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О *P*-СВОЙСТВЕ ПОДГРУПП КОНЕЧНЫХ ГРУПП

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ON *P*-PROPERTY OF SUBGROUPS OF FINITE GROUPS

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Пусть H – подгруппа группы G. Мы говором, что H имеет *P*-свойство в G, если $|G/K : N_{G/K}(HK/K \cap L/K)|$ является *p*-числом для любого *pd*-главного фактора L/K группы G. Используя это понятие, найдены некоторые новые критерии *p*-нильпотентности групп.

Ключевые слова*: коненая группа, p-нильпотентная группа, P-свойство подгруппы.*

Let H be a subgroup of a group G. We say that H has *P*-property in G if $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a *p*-number for any *pd*-chief factor L/K of G. Using this property of subgroups, some new criterions of *p*-nilpotency of groups are obtained.

Keywords*: finite group, p-nilpotent group, P-property of subgroup.*

Introduction

Throughout this paper, all groups considered are finite and G always denotes a finite group. We use $\pi(G)$ to denote the set of all prime divisors of the order $|G|$ of G, π denotes a set of some primes and π' is the complement of π in the set $\mathbb{P}$ of all primes. An integer n is called a π-number if all its prime divisors belong to π. G is said to be a π-group if $\pi(G) \subseteq \pi$. G is called a πd-group if $\pi(G) \cap \pi \neq \varnothing$. A class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic image and subdirect product. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. All unexplained notions and terminology are standard, as in [1], [2] or [3].

Recall that a subgroup A of G is said to permute with a subgroup B if $AB = BA$. It is known that AB is a subgroup of G if and only if A permutes with B. Thus the permutability of subgroups is very important. A subgroup H of G is called quasinormal [4] or permutable [3] in G if H permutes with all subgroups of G. If H permutes with all Sylow subgroups of G, then H is called s-permutable in G [5]. After the work in [5], [6], many authors attempted to study and apply other kinds of embedding properties of subgroups. For instance, a subgroup H in G is called seminormal [7], [8] if H is permutable with all subgroups of some supplement of H in G; A subgroup H of G is called semipermutable in G [9] if it is permutable with every subgroup K of G with $(|H|,|K|) = 1$; A subgroup H of a group G is said to be S-quasinormally embedded in G [10] if for each prime p dividing $|H|$, a Sylow *p*-subgroup of H is also a Sylow *p*-subgroup of some s-permutable subgroup of *G*; A subgroup H of G is said be conditionally permutable in G if H permutes with some conjugate of any subgroup of G [11]. More recently, W. Guo and A.N. Skiba [12] studied the structure of group by s-embedded and n-embedded subgroups.

It is known that if a *p*-group H is s-permutable in G then $|G : N_G(H)|$ is a *p*-number. Generalizing this property we give the following definition:

***Definition* 0.1.** *Let H be a subgroup of G. We call that H has P-property in G if for any pd-chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a p-number.*

In any finite soluble group G, it is easy to see that all permutable and s-permutable subgroups have *P*-property in G. Also, suppose that H is a *p*-subgroup of a group G (G is not necessary soluble), then H has *P*-property if H is seminormal, semipermutable, s-quasinormal embedded or so on in G (see proofs in Section 1).

The subgroups having *P*-property deeply influence the structure of a group. In this paper, we shall give some new criteria of *p*-nilpotency of groups.

2 Elementary Properties

In this section, we give some elementary results of *P*-property of subgroups, some of which are also lemmas of our main results.

***Lemma* 1.1.** *Let H be a subgroup of G and N a normal subgroup of G. If H has P-property in G, then HN/N has P-property in G/N.*

Proof. (1) Let $(L/N)/(K/N)$ be a *pd*-chief factor of G/N. Then L/K is a *pd*-chief factor of G and hence $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a *p*-number. It follows directly that

$$|(G/N)/(K/N):$$
$$N_{(G/N)/(K/N)}((HK/N)/(K/N) \cap (L/N)/(K/N))|$$

is a *p*-number and hence the lemma holds.

