory equations describing scattering  $s$ -states of two scalar particles are found in cases of a one-boson exchange potential and the Yukawa potential. Scattering amplitudes, scattering lengths and phase shifts are calculated on the basis of the solutions obtained. It is shown that the scattering amplitudes found satisfy a unitarity condition. The comparison of the results obtained in this simple model with experimental data for neutron-proton systems is carried out.

**Keywords:** two-particle equations, relativistic configurational representation, one-boson exchange potential, scattering amplitude, scattering length, phase shift, composite Gaussian quadrature, deuteron.

### Introduction

Equations of the quasipotential type [1], [2] have found wide application in the study of relativistic composite systems. Especially successfully they have been used in the study of two-particle system bound states. One of the advantages of this approach is the possibility of simple quantum-mechanical analogy. Two-particle equations of the quasipotential type can be formulated in the momentum representation in the integral form and in the relativistic configurational representation (RCR) in the difference or integral forms [3]. In our previous papers [4], [5] the advantages of integral equations for bound states in the RCR were demonstrated. For example, in article [5] numerical solutions of two-particle equations were obtained by simple methods for the potentials in the RCR, equations for which in the momentum representation are singular. Numerical solution of singular integral equations is a complicated problem which requires using of special methods [6], [7].

In this paper numerical solutions of relativistic two-particle integral equations for  $s$ -scattering states are found in cases of a one-boson exchange potential and the Yukawa potential in the RCR. The scattering amplitudes, scattering lengths and phase shifts are calculated on the basis of the wave functions found. Comparison of these values with experimental data

for neutron-proton scattering is carried out. It should be noted that numerical solving of the two-particle equations for wave functions of scattering states in the momentum representation, as well as of the equations for the scattering amplitudes in the momentum representation causes serious difficulties since the Green functions (GF) in these all equations are singular.

### 1 Relativistic two-particle equations

Relativistic equations for the scattering  $s$ -states of two particles with equal masses  $m$  have in the RCR the form [8], [9]

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q mr) - \lambda \int_0^\infty dr' G_{(j)}(\chi_q, r, r') V(r') \psi_{(j)}(\chi_q, r'), \quad (1.1)$$

where index  $j = 1, 2, 3, 4$  corresponds to one of the four variants of equations of the quasipotential type [1–3]:  $j = 1$  ( $j = 3$ ) – the Logunov – Tavkhelidze equation (modified),  $j = 2$  ( $j = 4$ ) – the Kadyshvsky equation (modified). The value  $\psi_{(j)}(\chi_q, r)$  in equations (1.1) is the relative motion wave function,  $r$  is the modulus of radius-vector in the RCR,  $\chi_q \geq 0$  is the rapidity connected with the energy of two-particle system  $2E_q$  by relation

$2E_q = 2m \operatorname{ch} \chi_q$ ,  $\lambda$  is the coupling constant,  $V(r)$  is the potential. Green functions  $G_{(j)}(\chi_q, r, r')$  have the following form [8], [9]:

$$G_{(j)}(\chi_q, r, r') = G_{(j)}(\chi_q, r - r') - G_{(j)}(\chi_q, r + r'), \quad (1.2)$$

where

$$G_{(1)}(\chi_q, r) = \frac{-i \operatorname{sh}[(\pi/2 + i\chi_q)mr]}{K_q^{(1)} \operatorname{sh}(\pi mr/2)}, \quad (1.3)$$

$$G_{(2)}(\chi_q, r) = \frac{(4m \operatorname{ch} \chi_q)^{-1}}{\operatorname{ch}(\pi mr/2)} - \frac{i \operatorname{sh}[(\pi + i\chi_q)mr]}{K_q^{(2)} \operatorname{sh}(\pi mr)},$$

$$G_{(3)}(\chi_q, r) = \frac{-i \operatorname{ch}[(\pi/2 + i\chi_q)mr]}{K_q^{(3)} \operatorname{ch}(\pi mr/2)},$$

$$G_{(4)}(\chi_q, r) = \frac{-i \operatorname{sh}[(\pi + i\chi_q)mr]}{K_q^{(4)} \operatorname{sh}(\pi mr)}.$$

