# Criteria for analyticity of subordinate semigroups

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**Abstract**

Let $\psi$ be a Bernstein function. A. Carasso and T. Kato obtained necessary and sufficient conditions for $\psi$ to have a property that $\psi(A)$ generates a quasibounded holomorphic semigroup for every generator $A$ of a bounded $C_0$-semigroup in a Banach space, in terms of some convolution semigroup of measures associated with $\psi$. We give an alternative to Carasso-Kato's criterium, and derive several sufficient conditions for $\psi$ to have the above-mentioned property.

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## 1. Introduction

The well known theorem due to Yosida [17] states that for every generator $A$ of a bounded $C_0$-semigroup on a Banach space $X$ its fractional power $-(-A)^{\alpha}, 0 < \alpha < 1$ is a generator of a holomorphic semigroup on $X$. The present paper is devoted to some generalizations and analogs of Yosida's Theorem in terms of so-called Bochner-Phillips calculus [1, 14] (see also [5, Chap. XIII]; [8, 15, 11, 2]). Though the majority of works on Bochner-Phillips calculus use the class $\mathcal{B}$ of (positive) Bernstein functions, we prefer the class $\mathcal{T}$ of negative one. The corresponding reformulation of Bochner-Phillips calculus is trivial in view of the fact that $\phi(x) \in \mathcal{B}$ if and only if $-\phi(-s) \in \mathcal{T}$.

We say that the function $\psi : (-\infty, 0] \to (-\infty, 0]$ belongs to the class $\mathcal{T}$ of *negative Bernstein functions* if $\psi \in C^{\infty}((-\infty, 0)) \cap C((-\infty, 0])$ and its derivative is absolutely monotonic, i.e. $\psi^{(n)} \geq 0$ for all $n \in \mathbb{N}$. It is known that in this case $\psi$ extends analytically to the left half-plane $\Pi_- = \{\mathrm{Re} z < 0\}$, the extension is continuous on $\{\mathrm{Re} z \leq 0\}$, and has the following integral representation

$$\psi(z) = c_0 + \int\limits_{\mathbb{R}_+} (e^{zu} - 1) u^{-1} d\rho(u), \quad Re z \leq 0 \tag{1}$$

where $c_0 = \psi(0)$, the positive measure $\rho$ on $\mathbb{R}_+$ is uniquely determined by $\psi$ and $\int_{[0,1]} d\rho < \infty, \int_{[1,\infty)} u^{-1} d\rho(u) < \infty$; the integrand in (1) is defined for $u = 0$ to be equal to $z$ .

Moreover, there is a convolution semigroup $(\nu_t)_{t \geq 0}$ of sub-probability measures on $\mathbb{R}_+$ with the Laplace transform

$$g_t(z) := e^{t\psi(z)} = \int\limits_{\mathbb{R}_+} e^{zu} d\nu_t(u), \quad Re z \leq 0 \tag{2}$$

(see [16], [5, Chap. XIII]).

The class $\mathcal{T}$ is a cone which is closed with respect to compositions and pointwise convergence on $(-\infty, 0]$, and contains a number of important functions, including (up to affine changes of variable) fractional powers, the logarithm, the inverse hyperbolic cosine, and polylogarithms $Li_p$ of all orders $p \in \mathbb{N}$ [12].

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For a negative Bernstein function $\psi$ with integral representation (1) and a generator $A$ of a bounded $C_0$-semigroup $T$ on a complex Banach space $X$ the value of $\psi$ at $A$ for $x \in D(A)$, the domain of $A$, is defined by the Bochner integral

$$\psi(A)x = c_0 x + \int\limits_{\mathbb{R}_+} (T(u) - I)xu^{-1} d\rho(u).$$

The closure of this operator, which is also denoted by $\psi(A)$, is a generator of a bounded $C_0$-semigroup $g_t(A)$ on $X$ (the "subordinate semigroup"), too. (For the multidimensional version of this calculus see, e.g., [9], [10], [11].)

In the following, without loss of generality we shall assume that $c_0 = 0$. The corresponding subclass of $\mathcal{T}$ will be denoted by $\mathcal{T}_0$. We shall denote also by $\mathcal{M}^b(\mathbb{R}_+, \mathbb{C})(\mathcal{M}(\mathbb{R}_+, \mathbb{R}_+))$ the space of all bounded complex valued (respectively positive) measures on $\mathbb{R}_+$, and by $C_0(\mathbb{R}_+)$ the space of all continuous complex valued functions on $\mathbb{R}_+$ which vanish at infinity; $X$ stands for a complex Banach space.

Another result by Yosida [18] asserts that if the bounded $C_0$-semigroup $T$ with generator $A$ on $X$ satisfies

$$T(t)X \subset D(A), t > 0, \quad \text{and} \quad \limsup_{t \downarrow 0} (t\|AT(t)\|) < \infty, \qquad (Y)$$

then for any $\beta > 0$, $e^{-\beta t}T(t)$ can be extended to a bounded holomorphic semigroup on $X$.

We shall denote by $\mathcal{T}_Y$ the set of all $\psi \in \mathcal{T}$ such that $\psi(A)$ generates a bounded $C_0$-semigroup with property (Y) for every generator $A$ of a bounded $C_0$-semigroup in a Banach space. The class $\mathcal{T}_Y$ is a cone [3, Theorem 6]. Moreover, it is clear that the composition $\psi_1 \circ \psi_2 \in \mathcal{T}_Y$ if $\psi_1 \in \mathcal{T}_Y$, $\psi_2 \in \mathcal{T}$. But the class $\mathcal{T}_Y$ is not closed with respect to pointwise convergence.

