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The Markov–Stieltjes transform on Hardy and Lebesgue spaces

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ABSTRACT

We prove that the Markov–Stieltjes transform is a bounded non compact Hankel operator on Hardy space H^p with Hilbert matrix with respect to the standard Schauder basis of H^p and a bounded non-compact operator on Lebesgue space $L^p[0, 1]$ for $p \in (1, \infty)$ and obtain estimates for its norm in this spaces. It is shown that the Markov–Stieltjes transform on $L^2(0, 1)$ is unitary equivalent to the Markov–Stieltjes transform on H^2 . Inverse formulas and operational properties for this transform are obtained.

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1. Introduction

Much work has been done during last years on the theory of integral transforms of functions of one real variable and in particular on convolution and inversion theorems for such transforms and their applications to integral equations (see, e.g. [1–7] and the bibliography cited therein). This paper is devoted to the Markov–Stieltjes transform S of functions on $(0, 1)$. The last transform was introduced in [8, Chapter 6] as a special case of the Stieltjes transform of measures on general semigroups. The terminology goes back to approximation theory, see, e.g. [9, p. 14, 10, 11]. We give inverse formulas for this transform and formulate its operational properties. The main goal of this paper is to study the Markov–Stieltjes transform as an operator on Hardy spaces H^p for $p \in (1, \infty]$ and Lebesgue spaces $L^p(0, 1)$ for $p \in (1, \infty)$. We prove that S is a bounded non-compact Hankel operator on Hardy space H^p with Hilbert matrix with respect to the standard Schauder basis of H^p for $p \in (1, \infty)$, a bounded non compact operator from H^∞ to BMOA, and a bounded operator in ℓ_A^p and give estimates for the norm of S in this cases. We show also that S is a bounded non compact operator on Lebesgue space $L^p(0, 1)$ for $p \in (1, \infty)$ and obtain estimates for its norm in this spaces, too. It is shown also that the Markov–Stieltjes transform on $L^2(0, 1)$ is unitary equivalent to the Markov–Stieltjes transform on H^2 . As a corollary the norm and the spectrum of S as an operator on $L^2(0, 1)$ are obtained.

Definition 1.1 (8, Chapter 6, 7): The Markov–Stieltjes transform of a function $f \in L^1(0, 1)$ is defined by the formula

$$Sf(z) := \int_0^1 \frac{f(t)}{1-tz} dt \quad (1.1)$$

(we write also $S_{t \rightarrow z}\{f(t)\}$ instead of $Sf(z)$). Obviously, for $z \notin [1, \infty)$ this Lebesgue integral exists and represents an analytic function f^* in the domain $\mathbb{C} \setminus [1, \infty)$. For $z \in [1, \infty)$ the integral in (1.1) is understood as a Cauchy principle value integral, i.e.

$$Sf(z) := V.P. \int_0^1 \frac{f(t)}{1-tz} dt := \lim_{\varepsilon \rightarrow 0^+} \int_{\{t \in (0,1): |t-1/z| > \varepsilon\}} \frac{f(t)}{1-tz} dt. \quad (1.2)$$

The limit in the right-hand side of (1.2) exists for almost all $z \in [1, \infty)$. In fact, $Sf(z) = (\pi/z)(Hf_1)(1/z)$, where Hf_1 stands for the Hilbert transform of the function $f_1(x) := f(x)$ for $x \in (0, 1)$ and $f_1(x) := 0$ otherwise. So, the application of Loomis theorem (see, e.g. [12, p. 239]) proves the assertion.

The following example shows that in general $\mathbb{C} \setminus [1, \infty)$ is the domain of holomorphy of Sf .

Example 1.2: Putting $t = x^2/(1+x^2)$ it is easy to verify that $S_{t \rightarrow z}\{(t(1-t))^{-1/2}\}$ equals to $\pi(1-z)^{-1/2}$ for $z \notin [1, \infty)$ and equals to zero otherwise.

As was mentioned above the study of Markov–Stieltjes transform as a function is important in approximation theory ([9, p. 226, 11]).

This transform is useful in solving some singular integral equations, too. In [7] for $f \in L^p(0, 1), g \in L^q(0, 1)$ ($1 < p, q < \infty, 1/p + 1/q < 1$) the following binary operation was considered (this operation was introduced for the first time in [1, p. 220, formula (24.38)])

$$f \otimes g(t) = tf(t) \int_0^1 \frac{f(u)}{t-u} du + tg(t) \int_0^1 \frac{g(u)}{t-u} du,$$

where the integrals are understood as their Cauchy principal values, and using methods developed by Srivastava and Vu Kim Tuan [2] a convolution theorem for the Markov–Stieltjes transform in the form

$$S(f \otimes g) = (Sf) \cdot (Sg)$$

was proved. Arguing as in [2] it was also shown in [7] that the equation

$$x(t) + \lambda \int_0^1 \frac{x(u)}{t-u} du = g(t) \quad (\lambda \neq 0)$$

where g is prescribed and x is an unknown function to be determined has (for appropriate g) the unique solution

$$x(u) = \cos(\alpha\pi) S_{s \rightarrow u}^{-1} \left\{ (1-s)^\alpha S_{t \rightarrow s} \left\{ \frac{g(t)t^\alpha}{(1-t)^\alpha} \right\} \right\},$$

α being a (unique) root of the equation $\tan(\alpha\pi) = \lambda\pi$, $0 < \operatorname{Re} \alpha < 1$ (for inversion formulas for the Markov–Stieltjes transform see the following section).