In expressions (1.3) we use the notations

$$K_q^{(1)} = K_q^{(2)} = m \operatorname{sh} 2\chi_q;$$

$$K_q^{(3)} = K_q^{(4)} = 2m \operatorname{sh} \chi_q.$$

Equations (1.1) for bound states become homogeneous, and rapidity becomes imaginary, namely  $\chi_q = iw_q$ , where  $0 < w_q \leq \pi/2$  ( $2E_q = 2m \cos w_q$ ,  $2E_q < 2m$ ):

$$\psi_{(j)}(iw_q, r) = -\lambda \int_0^\infty dr' G_{(j)}(iw_q, r, r') V(r') \psi_{(j)}(iw_q, r'). \quad (1.4)$$

In what follows we need the asymptotic behavior of GFs (1.2) at  $r \rightarrow \infty$ :

$$G_{(j)}(\chi_q, r, r') \Big|_{r \rightarrow \infty} \cong \frac{-2}{K_q^{(j)}} \exp(i\chi_q mr) \sin(\chi_q mr'). \quad (1.5)$$

Let us find the wave functions asymptotics at  $r \rightarrow \infty$  using equations (1.1) and expressions (1.5):

$$\psi_{(j)}(\chi_q, r) \Big|_{r \rightarrow \infty} \cong \sin(\chi_q mr) + q f_{(j)}(\chi_q) \exp(i\chi_q mr),$$

where  $f_{(j)}(\chi_q)$  is used for the scattering amplitude which is determined by analogy with quantum mechanics as the coefficient at the scattered wave  $\exp(i\chi_q mr)$  divided by  $q = m \operatorname{sh} \chi_q$  [10], [11]:

$$f_{(j)}(\chi_q) = \frac{2\lambda}{qK_q^{(j)}} \int_0^\infty dr' \sin(\chi_q mr') V(r') \psi_{(j)}(\chi_q, r'). \quad (1.6)$$

The scattering amplitude is connected with the partial scattering cross section  $\sigma_{0(j)}(\chi_q)$  and unitary  $S$ -matrix  $S_{(j)}(\chi_q)$  by the relations

$$\sigma_{0(j)}(\chi_q) = 4\pi |f_{(j)}(\chi_q)|^2;$$

$$S_{(j)}(\chi_q) = 1 + 2iqf_{(j)}(\chi_q).$$

The unitarity of the  $S$ -matrix is reflected in the representation  $S_{(j)}(\chi_q) = \exp(2i\phi_{(j)}(\chi_q))$ , where  $\phi_{(j)}(\chi_q)$  is the phase shift.

Let us determine the scattering length by analogy with quantum mechanics as  $a_{(j)} = -f_{(j)}(0)$  [10], [11]. It is not difficult to obtain the expression for scattering length via wave function from formula (1.6):

$$a_{(j)} = -\lambda \int_0^\infty dr' r' V(r') \tilde{\psi}_{(j)}(r'), \quad (1.7)$$

where the denotation  $\tilde{\psi}_{(j)}(r) = \lim_{\chi_q \rightarrow 0} \psi_{(j)}(\chi_q, r)/q$  is introduced. It is possible to find equations for functions  $\tilde{\psi}_{(j)}(r)$ . One can divide equations (1.1) by  $q$  and consider then their limit at  $\chi_q \rightarrow 0$ , it yields

$$\tilde{\psi}_{(j)}(r) = r - \lambda \int_0^\infty dr' G_{(j)}(0, r, r') V(r') \tilde{\psi}_{(j)}(r'). \quad (1.8)$$