A. Carasso and T. Kato [3, Theorem 4] obtained necessary and sufficient conditions for a function $\psi$ to be in $\mathcal{T}_Y$ in terms of the semigroup $(\nu_t)_{t \geq 0}$. They also gave two necessary conditions in terms of $\psi$ itself. Y. Fujita [6] obtained sufficient conditions for $\psi$ to be in $\mathcal{T}_Y$ in terms of analytical continuation of $\psi$ and regular variation.

We proceed as follows. First we prove the multiplication rule which connects the Bochner-Phillips and Hille-Phillips calculi and then derive the alternative to [3] necessary and sufficient conditions for the inclusion $\psi \in \mathcal{T}_Y$ (see Theorem 2 below; the variant of this theorem with $C_{00}(\mathbb{R}_+)$ instead of $E(\mathbb{R}_+)$ (for the definition of the last class see below) first appeared in [13]). Then we deduce two theorems from this criterium that give sufficient conditions for $\psi$ to be in $\mathcal{T}_Y$ in terms of $\psi$. It should ne noted that the assumptions of Theorem 4 below contain necessary conditions, obtained by Carasso and Kato (the idea to employ the Hausdorff-Young inequality in this context belongs to Carasso and Kato, too). Finally, we give one more condition, that is sufficient for the inclusion $\psi \in \mathcal{T}_Y$. Several examples have been considered.

## 2. The multiplication rule for the Bochner-Phillips and Hille-Phillips calculi, and the criterium for $\psi$ to be in $\mathcal{T}_Y$

In [7, Chap.XV] the functional calculus (the Hille-Phillips calculus) of generators of $C_0$-semigroups have been constructed. In particular let $a \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$ and

$$g(s) = La(s) := \int\limits_{\mathbb{R}_+} e^{su} da(u) \quad (s \leq 0)$$

be the *Laplace transform of* $a$. Then for a generator $A$ of a bounded $C_0$-semigroup $T$ on a complex Banach space $X$ the value of $g$ at $A$ is the bounded operator on $X$ defined by the Bochner integral

$$g(A)x = \int\limits_{\mathbb{R}_+} T(u)x da(u), \quad x \in X.$$

Our Theorem 1 connects the Bochner-Phillips and Hille-Phillips calculi. It is a generalization of Lemma 1 in [13]. But first we need the following approximation lemma. We shall denote by $E(\mathbb{R}_+)$ the complex space of exponential polynomials of the form

$$p(t) = \sum_{j=1}^{n} c_j e^{s_j t}, \quad c_j \in \mathbb{C}, s_j < 0,$$

endowed with sup-norm on $\mathbb{R}_+$.

**Lemma 1**. *For every bounded function $q \in C^1(\mathbb{R}_+)$ with bounded derivative there exists a sequence $q_n \in E(\mathbb{R}_+)$ such that*
*1) $q_n \to q$, and $q'_n \to q'$ pointwise on $\mathbb{R}_+$;*
*2) $(q_n)$ and $(q'_n)$ are uniformly bounded on $\mathbb{R}_+$.*

*Proof.* Let us pick a sequence $\tilde{q}_n \in C^1(\mathbb{R}_+)$ such that $\tilde{q}_n(t) = q(t)$ for $t \in [0, n]$, $\tilde{q}_n(t) = 0$ for $t \in [n+1, \infty)$, and $(\tilde{q}_n)$ and $(\tilde{q}'_n)$ are uniformly bounded, $|\tilde{q}_n| < C_1$, $|\tilde{q}'_n| < C_1$. Define $f_n(x) = \tilde{q}_n(-\log x)$ for $x \in [0,1]$ $\quad (f_n(0) = 0)$. Then $f_n \in C^1([0,1])$, $\quad |f_n(x)| < C_1$ for $x \in [0,1]$, and $|f'_n(x)| < C_1 x^{-1}$ for $x \in (0,1]$. It is well known (see, e. g., [4, Theorem 8.4.1]) that for every natural $n$ the algebraic polynomial $p_n$ exists such that

$$|f_n(x) - p_n(x)| < n^{-1}, \quad \text{and} \quad |f'_n(x) - p'_n(x)| < n^{-1}, \quad x \in [0,1].$$

Then $|p_n(0)| < n^{-1}, |p_n(x)| < C_1 + 1$, and $|p'_n(x)| < C_1 x^{-1} + 1$ for $x \in (0,1]$. Since $f_n(x) = q(-\log x)$ for $x \in (0,1]$, and $n > -\log x$, we have

$$|q(-\log x) - p_n(x)| < n^{-1}, \quad x \in (0,1], \quad n > -\log x.$$

Let $q_n(t) := p_n(e^{-t}) - p_n(0)$. Then $q_n \in E(\mathbb{R}_+), q_n \to q$ on $\mathbb{R}_+$, and $(q_n)$ and $(q'_n)$ are uniformly bounded on $\mathbb{R}_+$. Finally

$$|q'(-\log x)(-x^{-1}) - p'_n(x)| < n^{-1}, \quad x \in (0,1], n > -\log x.$$

Putting hear $x = e^{-t}$ we have for all natural $n > t \quad (t \in \mathbb{R}_+)$ that $|q'(t) - q'_n(t)| < n^{-1}$. This completes the proof.