2. Inversion formulas

A complex inversion formula for the Markov–Stieltjes transform looks as follows.

Theorem 2.1: *Let $f \in L^1(0, 1)$, $0 < t < 1$, and $f(t \pm 0)$ exist. If $ff^* = Sf$, then*

$$\frac{f(t+0) + f(t-0)}{2} = \frac{1}{2\pi i} \lim_{\eta \rightarrow 0^+} \left(\frac{1}{t - i\eta} f^* \left(\frac{1}{t - i\eta} \right) - \frac{1}{t + i\eta} f^* \left(\frac{1}{t + i\eta} \right) \right).$$

Proof: It is easy to verify that

$$\frac{1}{2\pi i} \left(\frac{1}{t - i\eta} f^* \left(\frac{1}{t - i\eta} \right) - \frac{1}{t + i\eta} f^* \left(\frac{1}{t + i\eta} \right) \right) = \frac{1}{\pi} \int_0^1 \frac{\eta}{(t-s)^2 + \eta^2} f(s) ds.$$

Application of Widder [13, p. 338, Lemma 7.2] completes the proof. ■

Now we shall formulate also a real inversion formula for the Markov–Stieltjes transform.

Theorem 2.2 ([7]): *Let $f \in L^p(0, 1)$, $1 < p < \infty$. The Markov–Stieltjes transform $f^*(x) = Sf(x)$ exists for a.e. $x \in \mathbb{R}$ and*

$$f(t) = \frac{1}{\pi^2} \text{V.P.} \int_{-\infty}^{\infty} \frac{f^*(x)}{1 - tx} dx.$$

Proof: It follows, e.g. from the above-mentioned equality $Sf(z) = (\pi/z)(Hf_1)(1/z)$ and the inversion formula for the Hilbert transform (see [7] for details). ■

3. Operational properties of the Markov–Stieltjes transforms

The following properties hold for the Markov–Stieltjes transform (cf., e.g. [3, p. 394]).

If $ff^* = Sf$, then

(1)

$$S_{t \rightarrow z} \{f(1-t)\} = \frac{1}{1-z} f^* \left(\frac{z}{z-1} \right);$$

(2)

$$S_{t \rightarrow z} \{f(at)\} = \frac{1}{a} f^* \left(\frac{z}{a} \right) \quad (a > 1);$$

(3)

$$S_{t \rightarrow z} \{tf(t)\} = \frac{1}{z} \left(f^*(z) - \int_0^1 f(t) dt \right) \quad (f \in L^1(0, 1)).$$

In particular, $S_{t \rightarrow z} \{tf(t)\} = (1/z)f^*(z)$ if $\int_0^1 f(t) dt = 0$;

(4)

$$S_{t \rightarrow z} \left\{ \frac{f(t)}{t+a} \right\} = \frac{z}{1+az} f^*(z) - \frac{1}{1+az} \int_0^1 \frac{f(t)}{t+a} dt \quad \left(\frac{f(t)}{t+a} \in L^1(0, 1) \right).$$

In particular, $S_{t \rightarrow z} \{f(t)/(t+a)\} = (z/(1+az))f^*(z)$ if $\int_0^1 f(t)/(t+a) dt = 0$;

(5)

$$S_{t \rightarrow z} \left\{ \frac{d}{dt} f(t) \right\} = -\frac{d}{dz} f^*(z) + \frac{f(1)}{1-z} - f(0) \quad \text{if } f \in C^1[0, 1], z \notin [1, \infty).$$

In particular, $S_{t \rightarrow z} \{(d/dt)f(t)\} = -(d/dz)f^*(z)$ if in addition $f(0) = f(1) = 0$;

(6)

$$S_{t \rightarrow z} \left\{ \int_0^t f(t) dt \right\} = -\int_0^z f^*(z) dz - \left(\int_0^1 f(t) dt \right) \log(1-z) + \int_0^1 (1-t)f(t) dt.$$

In particular, $S_{t \rightarrow z} \{ \int_0^t f(t) dt \} = -\int_0^z f^*(z) dz$ if $\int_0^1 f(t) dt = \int_0^1 tf(t) dt = 0$.

We omit simple proofs of this properties.

4. Markov–Stieltjes transform as an operator on Hardy spaces

In this section we identify the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ (\mathbb{D} stands for the open unit disk and \mathbb{T} for the unit circle ; see, e.g. [14]) and frequently use the notation H^p for this space, we denote also by $\chi_n(z) := z^n$ ($n \in \mathbb{Z}_+$) the standard (Schauder) basis of $H^p(\mathbb{D})$.