Green functions at zero rapidity  $\chi_q$  have the form

$$G_{(j)}(0, r, r') = G_{(j)}(0, r - r') - G_{(j)}(0, r + r'),$$

where

$$G_{(1)}(0, r) = \frac{1}{2} r \operatorname{coth} \frac{\pi mr}{2},$$

$$G_{(2)}(0, r) = \frac{1}{4m \operatorname{ch}(\pi mr/2)} + \frac{1}{2} r \operatorname{coth} \pi mr,$$

$$G_{(3)}(0, r) = \frac{1}{2} r \tanh \frac{\pi mr}{2},$$

$$G_{(4)}(0, r) = \frac{1}{2} r \operatorname{coth} \pi mr.$$

Taking into account expressions (1.7) one can represent the asymptotics of equations (1.8) at  $r \rightarrow \infty$  in the form

$$\tilde{\psi}_{(j)}(r) \Big|_{r \rightarrow \infty} \cong r - a_{(j)}. \quad (1.9)$$

Formula (1.9) is much easier to use for the scattering length finding than the expression (1.7) since it does not contain integrals.

It is not difficult to see that the non-relativistic limit ( $m \rightarrow \infty$ ,  $\chi_q \rightarrow 0$ ) of all equations and formulas discussed gives corresponding equations and expressions of quantum mechanics [10], [11].

## 2 Methods of solving

To solve integral equations (1.1), (1.4) and (1.8) we used the composite Gaussian quadrature method [12]. Let us describe the essence of this method in case of equations (1.1). After replacing the upper infinite limit in the equations by a large value  $R$ , which can be chosen on the basis of the accuracy requirements for the solutions, and after presenting the integral as a sum of  $N$  integrals, we obtain

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q mr) - \lambda \sum_{k=1}^N \int_{r_{k-1}}^{r_k} dr' G_{(j)}(\chi_q, r, r') V(r') \psi_{(j)}(\chi_q, r'), \quad (2.1)$$

where  $r_k = r_{k-1} + h_k$ ,  $r_0 = 0$ ,  $r_N = R$  and  $h_k$  is the step. Let us reduce every interval  $r \in [r_{k-1}; r_k]$  in equations (2.1) to the interval  $x \in [-1; 1]$  by the variable substitution

$$r = u_k(x) = \frac{h_k x}{2} + \frac{r_k + r_{k-1}}{2}$$

and let us then apply the Gaussian quadrature formula for  $M$  nodes for integrals obtained [12], [13]. As a result we yield the following expressions:

$$\begin{aligned} \psi_{(j)}(\chi_q, r) = & \sin(\chi_q m r) - \\ & - \lambda \sum_{k=1}^N \sum_{i=1}^M h_k \frac{C_i}{2} G_{(j)}(\chi_q, r, u_k(x_i)) \times \\ & \times V(u_k(x_i)) \psi_{(j)}(\chi_q, u_k(x_i)), \end{aligned} \quad (2.2)$$

where  $x_i$ ,  $C_i$  are nodes and weights of the Gaussian quadrature on the interval  $x \in [-1; 1]$ . Taking formula (2.2) in the points

$$r_{s,l} = u_s(x_l) = \frac{h_s x_l}{2} + \frac{r_s + r_{s-1}}{2}$$

one can obtain linear algebraic systems ( $\alpha = M(s-1) + l$ ,  $\beta = M(k-1) + i$ ):

$$\begin{aligned} \sum_{\beta=1}^{N \cdot M} [\delta_{\alpha\beta} + \lambda K_{\alpha\beta}] \psi_{\beta} &= b_{\alpha}; \\ K_{\alpha\beta} &= A_{\beta} G_{\alpha\beta}; \\ G_{\alpha\beta} &= G_{(j)}(\chi_q, u_s(x_l), u_k(x_i)); \\ A_{\beta} &= h_k C_i V \frac{u_k(x_i)}{2}; \\ b_{\alpha} &= \sin(\chi_q m u_s(x_l)); \\ \psi_{\beta} &= \psi_{(j)}(\chi_q, u_k(x_i)). \end{aligned} \quad (2.3)$$

Application of standard solving methods [12], [13] for algebraic systems (2.3) gives the values of the wave function for every particular energy value  $2E_q$ . In articles [4], [5] this method was used to solve the two-particle relativistic equations in the RCR for the bound states for which the corresponding systems of equations were homogeneous.