For measures $a \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$, and $\rho \in \mathcal{M}(\mathbb{R}_+, \mathbb{R}_+)$ let

$$K(a, \rho) = \sup_{\phi \in S} \left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \phi(r) d_r (a(r-u) - a(r)) u^{-1} d\rho(u) \right|$$

(if the right hand side exists), where $S$ is the unit sphere of the space $E(\mathbb{R}_+)$ with respect to sup-norm on $\mathbb{R}_+$ . Here we assume that $a = 0$ on $(-\infty; 0)$. See the proof of Theorem 5 for an estimate for $K(a, \rho)$ with bounded positive measure $a$, but $K(\delta, \delta) = \infty$.

**Theorem 1**. *Let $g = La$, $\quad a \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$, and $\quad \psi \in \mathcal{T}_0$ has integral representation* (1). *If $K(a, \rho) < \infty$, then*
*1) the function $h := \psi g$ has the form $h = Lb$, where $b \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$, $\|b\| = K(a, \rho)$;*
*2) $g(A)X \subset D(\psi(A))$, $h(A) = \psi(A)g(A)$, and $\|h(A)\| \leq MK(a, \rho)$ for every operator $A$ in a Banach space $X$, which generates a bounded $C_0$-semigroup $T$ with $\|T(t)\| \leq M$.*

*Proof.* Let $a(r)$ denotes the distribution function for $a$, $a(r) = 0$ for $r \in (-\infty, 0]$. Then for $s < 0$

$$g(s) = \int_{\mathbb{R}_+} e^{sr} da(r) = (-s) \int_{\mathbb{R}_+} e^{sr} a(r) dr.$$

Thus for $u \geq 0$ and $s < 0$ we have

$$(e^{su}-1)g(s) = (e^{su}-1)(-s)\int_{\mathbb{R}_+} e^{sr}a(r)dr$$

$$= (-s)\left(\int_{\mathbb{R}_+} e^{s(r+u)}a(r)dr - \int_{\mathbb{R}_+} e^{sr}a(r)dr\right) = (-s)\int_{\mathbb{R}_+} e^{sr}(a(r-u)-a(r))dr = Lb^u(s),$$

where $b^u(r) = a(r-u) - a(r)$ has bounded variation and is concentrated on $\mathbb{R}_+$. Therefore for $\psi \in \mathcal{T}_0$ with integral representation (1) we get

$$h(s) = \int_{\mathbb{R}_+} (e^{su}-1)g(s)u^{-1}d\rho(u) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{sr}db^u(r)u^{-1}d\rho(u). \tag{3}$$

For $\phi \in E(\mathbb{R}_+)$ let

$$b(\phi) := \int_{\mathbb{R}_+} b^u(\phi)u^{-1}d\rho(u) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \phi(r)d_r(a(r-u)-a(r))u^{-1}d\rho(u)$$

be the linear functional on $E(\mathbb{R}_+)$ (we use the notation $b^u(\phi)$ for $\int \phi db^u$). By the hypothesis of the theorem $\|b\| = K(a,\rho) < \infty$, and since $E(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+)$ by Stone-Weierstrass Theorem, $b$ extends to a measure $b \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$. Furthermore,

$$b = \int_{\mathbb{R}_+} b^u u^{-1}d\rho(u)$$

(the weak integral; $\mathcal{M}^b(\mathbb{R}_+,\mathbb{C})$ is endowed with vague topology).

We claim that for every bounded function $q \in C^1(\mathbb{R}_+)$ with bounded derivative the following equality holds (we write $b(q)$ instead of $\int_{\mathbb{R}_+} qdb$ in the rest of the proof)

$$b(q) = \int_{\mathbb{R}_+} b^u(q)u^{-1}d\rho(u). \tag{4}$$

In fact, let $(q_n)$ be as in Lemma 1, and $|q_n| < C$, $|q_n'| < C$ for some constant $C > 0$. Putting $p_n(u) := b^u(q_n)$ we have

$$p_n(u) = \int_{\mathbb{R}_+} q_n(r)d_r(a(r-u)-a(r)) = \int_{\mathbb{R}_+} (q_n(r+u)-q_n(r))da(r). \tag{5}$$

Now let $p(u) := b^u(q)$. Then $p_n(u) \to p(u) \quad (n \to \infty)$ pointwise by Lebesgue Theorem. We have $|q_n(r+u) - q_n(r)| \leq Cu$, and $\leq 2C$. If we take $w(u) = \min\{u, 1\}$, then $w \in L^1(u^{-1}d\rho(u))$ and (5) implies that $|p_n(u)| \leq 2\|a\|w(u)$. Thus by the Lebesgue Theorem

$$\int_{\mathbb{R}_+} p_n(u)u^{-1}d\rho(u) \to \int_{\mathbb{R}_+} p(u)u^{-1}d\rho(u)(n \to \infty).$$

On the other hand,

$$\int_{\mathbb{R}_+} p_n(u)u^{-1}d\rho(u) = \int_{\mathbb{R}_+} b^u(q_n)u^{-1}d\rho(u) = b(q_n) \to b(q) \quad (n \to \infty).$$

Then $b(q) = \int_{\mathbb{R}_+} p(u)u^{-1}d\rho(u)$, i. e. (4) holds. In particular, for $q(r) = e^{sr} \quad (s \le 0)$ (4) and (3) imply the equality $h = Lb$ which proves the first statement of the theorem.