Let X be a subset of G, a subgroup H is said *X*-permutable with T if there is an element $x \in X$ such that $HT^x = T^x H$ (cf. [13]).

***Proposition* 1.2.** *Let H be a p-subgroup of G. Then H has P-property in G if one of the following holds:*

(1) *H is normal in G*;

(2) *H is permutable in G*;

(3) *H is s-permutable in G*;

(4) *H is X-permutable with all Sylow subgroups of G, where X is a soluble normal subgroup of G.*

Proof. Since all normal subgroups, all permutable subgroups and all s-permutable subgroups in G are *X*-permutable with all Sylow subgroups in G for any subgroup X of G, we only need to prove that H satisfies *P*-property in G when (4) holds. Let L/K be any *pd*-chief factor of G. Then HK/K is XK/K-permutable with all Sylow subgroups of G/K and XK/K is a soluble normal subgroup of G/K. If $K \neq 1$, then, by induction on $|G|$, we can assume that HK/K satisfies *P*-property in G/K and hence

$$|G/K : N_{G/K}(HK/K \cap L/K)|$$

is a *p*-number. Assume that $K = 1$. Then L is minimal normal in G. Assume that L is abelian. Then L is a *p*-subgroup. If $H \cap L = 1$, then it holds clearly that $|G : N_G(H \cap L)| = 1$ is a *p*-number. Assume that $H \cap L \neq 1$. We claim that $|G : N_G(H \cap L)|$ is a *p*-number. Let q be any prime divisor of $|G|$ with $q \neq p$ and G_q a Sylow *q*-subgroup of *G*. Then there is an element $x \in X$ such that $HG_q^x = G_q^x H$. Since L is a normal *p*-group, we have that $L \cap HG_q^x = L \cap H$ is a normal subgroup of HG_q^x. Hence $G_q^x \subseteq N_G(H \cap L)$ and so $|G : N_G(H \cap L)|$ is a q'-number. Now, by the choice of q, we have obtained that $|G : N_G(H \cap L)|$ is a *p*-number when L is abelian. Assume that L is nonabelian. Then $L \cap X = 1$ and hence $X \subseteq C_G(L)$. Let q be any prime divisor of $|L|$ and Q a Sylow *q*-subgroup of L. Put G_q be a Sylow *q*-subgroup of G such that $Q \leq G_q$. By hypotheses, there is an element $x \in X$ such that $HG_q^x = G_q^x H$. Clearly,

$$(|HG_q^x : H|, |HG_q^x : G_q^x|) = 1$$

and $L \cap HG_q^x \trianglelefteq HG_q^x$. By [2, Lemma 3.8.2],

$$L \cap HG_q^x = (L \cap HG_q^x \cap H)(L \cap HG_q^x \cap G_q^x) =$$
$$= (H \cap L)(L \cap G_q^x) = (H \cap L)Q^x.$$

Since $X \subseteq C_G(L)$, $Q^x = Q$ and so $H \cap L$ permutes with Q. By the choice of Q, we see that $H \cap L$ is s-permutable in L and so is subnormal in L. Since L is minimal normal in G, L is a product of some simple groups. Thus $H \cap L \trianglelefteq L$. Now consider that q is a prime different from p. Then $H \cap L$ char $(H \cap L)Q$ since $(H \cap L)Q \leq L$. On the other hand, $(H \cap L)Q = L \cap HG_q^x \trianglelefteq HG_q^x$ by the above argument. Thus $G_q^x \subseteq N_G(H \cap L)$ and hence $|G : N_G(H \cap L)|$ is a q'-number for any $q \notin \pi(H \cap L)$. Therefore, $|G : N_G(H \cap L)|$ is a *p*-number and the proposition holds.