### 3 Results of numerical calculations

Let us consider solving two-particle equations (1.1) and (1.8) in case of the one-boson exchange potential [3]

$$V(r) = \frac{\text{ch}(\pi - \alpha)mr}{r \text{sh } \pi mr} \quad (3.1)$$

and in case of the Yukawa potential

$$V(r) = \frac{\exp(-\mu r)}{r}, \quad (3.2)$$

where parameter  $\alpha$  is connected to the exchange scalar boson mass  $\mu$  as  $\cos \alpha = 1 - \mu^2 / (2m^2)$ . Potential (3.1) is a variant of relativistic generalization of the Yukawa potential. The main difference of the relativistic potential from the non-relativistic one is in its stronger singularity at  $r = 0$ .

Results of the scattering lengths calculations for the Logunov – Tavkhelidze and modified Kadyshvsky equations with potential (3.1) at  $m = 1$ ,  $\mu = 0.2$  are demonstrated in figure 3.1. It is shown in the figure that the scattering lengths increase sharply at certain values of the coupling constants. A similar behavior of the scattering length depending on the coupling constant is well known in the non-relativistic theory. For instance, the scattering length in case of the potential well has similar property [10]. We do not demonstrate scattering lengths for other equations with potential (3.1) as well as for all equations with potential (3.2) because they have similar behaviour.

The results of the scattering cross sections and phase shifts calculating for potential (3.1) at  $m = 1$ ,  $\mu = 0.2$ ,  $\lambda = 1$  are demonstrated in figure 3.2.

Numerical calculations show that all the scattering amplitudes obtained satisfy the unitarity condition

$$\text{Im } f_{(j)}(\chi_q) = q |f_{(j)}(\chi_q)|^2, \quad (3.3)$$

which in case of superposition of two delta-function potentials in the RCR was proved exactly [14].

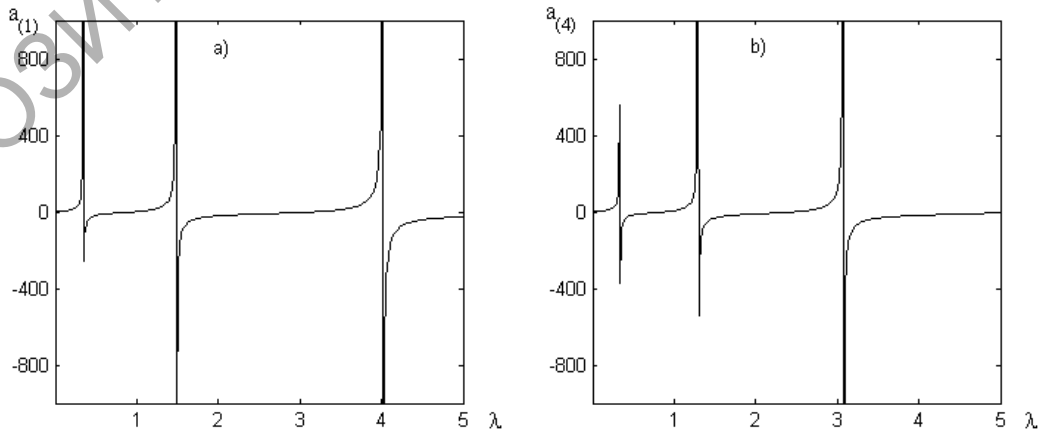


Figure 3.1 – Scattering lengths dependence on the coupling constant for the one-boson exchange potential: a)  $j = 1$ , b)  $j = 4$