To prove the second one, fix a bounded linear functional $f \in X'$, vector $x \in D(A)$, and let $q(r) = f(T(r)x)$. Then $q \in C^1(\mathbb{R}_+)$ and $q$ is bounded together with the derivative $q'(r) = f(T(r)Ax) \quad (r \ge 0)$. For such $q$ equation (4) implies that

$$f\left(\int_{\mathbb{R}_+} Txdb\right) = \int_{\mathbb{R}_+} f(T(r)x)db(r) = \int_{\mathbb{R}_+} f\left(\int_{\mathbb{R}_+} T(r)xdb^u(r)\right) u^{-1}d\rho(u).$$

So by the definition of the weak integral

$$\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} T(r)xdb^u(r)\right) u^{-1}d\rho(u) = \int_{\mathbb{R}_+} Txdb.$$

In addition, the interior integral in the left hand side here exists in the sense of Bochner, and

$$\int_{\mathbb{R}_+} T(r)xdb^u(r) = \int_{[u,\infty)} T(r)xd_r a(r-u) - \int_{\mathbb{R}_+} T(r)xda(r)$$

$$= \int_{\mathbb{R}_+} T(r+u)xda(r) - \int_{\mathbb{R}_+} T(r)xda(r) = (T(u)-I)g(A)x.$$

Therefore for $x \in D(A)$ we have

$$h(A)x = \int_{\mathbb{R}_+} T(r)xdb(r) = \int_{\mathbb{R}_+} (T(u)-I)g(A)xu^{-1}d\rho(u) = \psi(A)g(A)x.$$

Since the operator $h(A)$ is bounded, and, on the other hand, the operator $\psi(A)g(A)$ is closed (as the product of a closed and a bounded operators), the last equality holds for all $x \in X$. In particular, $g(A)X \subset D(\psi(A))$. Finally

$$\|h(A)\| \le \int_{\mathbb{R}_+} \|T(r)\|d|b|(r) \le M\|b\| = MK(a,\rho).$$

The theorem is proved.

**Theorem 2**. *Let $\psi \in \mathcal{T}_0$. Then $\psi \in \mathcal{T}_Y$ if and only if*

$$K(\nu_t, \rho) = O(t^{-1}), \quad t \downarrow 0 \tag{6}$$

*holds (see formulas (1) and (2) for the definitions of $\rho$ and $\nu_t$).*

*Proof.* Let (6) holds. Putting $a = \nu_t$ in Theorem 1 we get that for sufficiently small $t > 0$ the function $h_t = \psi g_t$ has the form $h_t = Lb_t$, where $b_t$ is a bounded measure on $\mathbb{R}_+$, $\|b_t\| = K(\nu_t, \rho)$. In addition, $g_t(A)X \subset D(\psi(A))$ for all $t > 0$ ($\psi(A)$ = generator of the semigroup $g_t(A)$) and

$$\|h_t(A)\| = \|\psi(A)g_t(A)\| \le MK(\nu_t, \rho).$$

Now (6) implies (Y) with $g_t(A)$ instead of $T(t)$.

To prove the converse, consider $X = C_0(\mathbb{R}_+)$ with sup-norm, let $\psi \in \mathcal{T}_Y$, and let $T$ be the $C_0$-semigroup of shifts on $X$, $(T(r)x)(v) = x(v+r)$ (in this concrete situation $A$ is a derivation with appropriate domain). Then, for each $x \in C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+), \quad t > 0$ integration by parts gives

$$y(v) := g_t(A)x(v) = \int_{\mathbb{R}_+} x(v+r)d\nu_t(r) = -\int_{\mathbb{R}_+} x'(v+r)\nu_t(r)dr.$$

Therefore

$$\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} (y(v+u) - y(v))u^{-1}d\rho(u)$$

$$= \int_{\mathbb{R}_+} \left( -\int_{\mathbb{R}_+} x'(v+u+r)\nu_t(r)dr + \int_{\mathbb{R}_+} x'(v+r)\nu_t(r)dr \right) u^{-1}d\rho(u).$$

Since $\nu_t$ is concentrated on $\mathbb{R}_+$, we get

$$\int_{\mathbb{R}_+} x'(v+u+r)\nu_t(r)dr = \int_{\mathbb{R}_+} x'(v+r)\nu_t(r-u)dr,$$

and thus

$$\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} (\nu_t(r) - \nu_t(r-u))x'(v+r)dr \right) u^{-1}d\rho(u).$$

But integration by parts gives since $\nu_t(0) = \nu_t(-u) = 0$,

$$\int_{\mathbb{R}_+} (\nu_t(r) - \nu_t(r-u))x'(v+r)dr = \int_{\mathbb{R}_+} x(v+r)d_r(\nu_t(r-u) - \nu_t(r)).$$

Finally, for each $x \in C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+), \quad v \geq 0$

$$\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} x(v+r)d_r(\nu_t(r-u) - \nu_t(r)) \right) u^{-1}d\rho(u).$$

Taking into account that $t\|\psi(A)g_t(A)\| \leq C$ for some $C > 0$ and all $t \in (0,1]$ we have for our $x$ with $\|x\| = 1$ that $|\psi(A)g_t(A)x(v)| \leq Ct^{-1}$. So for each $v \geq 0$, $t \in (0,1]$

$$\left| \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} x(v+r)d_r(\nu_t(r-u) - \nu_t(r)) \right) u^{-1}d\rho(u) \right| \leq Ct^{-1}.$$

Since $C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+)$, it follows for $v = 0$ that $K(\nu_t, \rho) = O(t^{-1}), \quad t \downarrow 0$, as desired.