***Proposition* 1.3.** *Let H be a p-subgroup of G. If H is seminormal or semipermutable in G, then H satisfies P-property in G.*

Proof. Assume that H is seminormal in G and let T be a supplement of H in G such that H permutes with all subgroup of T. Let q be any prime divisor of $|G|$ different from p and G_q an arbitrary Sylow *q*-subgroup of G. Then $G_q^x \subseteq T$ for some $x \in H$ since $G = HT$ and H is a *p*-subgroup. Thus H permutes with G_q^x and so $(HG_q)^x = HG_q^x$ is a subgroup of G. It follows that H permutes with G_q. Let L/K be any *pd*-chief factor of G. Clearly, HK/K is seminormal in G/K. If $K \neq 1$, then, by induction on $|G|$, we can assume that HK/K satisfies *P*-property in G/K and hence $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a *p*-number. Assume that $K = 1$. Then L is minimal normal in G. If L is abelian, then it can be obtained that $|G : N_G(H \cap L)|$ is a *p*-number by an argument as in Proposition 1.2. Assume that L is nonabelian. We claim that $H \cap L = 1$. Let Q be any Sylow *q*-subgroup of L and choose G_q to be a Sylow *q*-subgroup of G containing Q. Then $HG_q = G_q H$ and, by [2, Lemma 3.8.2],

$$L \cap HG_q = (L \cap H)(L \cap G_q) = (H \cap L)Q.$$

This induce that $H \cap L$ permutes with all Sylow *q*-subgroup of L. Since L is nonabelian, $(H \cap L)Q \neq L$ by Burnside $p^a q^b$-Theorem. It follows from [14, Theorem 3] that there is a proper normal subgroup N of L such that $H \cap L \subseteq N$ or $Q \subseteq N$. But $Q \subseteq N$ is nonsense since Q is a Sylow

q-subgroup of L and L is a direct product of some nonabelian simple groups which are isomorphic to each other. Hence $H \cap L \subseteq N$. Repeat this argument, we can find finally that $H \cap L$ is subnormal in L. Thus $H \cap L \subseteq O_p(L) = 1$. It follows that $H \cap L \trianglelefteq G$ and, certainly $|G : N_G(H \cap L)|$ is a p-number. Similarly, one can prove that H satisfies P-property in G if H is semipermutable in G and the proposition holds.

***Lemma* 1.4.** *Let H be a p-subgroup of G and N a minimal normal subgroup of G. Assume that H has P-property in G. If there is a Sylow p-sub-group* G_p *of G such that* $H \cap N \trianglelefteq G_p$*, then* $H \cap N = N$ *or 1.*

Proof. Since H has P-property in G, $|G : N_G(H \cap N)|$ is a p-number. On the other hand, since $H \cap N \trianglelefteq G_p$, $|G : N_G(H \cap N)|$ is a p'-number. Thereby, $H \cap N$ is normal in G and it follows that $H \cap N = N$ or 1 since N is minimal normal in G and the lemma holds.

***Lemma* 1.5.** *Let H be a p-subgroup of G for some prime divisor* p *of* $|G|$ *and assume that* H *has P-property in* G*. Then any G-chief factor* L/K *which does not avoid by H is a p-factor and hence is abelian.*

Proof. Assume that L/K is a G-chief factor which does not avoid by H. Then $(L/K)/(K/K)$ is a chief factor of G/K and does not avoid by HK/K. Since HK/K satisfies P-property in G/K by lemma 1.1, we can obtain that $L/K \cong (L/K)/(K/K)$ is a p-factor by induction on $|G|$ if $K \neq 1$. Assume that $K = 1$. Since H has P-property in G, $|G : N_G(H \cap L)|$ is a p-number. This induces that $G = G_p N_G(H \cap L)$, where G_p is a Sylow p-subgroup of G contained $H \cap L$. By [2, Lemma 3.4.9], $(H \cap L)^G \subseteq G_p$ and hence $(H \cap L)^G \subseteq O_p(G)$. It follows that $L \subseteq O_p(G)$ since $L \cap O_p(G) \supseteq H \cap L \neq 1$ and L is minimal normal in G. Thus the lemma holds.

Recall that a group is called a C_π-group if it has at least one Hall π-subgroups and all its Hall π-subgroups are conjugate.