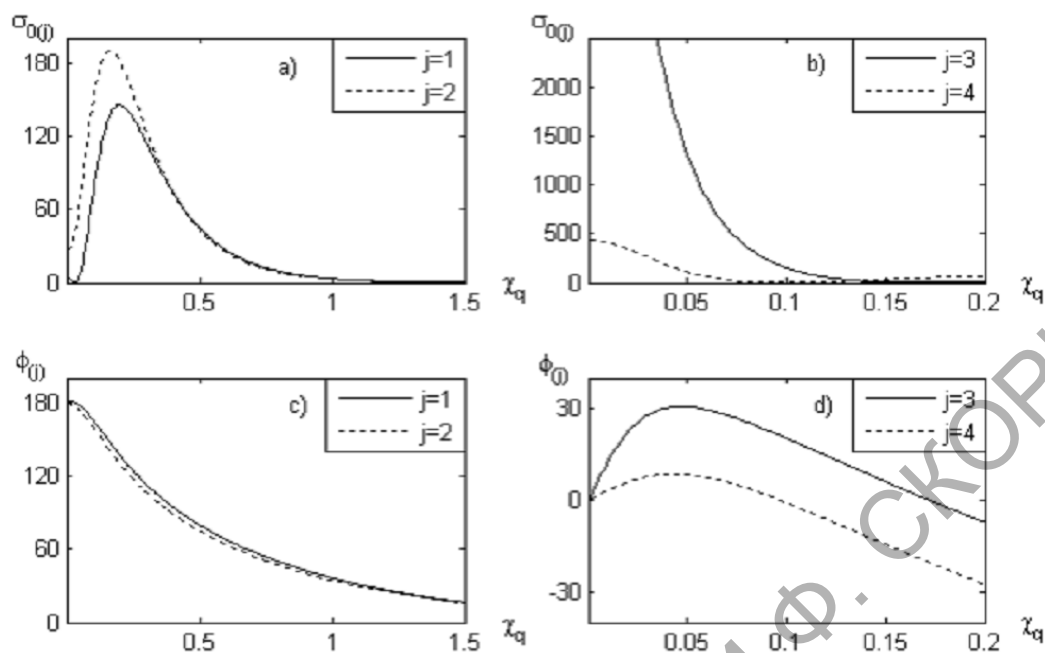


Figure 3.2 – Scattering cross sections (a, b) and respective phase shifts (c, d) dependence on the rapidity

Table 4.1 – Coupling constants for the deuteron binding energy and scattering lengths

Potential	Equation	Coupling constant $\lambda$	Scattering length $a_{(j)}$ , Fm
Yukawa	$j = 0$	0.348536	5.43637
	$j = 1$	0.357978	5.47843
	$j = 2$	0.363810	5.48804
	$j = 3$	0.344018	5.43430
	$j = 4$	0.351273	5.44762
One-boson exchange	$j = 1$	0.354172	5.46913
	$j = 2$	0.360286	5.47941
	$j = 3$	0.338462	5.42058
	$j = 4$	0.346667	5.43618

#### 4 A simplest model of two-particle system of the deuteron type

Let us consider as an example of application of two-particle equations with one-boson exchange potential some results of calculations for neutron-proton scattering and for neutron-proton bound state (deuteron). The two-particle relativistic equations under consideration and one-boson exchange potential (3.1) were obtained under the assumption of equal masses of both particles in the system. However proton and neutron have different masses. In addition the nucleons are spinor particles but equations (1.1) and potentials (3.1), (3.2) describe the system of two scalar particles. Nevertheless, not pretending to an excellent agreement with the experimental results we consider extremely simple scalar model of the deuteron type in which the mass  $m$  is the double reduced mass of the proton and neutron:

$$m = 2m_p m_n / (m_p + m_n),$$

where

$$m_p = 938.272013 \text{ MeV}, \quad m_n = 939.565346 \text{ MeV}$$

[15], when  $m = 938.918234 \text{ MeV}$ . We choose the mass of scalar exchange boson equal to the mass of  $\pi^0$ -meson:  $\mu = 134.9766 \text{ MeV}$  [15].