### 3. Sufficient conditions for $\psi$ to be in $\mathcal{T}_Y$ in terms of $\psi$

In the following we shall denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}$,

$$\mathcal{F}f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda t} f(t)dt,$$

and by $\mathcal{F}^{-1}$ the inverse of $\mathcal{F}$. Let

$$F_t(\lambda) = e^{t\psi(i\lambda)}\psi(i\lambda) \quad (\mathrm{Im}\lambda \geq 0, t > 0).$$

The restriction $F_t|\mathbb{R}$ will be also denoted by $F_t$.

**Theorem 3**. *Let $\psi \in \mathcal{T}_0$. Assume that*

(i) *the derivative $\partial/\partial y F_t(y)$ exists for a.e. $y \in \mathbb{R}$ and each sufficiently small $t > 0$;*

(ii) *for some $p \in (1, 2]$ functions $F_t$ and $\partial/\partial y F_t$ both belong to $L^p(\mathbb{R})$ for each sufficiently small $t > 0$;*

(iii) *$\mathcal{F}F_t$ is concentrated on $\mathbb{R}_+$ for each sufficiently small $t > 0$;*

(iv) $\|F_t\|_p^{1/q}\|\partial/\partial y F_t\|_p^{1/p} = O(t^{-1})$ *as* $t \downarrow 0$ $(p^{-1} + q^{-1} = 1)$.

*Then $\psi \in \mathcal{T}_Y$.*

*Proof.* First we prove that $f_t := \mathcal{F}F_t \in L^1(\mathbb{R}_+)$, and $F_t = \mathcal{F}^{-1}f_t$. Indeed, $f_t \in L^q(\mathbb{R})$, and $\mathcal{F}(\partial/\partial y F_t)(y) = iyf_t(y) \in L^q(\mathbb{R})$. By Hölder's inequality $f_t(y) = (iyf_t(y))(iy)^{-1} \in L^1(\{|y| > 1\})$, and so $f_t \in L^1(\mathbb{R})$. Now by the Inverse Theorem for the Fourier transform, $F_t(y) = \mathcal{F}^{-1}f_t(y)$ a.e. $y \in \mathbb{R}$, and by the continuity the last equality holds for all $y \in \mathbb{R}$. Therefore we have for the Laplace transform

$$Lf_t(z) = \int\limits_{\mathbb{R}_+} e^{zr}f_t(r)dr = \sqrt{2\pi}e^{t\psi(z)}\psi(z), \quad \mathrm{Re}z \leq 0,$$

because both sides here are analytic on the left half-plane $\Pi_-$, continuous on its closure, and coincide on its boundary $i\mathbb{R}$. In particular, $Lf_t(s) = \sqrt{2\pi}e^{t\psi(s)}\psi(s)$ for all $s \leq 0$. It follows that for an arbitrary exponential polynomial $\phi \in E(\mathbb{R}_+)$, $\phi(r) = \sum_j c_j e^{s_j r}$ $(c_j \in \mathbb{C}, s_j < 0)$ we have

$$\int\limits_{\mathbb{R}_+} \phi(r)f_t(r)dr = \sqrt{2\pi}\sum_j c_j e^{t\psi(s_j)}\psi(s_j).$$

On the other hand,

$$\int\limits_{\mathbb{R}_+} \phi(r)d_r(\nu_t(r-u) - \nu_t(r)) = \int\limits_{[-u,\infty)} \phi(r+u)d\nu_t(r) - \int\limits_{\mathbb{R}_+} \phi(r)d\nu_t(r)$$

$$= \int\limits_{\mathbb{R}_+} (\phi(r+u) - \phi(r))d\nu_t(r) = \sum_j c_j(e^{s_j u} - 1)e^{t\psi(s_j)},$$

and thus

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \phi(r)d_r(\nu_t(r-u) - \nu_t(r))u^{-1}d\rho(u) = \sum_j c_j e^{t\psi(s_j)}\psi(s_j).$$

Now we conclude that ($E(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+)$)

$$K(\nu_t, \rho) = \frac{1}{\sqrt{2\pi}}\sup_{\phi \in S}\left|\int\limits_{\mathbb{R}_+} \phi(r)f_t(r)dr\right| = \frac{1}{\sqrt{2\pi}}\|f_t\|_1.$$

Let $k_t(u) := iuf_t(u)$. Then $k_t = \mathcal{F}(\partial/\partial y F_t)$, and using the Hausdorff-Young inequality we obtain

$$\|f_t\|_q \leq \|F_t\|_p, \quad \|k_t\|_q \leq \|\partial/\partial y F_t\|_p.$$

Next, for any $v > 0$ Hölder's inequality gives

$$\int\limits_{[0,v]} |f_t(u)|du \leq \|f_t\|_q v^{1/p},$$

$$\int\limits_{[v,\infty)} (u|f_t(u)|)u^{-1}du \leq \|k_t\|_q (p-1)^{-1/p} v^{-1/q}.$$