***Lemma* 1.6.** *Let* G *be a* C_π*-group and* $p \notin \pi$*. Assume that P is a Sylow p-subgroup of G. If every maximal subgroup of P (except one) has a* π*-closed supplement in* G*, then* G *is* π*-closed.*

Proof. It can be proved similar to Lemma 2.2 in [15] by choose Q to be a Hall π-subgroup of G.

2 Some Results

***Theorem* 2.1.** *Let p be a minimal prime divisor of* $|G|$*,* E *be a normal subgroup of* G *such that* G/E *is p-nilpotent. Suppose that P is a Sylow p-subgroup of* E*. If there is a subgroup* D *of* P *with* $1 < |D| < |P|$ *such that every subgroup* H *in* $\Sigma_D = \{H \subseteq P \mid |H| = |D|,$ *or* $|H| = 2|D|$ *and* $exp(H) > 2$ *whenever* P *is nonabelian 2-group and* $|P : D| > 2\}$ *either has a p-nilpotent supplement in* G *or P-property in* G*, then* G *is p-nilpotent.*

Proof. Assume that the theorem does not hold and choose G to be a counter example of minimal order. We divide the proof into the following steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then the hypotheses still hold on $G/O_{p'}(G)$ by Lemma 1.1. It follows from the choice of G that $G/O_{p'}(G)$ is p-nilpotent and so is G.

(2) $O_p(E) = O_p(G) \cap E \neq 1$.

Let N be a minimal normal subgroup of G with $N \subseteq E$. Then $p \mid |N|$ by (1). Let G_p be a Sylow p-subgroup of G with $P \subseteq G_p$. If there is a subgroup H of order $|D|$ such that $1 \neq H \cap N \trianglelefteq G_p$ and H has P-property in G, then N is a p-group by lemma 1.5 and hence $O_p(E) \neq 1$. Assume that any such subgroup H is not P-normal in G. Then H have a p-nilpotent supplement in G. Let P_1 be any maximal subgroup of G_p. If $P_1 \cap N \neq 1$, then there must be a subgroup $H \subseteq P \cap P_1$ of order $|D|$ with $1 \neq H \cap N \trianglelefteq G_p$. In fact, if $|D| < P_1 \cap N$, then we can choose H to be a subgroup of $P_1 \cap N$, otherwise, we can choose H with $P_1 \cap N \leq H \leq P \cap P_1$ since $|D| \leq |P \cap P_1|$. Thereby, H and so P_1 has a p-nilpotent supplement in G. If $N_p = G_p \cap N$, which is a Sylow p-subgroup of N, is not of order p, then, for every maximal subgroup P_1 of G_p, we have $P_1 \cap N = P_1 \cap N_p \neq 1$. By the above argument, every maximal subgroup P_1 of G_p has a p-nilpotent supplement in G. If $p > 2$, then G is soluble and so is a $C_{p'}$-group. If $p = 2$, then G is a $C_{p'}$-group by [16, Main Theorem]. Now, by Lemma 1.6, G is p'-closed and so is p-nilpotent, a contradiction. Assume that N_p is cyclic of order p. Then N is simple since N is minimal normal in G. But p is the minimal prime divisor of $|G|$ and so is of $|N|$. Thus N is soluble since N_p is cyclic. It follows that $|N| = p$ and hence $O_p(E) \supseteq N \neq 1$.

$O_p(E) = O_p(G) \cap E$ is clear and (2) holds.

(3) *Let* N *be a minimal normal subgroup of* G *with* $N \subseteq O_p(E)$*. Then* $|N| = |D|$.

If $|N| > |D|$, then there is a subgroup H of N

such that $|H|=|D|$ and $H \trianglelefteq G_p$. If H has P-property in G, then $H = H \cap N = N$ or 1 by lemma 1.4, a contradiction. Assume that H has a p-nilpotent supplement U in G. Then $G = HU = NU$ and it follows that $N \cap U \neq 1$. Clearly, $N \cap U \trianglelefteq G$, so $N \subseteq U$ and $G = U$ is p-nilpotent, a contradiction. Thus we can assume that $|N| \leq |D|$. If $|N| < |D|$, then, by lemma 1.1, we see that the hypotheses hold on G/N and, by the choice of G, G/N is p-nilpotent. If $N \subseteq \Phi(G)$, then G is p-nilpotent since the formation of all p-nilpotent group is saturated. Assume that $N \not\subseteq \Phi(G)$. Then N is complemented in G. Let X be a complement of N in G and Q a Hall p'-subgroup of $X \cap E$. Then, since