It is known from the experiments that binding energy of the deuteron is  $2.224575(9) \text{ MeV}$  [16] and the scattering length for the proton-neutron triplet state is  $5.424(4) \text{ Fm}$  [17]. Substituting this value of energy in equations (1.4) with potentials (3.1), (3.2) we find the eigenvalues of the coupling constants  $\lambda$  using the method discussed in [5]. Then we use the obtained values of the coupling constant for determining the scattering characteristics of the neutron-proton system: the scattering lengths, scattering cross sections and phase shifts. The results of calculation of the coupling constant and scattering lengths

are given in Table 4.1 for the four relativistic equations and for the Schrödinger equation ( $j = 0$ ). A few values of the coupling constant in each case were found. They correspond to different states, i.e. this model does not match the experimentally established possibility of the existence of only one state of the deuteron. The coupling constants corresponding to the ground state of the system are given in table 4.1.

It is seen in the table that the best correspondence with experiment is given by the value of the scattering length, obtained by solving the modified Logunov – Tavkhelidze equation ( $j = 3$ ) with the one-boson exchange potential.

The results of calculations of the phase shift for equation  $j = 3$  with one-boson exchange potential and the experimental values in the dependence of the scattering energy [18] are demonstrated in figure 4.1. The figure shows that the results of calculations are located very close to the experimental values. Thus, this simple model gives good correspondence with the experimental value of the scattering length but gives not very good results for the phase shifts. In figure 4.2 the results of calculations of partial cross section also are represented corresponding to the phase shift in figure 4.1 for different values of energy.

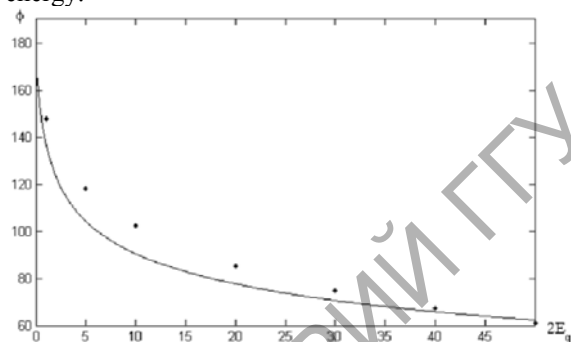


Figure 4.1 – Phase shifts in degrees dependence on energy (MeV): the experimental values are denoted by dots, calculated values by solid line

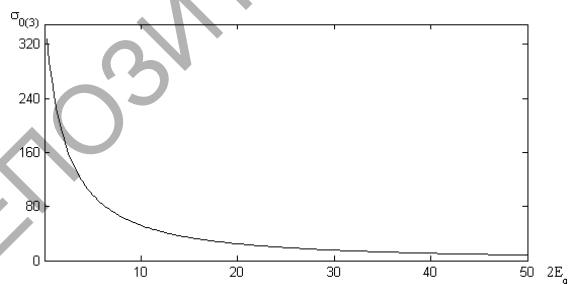


Figure 4.2 – Scattering cross section in ( $\text{Fm}^2$ ) dependence on energy (MeV) for  $j = 3$  equation

We do not represent figures with the results of calculations of the phase shifts and scattering cross sections for the other cases since they have similar

behavior and other phase shifts do not give better correspondence with experiment.

### Conclusion

In this paper we found numerical solutions of the relativistic equations of quantum field theory describing the scattering  $s$ -states of two scalar particles with a variant of one-boson exchange potential and the Yukawa potential in the relativistic configurational representation. The scattering amplitudes, phase shifts and scattering lengths are calculated on the basis of the solutions obtained. It is shown that all numerical values of the scattering amplitudes found satisfy the unitarity condition. A comparison of the results obtained in this simple model with the experimental measurements for the triplet neutron-proton scattering was carried out. Comparison demonstrated good correspondence with the experimental results for the scattering length.

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