Then for any $v > 0$

$$\|f_t\|_1 \leq \|f_t\|_q v^{1/p} + \|k_t\|_q (p-1)^{-1/p} v^{-1/q} \leq \|F_t\|_p v^{1/p} + \|\partial/\partial y F_t\|_p (p-1)^{-1/p} v^{-1/q}.$$

Therefore, on choosing $v = (p-1)^{1/q} \|\partial/\partial y F_t\|_p / \|F_t\|_p$, it follows that

$$K(\nu_t, \rho) = \frac{1}{\sqrt{2\pi}} \|f_t\|_1 \leq \text{const} \|F_t\|_p^{1/q} \|\partial/\partial y F_t\|_p^{1/p} = O(t^{-1}) \quad \text{as} \quad t \downarrow 0.$$

Application of Theorem 2 completes the proof.

Before formulating the next theorem we note that by [3, Theorem 4] every $\psi \in \mathcal{T}_Y \cap \mathcal{T}_0$ maps $\Pi_-$ into a truncated sector

$$S(\theta, \beta) := (\beta + \{|\arg(-z)| < \theta\}) \cap \Pi_-$$

for some $\beta \geq 0, \theta \in (0, \pi/2)$, and there exist constants $k, k > 0$, and $\gamma, \gamma \in (0,1)$, such that $|\psi(z)| \leq k|z|^\gamma, \quad |z| \geq 1, \quad \text{Re} z \leq 0$. The problem is what one can add to this conditions to obtain (necessary and) sufficient conditions for $\psi$ to be in $\mathcal{T}_Y$. Now we shall deduce the partial answer to this question from Theorem 3.

**Theorem 4**. *Let $\psi \in \mathcal{T}_0$, and assume that the following conditions hold:*
(i) $\psi : \Pi_- \to S(\theta, \beta)$ *for some* $\beta \geq 0, \theta \in (0, \pi/2)$;
*there exist such positive constants $k, b, \alpha, \gamma$ and $R$, that $\alpha \leq \gamma < 1$, $R \geq 1$, and*
(ii) $b|z|^\alpha \leq |\psi(z)| \leq k|z|^\gamma$ *for* $z \in \Pi_-, \quad |z| \geq R$;
(iii) *the function $y \mapsto \psi(iy)$ is differentiable for a. e. $y \in \mathbb{R}$ and*

$$|\psi'(iy)| \leq k|y|^\delta, \quad a.e. \quad y \in \mathbb{R}, \quad |y| \geq R,$$

*for some $\delta \in (\alpha - \gamma - 1, 2\alpha - \gamma - 1)$ if $\alpha < \gamma$, and $\delta = \gamma - 1$ if $\alpha = \gamma$;*
(iv) $\psi'(iy) \in L^p([0,R])$ *for some $p \in (1,2]$ such that $p = \min\{2, (\alpha-\gamma-\delta)^{-1}, (\alpha-\delta-1)/(\gamma-\alpha)\}$ if $\alpha < \gamma$, and $p < \min\{2, (1-\gamma)^{-1}\}$ if $\alpha = \gamma$.*
*Then $\psi \in \mathcal{T}_Y$.*

*Proof.* We shall verify all the conditions of Theorem 3 for $\psi$. Let $a_1 = \max\{|\psi(z)| | z \in \Pi_-, |z| \leq R\}$, $m_1 = \min\{|\psi(z)| - b|z|^\alpha | z \in \Pi_-, |z| \leq R\}$. Then $b|z|^\alpha + a_2 \leq |\psi(z)| \leq k|z|^\gamma + a_1$ for $z \in \Pi_-$, where $a_2 = \min\{0, m_1\}$. Since $\psi(iy) - \beta \in S(\theta, 0)$, we have $-\text{Re}\psi(iy) + \beta \geq \cos\theta(|\psi(iy)| - \beta)$, and $\text{Re}\psi(iy) \leq -c_1|y|^\alpha + c_2$, where $c_1 = b\cos\theta > 0, c_2 \in \mathbb{R}$. It follows that

$$|F_t(y)| \leq e^{c_2 t} e^{-c_1 t|y|^\alpha} (k|y|^\gamma + a_1),$$

and $(p \geq 1)$

$$\|F_t\|_p \leq e^{c_2 t} 2^{1/p} \left( \int\limits_{\mathbb{R}_+} e^{-c_1 pt|y|^\alpha} (k|y|^\gamma + a_1)^p dy \right)^{1/p}.$$

Putting $x = ty^\alpha$ we get for some constant $c_3 > 0$

$$\|F_t\|_p \le c_3 e^{c_2 t} t^{-\gamma/\alpha - 1/\alpha p} \left( \int\limits_{\mathbb{R}_+} e^{-c_1 p x} (k x^{\gamma/\alpha} + a_1 t^{\gamma/\alpha})^p x^{1/\alpha - 1} dx \right)^{1/p}.$$

The integral converges for all $t \ge 0, p \ge 1$, and by B. Levi's Theorem

$$\|F_t\|_p = O(1) t^{-\gamma/\alpha - 1/\alpha p} \quad \text{as} \quad t \downarrow 0. \tag{7}$$