$$X \cap E \cong (X \cap E)N/N = E/N \leq G/N$$

is p-nilpotent, NQ/N char $E/N \trianglelefteq G/N$ and so $NQ \trianglelefteq G$. By Frattini Argument,

$$G = (NQ)N_G(Q) = NN_G(Q).$$

Since $O_{p'}(G) = 1$, $N_G(Q) \neq G$. Hence $G_p \not\subseteq N_G(Q)$ and so $G_p \cap N_G(Q) < G_p$. Assume that P_1 is a maximal subgroup of G_p containing $G_p \cap N_G(Q)$. If $N \subseteq P_1$, then

$$G_p = G_p \cap NN_G(Q) = N(G_p \cap N_G(Q)) \subseteq NP_1 = P_1,$$

a contradiction. Thus $N \not\subseteq P_1$. Since P_1 is maximal in G_p, $N \cap P_1$ is maximal in N and $N \cap P_1 \trianglelefteq G_p$. Clearly, $P \cap P_1$ is maximal in P and

$$|N \cap P_1| < |N| < |D| \leq |P \cap P_1|,$$

so there is a subgroup H of P_1 such that $N \cap P_1 < H \leq P \cap P_1$ and $|H|=|D|$. Thus $N \cap P_1 = N \cap H$. If H has P-property in G, then, by lemma 1.4, $N \cap P_1 = N \cap H = 1$ or N. If $N \cap P_1 = N$, then $N \subseteq P_1$, a contradiction. Thus $N \cap P_1 = 1$. It follows that N is cyclic of order p and therefore, G is p-nilpotent since G/N is p-nilpotent and p is a minimal prime divisor of $|G|$. This contradicts to the choice of G. By hypotheses, H has a p-nilpotent supplement in G and so has a p-nilpotent supplement in E. Assume that $E = HU$ for some p-nilpotent subgroup U of E. Then U contains some conjugate of Q since E is clearly p-soluble and U contains some Hall p'-subgroup of E. Without loss of generality, we can assume that $Q \subseteq U$. Hence $U \subseteq N_G(Q)$. Since $E = HU = (P \cap P_1)U$,

$$\begin{aligned} P = P \cap E = P \cap (P \cap P_1)U &= \\ = (P \cap P_1)(P \cap U) &= \\ = (P \cap P_1)(P \cap N_G(Q) \cap U) &\leq \\ \leq (P \cap P_1)(P_1 \cap U) &= P \cap P_1. \end{aligned}$$

It follows that $P \subseteq P_1$ and so $N \subseteq P \subseteq P_1$, a contradiction. This contradiction shows that (3) holds.

(4) *N is cyclic and so $|D|=|N|=p$.*

Assume that N is noncyclic. We claim that all minimal subgroup of E/N of order p either has p-nilpotent supplement in G or has P-property in G. Assume A/N is of order p and $A/N \subseteq E/N$. Clearly, A is noncyclic since N is. Thus there is a maximal subgroup H of A different from N. Therefore, $A/N = HN/N$ and $|H|=|N|=|D|$. If A/N has no p-nilpotent supplement in G/N, then clearly, H has no p-nilpotent supplement in G. Hence, by hypotheses, H has P-property in G. By lemma 1.1, $A/N = HN/N$ has P-property in G/N. Analogously, one can prove that if P is a nonabelian 2-group and $|P:D|>2$, then every cyclic subgroup of P/N of order 4 either has a p-nilpotent supplement in G/N or has P-property in G/N. Thus, if N is noncyclic, then the hypotheses hold on G/N. Thereby, G/N is p-nilpotent by the choice of G. By a similar argument as in (3), one can prove that G is p-nilpotent. This contradicts to the choice of G and hence N is cyclic and $|N|=|D|=p$.

(5) $O_p(E) \leq Z_\infty(G)$.

Since p is the minimal divisor of G, it is equivalent to prove that every G-chief factor L/K in $O_p(E)$ is of prime order.

Assume that there exists a G-chief factor in $O_p(G)$ which is not of prime order. Then we can choose a G-chief factor L/K in $O_p(G)$ such that $|L/K|$ is not a prime but $|U/V|$ is a prime for all chief factor U/V of G with $U \subseteq P$ and $|U|<|L|$.

Let $W = \bigcap_{U \subseteq K} C_G(U/V)$, where U/V is a G-chief factor. Then, by [3, A, (12.3)], all elements in W of p'-order act trivially on K since they act trivially on each G-chief factor of K. Let $C = C_G(K)$. Assume $L \not\subseteq C$. If $L \subseteq KC$, then

$$L \cap C / K \cap C \cong L/K$$

is a chief factor of G. By the choice of L/K, $|L/K| = |L \cap C/K \cap C|$ is a prime, a contradiction. If $L \not\subseteq KC$, then it is easily to see that

$$LC/K = L/K \times KC/K$$

and thereby, all p'-elements in C act trivially on L/K. It follows that all p'-elements in W act trivially on L/K. Hence $W \subseteq C_G(L/K)$. Since $G/W = G/\bigcap_{U \subseteq K} C_G(U/V)$ is an abelian group of exponent dividing $p-1$ and $W \subseteq C_G(L/K)$, $G/C_G(L/K)$ is an abelian group of exponent dividing $p-1$. Since L/K is G-irreducible, L/K is of prime order by [17, I, Lemma 1.3], a contradiction.

Now assume that $L \subseteq C$. Then $K \subseteq Z(L)$. Let a, b be elements of order p in L. Suppose $p>2$ or P is abelian. Then $(ab)^p = a^p b^p [b,a]^{\frac{p(p-1)}{2}} = 1$. Hence the product of elements of order p is still of order p and so $\Omega = \{a \in L \mid a^p = p\}$ is a subgroup of L. If $\Omega \subseteq K$, then all elements of W with p'-order act trivially on every element of L of order p since they act trivially on K. It follows from [18, IV, Satz 5.12] that all elements in W of p'-order act trivially on L. Thus $W \subseteq C_G(L/K)$ and, as above argument, L/K is of prime order, a contradiction. If $\Omega \nsubseteq K$, then $L = \Omega K$. Choose an element a in $\Omega \setminus K$ such that $\langle a \rangle K/K \subseteq L/K \cap Z(G_p/K)$. Let $H = \langle a \rangle$. If H has a p-nilpotent supplement U in G, then HK/K has a p-nilpotent supplement UK/K in G/K. Thus

$$G/K = (HK/K)(UK/K) = (L/K)(UK/K).$$

Since L/K is minimal normal in G/K and is abelian, $L/K \cap UK/K = 1$ or $L/K \subseteq UK/K$ and $UK/K = G/K$. If $L/K \cap UK/K = 1$, then

$$|L/K| = |G/K : UK/K| = \\ = |HUK/K : UK/K| \le |H| = p.$$

It follows that L/K is cyclic of order p, which contradicts to the choice of L/K. If $L/K \subseteq UK/K = G/K$, then L/K is cyclic since L/K is minimal normal in G/K and $G/K = UK/K \cong U/U \cap K$ is p-nilpotent. Hence H has no p-nilpotent supplement in G. Since a is of order p, by the hypotheses and (4), H has P-property in G and so HK/K satisfies P-property in G/K. It follows from lemma 1.4 that

$$L/K = HK/K \cap L/K = HK/K$$

is cyclic, a contradiction. This contradiction shows that (5) holds.