Let $\alpha < \gamma$, $p = \min\{2, (\alpha-\gamma-\delta)^{-1}, (\alpha-\delta-1)/(\gamma-\alpha)\}$, $\delta \in (\alpha-\gamma-1, 2\alpha-\gamma-1)$. Since

$$|\partial/\partial y F_t(y)| \le e^{c_2 t} e^{-c_1 t |y|^\alpha} (tk|y|^\gamma + t a_1 + 1) |\psi'(iy)|,$$

we have

$$\|\partial/\partial y F_t\|_p \le e^{c_2 t} 2^{1/p} \Big( \int\limits_{[0,R]} e^{-c_1 p t |y|^\alpha} (tk|y|^\gamma + t a_1 + 1)^p |\psi'(iy)|^p dy$$

$$+ k^p \int\limits_{[R,\infty)} e^{-c_1 p t |y|^\alpha} (tk|y|^\gamma + t a_1 + 1)^p y^{\delta p} dy \Big)^{1/p}.$$

Putting $x = t y^\alpha$ in the second integral we get

$$\|\partial/\partial y F_t\|_p \le e^{c_2 t} 2^{1/p} t^{-\frac{\gamma+\delta}{\alpha} - \frac{1}{\alpha p} + 1} \Big( t^{(\frac{\gamma+\delta}{\alpha} + \frac{1}{\alpha p} - 1)p} \int\limits_{[0,R]} e^{-c_1 p t |y|^\alpha} (tk|y|^\gamma + t a_1 + 1)^p |\psi'(iy)|^p dy$$

$$+ k^p \alpha^{-1} \int\limits_{[tR^\alpha,\infty)} e^{-c_1 p x} (k x^{\frac{\gamma}{\alpha}} + t^{\frac{\gamma}{\alpha} - 1}(t a_1 + 1))^p x^{\frac{\delta p + 1}{\alpha} - 1} dx \Big)^{1/p}. \tag{8}$$

The second integral in (8) converges for all $t \ge 0$ because $(\gamma+\delta)p/\alpha + 1/\alpha - 1 > -1$ for our $p$ and $\delta$. Note that $(\gamma+\delta)/\alpha + 1/\alpha p - 1 \ge 0$. Therefore (8) implies

$$\|\partial/\partial y F_t\|_p = O(1) t^{-(\gamma+\delta)/\alpha - 1/\alpha p + 1} \quad \text{as} \quad t \downarrow 0. \tag{9}$$

It follows from (7) and (9) that for our $\delta$ we have

$$\|F_t\|_p^{1/q} \|\partial/\partial y F_t\|_p^{1/p} = O(1) t^{-\gamma/\alpha - 1/\alpha p - (\delta/\alpha - 1)/p} = O(t^{-1}) \quad \text{as} \quad t \downarrow 0,$$

because $\gamma/\alpha + 1/\alpha p + (\delta/\alpha - 1)/p \le 1$.

The case $\gamma = \alpha, \delta = \gamma - 1, 1 < p < \min\{2, (1-\gamma)^{-1}\}$ can be examined in the same manner.

Finally since $\psi(i\lambda) - \beta \in S(\theta, 0)$ for $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda \ge 0$, we have for such $\lambda$ (as above)

$$|F_t(\lambda)| \le e^{c_2 t} e^{-c_1 t |\lambda|^\alpha} (k|\lambda|^\gamma + a_1).$$

Then for $t > 0$ ($\lambda = s + iy, y > 0$)

$$\int\limits_{\mathbb{R}} |F_t(s+iy)| ds \le 2 e^{c_2 t} \int\limits_{\mathbb{R}_+} e^{-c_1 t (s^2+y^2)^{\alpha/2}} (k(s^2+y^2)^{\gamma/2} + a_1) ds$$

$$\overset{[s^2+y^2=v]}{=} e^{c_2 t} \int\limits_{[y^2,\infty)} e^{-c_1 t v^{\alpha/2}} (k v^{\gamma/2} + a_1)(v - y^2)^{-1/2} dv.$$

But

$$\int\limits_{[y^2,y^2+1]} e^{-c_1tv^{\alpha/2}}(kv^{\gamma/2}+a_1)(v-y^2)^{-1/2}dv \le \max_{v\ge0} e^{-c_1tv^{\alpha/2}}(kv^{\gamma/2}+a_1)\int\limits_{[0,1]} u^{-1/2}du.$$

Furthermore

$$\int\limits_{[y^2+1,\infty)} e^{-c_1tv^{\alpha/2}}(kv^{\gamma/2}+a_1)(v-y^2)^{-1/2}dv \le \int\limits_{[1,\infty)} e^{-c_1tv^{\alpha/2}}(kv^{\gamma/2}+a_1)dv.$$

Thus $F_t$ belongs to the Hardy class $H^1(\{\mathrm{Im}\lambda>0\})$ for all $t>0$ and therefore $\mathcal{F}F_t$ is concentrated on $\mathbb{R}_+$. This completes the proof.

**Corollary 1**. *Let $\psi\in\mathcal{T}_0$, and assume that the following conditions hold:*
(i) *$\psi:\Pi_-\to S(\theta,\beta)$ for some $\beta\ge0,\theta\in(0,\pi/2)$;*
(ii) *$\psi(z)\asymp z^\gamma$ for some $\gamma\in(0,1)$ $(z\to\infty,\quad z\in\Pi_-)$;*
(iii) *the function $y\mapsto\psi(iy)$ is differentiable for a. e. $y\in\mathbb{R}$ and*

$$|\psi'(iy)|\le k|y|^{\gamma-1},\quad a.e.\quad y\in\mathbb{R}.$$

*Then $\psi\in\mathcal{T}_Y$.*

*Example 1* [17]. Let $\psi(z)=c^\alpha-(c-z)^\alpha,\quad \alpha\in(0,1),\quad c\ge0$. In this case, all the conditions of Corollary 1 (and hence of Theorems 3 and 4) are clear.