(6) *E is p-nilpotent.*

Assume that E is not p-nilpotent. Then E is not a p-group and so $O_p(E) < E$. Let $R/O_p(E)$ be a G-chief factor with $R \le E$. Then, clearly, R is not p-nilpotent. Let X be a minimal non-p-nilpotent subgroup of R. Then $X = A \rtimes B$, where A is a p-group of exponent p or 4 (when A is a nonabelian 2-group) and B is a p'-group. If $A \subseteq O_p(E)$, then B acts trivially on A by (5), a contradiction. Thus $A \nsubseteq O_p(E)$. Hence, if $p>2$ or P is abelian, then there are elements of order p in $R \setminus O_p(E)$, and, if P is a nonabelian 2-group, then there are elements of order 2 or 4 in $R \setminus O_p(E)$. Assume that there is an element a of order p in $R \setminus O_p(E)$ and let $H = \langle a \rangle$. Assume that H has a p-nilpotent supplement U in G. If $H \subseteq U$, then $G = U$ is p-nilpotent. If $H \nsubseteq U$, then $|G:U| = |H| = p$. Since p is the minimal prime divisor of $|E|$, $U \trianglelefteq G$. It follows that G is p-nilpotent since U is. Thus H has no p-nilpotent supplement in G. By hypotheses and (4), H has P-property in G. It follows that $HO_p(G)/O_p(G)$ has P-property in $G/O_p(G)$. Since $H \nsubseteq O_p(G)$, we have that

$$R/O_p(G) = HO_p(G)/O_p(G)$$

by Lemma 1.4. Thus $R = HO_p(G)$ is a p-group, a contradiction. Thus there is no element of order p in $R \setminus O_p(E)$. Hence $p=2$ and there is an element a of order 4 in $R \setminus O_p(E)$ and $a^2 \in O_p(G)$. By a similar argument as above, we can get a contradiction and hence E must be p-nilpotent.

(7) *The final contradiction*

By (1) and (6), we have that E is a p-group and hence $E \le Z_\infty(G)$ by (5). Again by (5), we see that G is p-nilpotent since G/E is. This is the final contradiction and the theorem holds.

By Theorem 2.1, we can obtain the following corollaries.

***Corollary* 2.2.** *Let p be the minimal prime divisor of $|G|$ and E be a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. If every maximal subgroup of P either has a p-nilpotent supplement in G or has P-property in G, then G is p-nilpotent.*

Proof. If P is of order p, then E is clear p-nilpotent. By induction, we can assume that $O_{p'}(G) = 1$. It follows that E is a p-group and so is of order p. By N/C-Theorem, $G/C_G(E)$ is isomorphic to a subgroup of $\mathrm{Aut}(E)$ and hence is of order dividing $p-1$. Since p is the minimal prime divisor of $|G|$, we see that $G/C_G(E) = 1$ and so $E \le Z(G)$. Thus G is p-nilpotent since G/E is. If P is not of order p, then its maximal subgroup is not trivial and hence G is p-nilpotent by Theorem 2.1.

***Corollary* 2.3.** *Let p be the minimal prime divisor of $|G|$ and E be a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. If every subgroup of P of order 2 and 4 (if P is a nonabelian 2-group) either has a p-nilpotent supplement in G or P-property in G, then G is p-nilpotent.*

Proof. It can be proved similar as Corollary 2.2.

The following corollary is direct from Propositions 1.2, 1.3 and Theorem 2.1.

***Corollary* 2.4.** *Let p be the minimal prime divisor of $|G|$ and E be a normal subgroup of G*

such that G/E *is p-nilpotent. Let* P *be a Sylow p-subgroup of* E. *If there is a subgroup* D *of* P *with* $1<|D|<|P|$ *such that for every subgroup* H *in* $\Sigma_D=\{H\subseteq P \mid |H|=|D|,$ *or* $|H|=2|D|$ *and* $exp(H)>2$ *whenever* P *is nonabelian 2-group and* $|P:D|>2\}$, *one of the following holds:*

(1) *H is s-permutable in* G;

(2) *H is seminormal in* G;

(3) *H is semipermutable in* G;

(4) *H is* X*-permutable in* G, *where* X *is a soluble normal subgroup of* G;

(5) *H has a p-nilpotent supplement in* G, *then* G *is p-nilpotent.*

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