Now we shall give an example of a function $\psi\in\mathcal{T}_0$ that satisfies all the conditions of Theorem 4, but conditions of the Theorem in [6] do not hold for $-\psi(-x)$.

*Example 2.* Let $0<\alpha<\beta<1$, and

$$\psi(z)=-(-z)^\alpha+(e^{-(-z)^\beta}-1).$$

Since the summands map $\Pi_-$ into a sector and into a truncated sector respectively, the condition (i) of Theorem 4 holds. It is easy to verify that $\psi(z)\sim z^\alpha$ as $z\to\infty, z\in\Pi_-$, $\psi'(iy)\sim\alpha|y|^{\alpha-1}$ as $y\to\infty$. Finally (iv) holds for $p\in(1,\min\{2,(1-\alpha)^{-1}\})$. At the same time, $-\psi(-x)$ is not regularly varying.

### 4. Further sufficient conditions for $\psi$ to be in $\mathcal{T}_Y$

In this section, we shall deduce further conditions from Theorem 2, that are sufficient for $\psi\in\mathcal{T}_Y$.

**Theorem 5**. *Let $\psi\in\mathcal{T}_0$ and the function $r\mapsto\nu_t([r-u,r))$ is monotone decreasing on $[u,+\infty)$ $(u\ge0)$ for each sufficiently small $t>0$. If*

$$\int\limits_{\mathbb{R}_+}\nu_t([0,u))u^{-1}d\rho(u)=O(t^{-1})\quad\text{as}\quad t\downarrow0,$$

*then $\psi\in\mathcal{T}_Y$.*

*Proof.* Let $a\in\mathcal{M}^b(\mathbb{R}_+,\mathbb{R}_+)$, and the function $r\mapsto a([r-u,r))$ is monotone decreasing on $[u,+\infty)$ $(u\ge0)$. Since $\lim_{r\to+\infty}a([r-u,r))=0$ for every $u>0$, for all and $\phi\in E(\mathbb{R}_+)$ with $\sup|\phi|\le1$ we find ($a(r)=a([0,r))$ for $r>0$, and $a(r)=0$ for $r\in(-\infty,0]$)

$$|\int\limits_{\mathbb{R}_+}\phi(r)d_r(a(r-u)-a(r))|\le Var_{r\in\mathbb{R}_+}(a(r-u)-a(r))=$$

$$Var_{r\in[0,u)}a(r)+Var_{r\in[u,+\infty)}(a(r-u)-a(r))=2a([0,u)).$$

Thus

$$K(a,\rho)\le 2\int_{\mathbb{R}_+} a([0,u))u^{-1}d\rho(u).$$

It remains to put here $a=\nu_t$ and to apply Theorem 2.

*Example 3* (cf. [3, Example 1]). Let for $b>0$

$$\psi(z)=\log b-\log(b-z).$$

It is well known that $d\rho(u)=e^{-bu}du$ and

$$e^{t\psi(s)}=b^t(b-s)^{-t}=b^t\Gamma(t)^{-1}\int_{\mathbb{R}_+} e^{sr}r^{t-1}e^{-br}dr.$$

So $d\nu_t(r)=b^t\Gamma(t)^{-1}r^{t-1}e^{-br}dr$, and $\nu_t$ has monotone decreasing density for $t\in(0,1)$. Therefore

$$\int_{\mathbb{R}_+}\nu_t([0,u))u^{-1}d\rho(u)=\int_{\mathbb{R}_+}\left(\int_{[0,u)} b^t\Gamma(t)^{-1}r^{t-1}e^{-br}dr\right)u^{-1}e^{-bu}du\le$$

$$b^t\Gamma(t)^{-1}\int_{\mathbb{R}_+}\left(\int_{[0,u)} r^{t-1}dr\right)u^{-1}e^{-bu}du=\frac{1}{t}.$$

Thus $\psi\in\mathcal{T}_Y$ by Theorem 5.

*Example 4* (cf. [13]). Let

$$\psi(s)=\mathrm{acosh}b-\mathrm{acosh}(b-s)\quad(b\ge 1, s\le 0).$$

Since $\psi\in\mathcal{T}_Y$ implies $-\psi(-c)+\psi(s-c)\in\mathcal{T}_Y$ for all $c\ge 0$, one can to restrict ourselves to the case $b=1$. In this case, $\psi\in\mathcal{T}_0$ with $d\rho(u)=e^{-u}I_0(u)du$ (the corresponding integral representation (1) can be verified by differentiation under the integral sign), and $e^{t\psi(s)}=Lf_t(s)$ with $f_t(r)=tr^{-1}e^{-r}I_t(r),\quad r>0$ ($I_t$ denotes the Bessel function of the first kind). Hence, $d\nu_t(r)=f_t(r)dr$, and $\nu_t$ has monotone decreasing density for $t\in(0,1)$ (see [13]). The calculations from Example 3 in [13] show, that the conditions of Theorem 5 hold. So, $\psi\in\mathcal{T}_Y$.

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