Definition 4.1: Following Böttcher and Silbermann [15, p. 52] for $b \in L^\infty(\mathbb{T})$ we define the Hankel operator $H(b)$ on $H^p(\mathbb{T})$ ($1 < p < \infty$) by

$$H(b) : H^p \rightarrow H^p : f \mapsto PM(b)(I - P)Jf,$$

where

$$P : \sum_{n=-N}^N f_n \chi_n \mapsto \sum_{n=0}^N f_n \chi_n,$$

$$M(b) : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T}) : f \mapsto bf,$$

$$J : f(t) \mapsto \frac{1}{t} f\left(\frac{1}{t}\right) = \sum_{n \in \mathbb{Z}} f_n \chi_{-n-1}(t) \quad (t \in \mathbb{T}),$$

where $f = \sum_{n \in \mathbb{Z}} f_n \chi_n$.

The function b is called the symbol of the Hankel operator $H(b)$.

The next theorem describes the properties of S as an operator on $H^p(\mathbb{D})$ (this means that S is defined by the formula (1.1), where $f \in H^p(\mathbb{D}), z \in \mathbb{D}$).

Theorem 4.2:

(1) *The Markov–Stieltjes transform S is a bounded non-compact Hankel operator on $H^p(\mathbb{D})$ ($1 < p < \infty$) and has Hilbert matrix with respect to the standard basis. Moreover, the following estimates hold:*

$$\pi \leq \|S\|_{H^p \rightarrow H^p} \leq \frac{\pi}{\sin \frac{\pi}{\max\{p,q\}}}. \tag{4.1}$$

In particular, if $p = 2$ then the norm and the essential norm of S equal to π , and the spectrum and the essential spectrum of S equal to $[0, \pi]$.

(2) The Markov–Stieltjes transform S is a bounded non compact Hankel operator from $H^\infty(\mathbb{T})$ to $BMOA(\mathbb{T})$, and

$$\|S\|_{H^\infty \rightarrow BMOA} \leq \pi \|P\|_{L^\infty \rightarrow BMOA}.$$

Proof: (1) First note, that for $f \in H^p(\mathbb{D})$ the Fejer–Riesz inequality (see, e.g. [14, Theorem 3.13]) implies that

$$\|f|(0, 1)\|_{L^p(0,1)} \leq \pi^{1/p} \|f\|_{H^p}. \quad (4.2)$$

It follows that the restriction $f|(0, 1)$ belongs to $L^p(0, 1)$ and therefore the Lebesgue integral in (1) exists for all $p \in (1, \infty)$ and $z \in \mathbb{D}$.

Next, since for all $z \in \mathbb{D}$

$$(S\chi_n)(z) = \int_0^1 \frac{t^n}{1-tz} dt = \sum_{m=0}^{\infty} z^m \int_0^1 t^{n+m} dt = \sum_{m=0}^{\infty} \frac{\chi_m(z)}{n+m+1}, \quad (4.3)$$

operator S is Hankel and has Hilbert matrix $\Gamma = (1/n+m+1)_{m,n=0}^{\infty}$ with respect to the standard basis $(\chi_n)_{n \in \mathbb{Z}_+}$ of $H^p(\mathbb{D})$. Indeed, by formula (4.3) and Parseval’s formula,

$$\begin{aligned} \left\| S\chi_n - \sum_{m=0}^M \frac{\chi_m}{n+m+1} \right\|_{H^2}^2 &= \left\| \sum_{m=M+1}^{\infty} \frac{\chi_m}{n+m+1} \right\|_{H^2}^2 \\ &= \sum_{m=M+1}^{\infty} \frac{1}{(n+m+1)^2} \rightarrow 0 \quad (M \rightarrow \infty). \end{aligned}$$

This implies that $S\chi_n = \sum_{m=0}^{\infty} \chi_m/(n+m+1)$ in the sense of H^2 and therefore $\langle S\chi_j, \chi_k \rangle = 1/(j+k+1)$ for all $j, k \geq 0$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in H^2 .

Now we use the following remark to the Nehari theorem. If the n th Fourier coefficient of a function $a \in H^p(\mathbb{D})$ equals to a_n for $n \in \mathbb{N}$, and the operator A in $H^p(\mathbb{T})$ satisfies $\langle A\chi_j, \chi_k \rangle = a_{j+k+1}$ for all $j, k \geq 0$ then A is bounded on $H^p(\mathbb{T})$ if (and only if) $a \in BMO$, the space of functions of bounded mean oscillation on \mathbb{T} [15, p. 55]. But

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_n(z),$$

and the function $a(z) := -\log(1-z)$ belongs to BMO (a is analytic in \mathbb{D} and its imaginary part belongs to $L^\infty(\mathbb{T})$, thus, a has the form $f + \tilde{g}$, where $f, g \in L^\infty(\mathbb{T})$ and \tilde{g} is the harmonic conjugate of g). It follows that $a \in H^p(\mathbb{D})$, since $BMO \subset L^p(\mathbb{T})$. Now by the previous remark to the Nehari theorem, S is bounded on $H^p(\mathbb{D})$.

Moreover, if b is the symbol of S the Nehari theorem [15, Theorem 2.11] implies

$$\text{dist}_{L^\infty}(b, \overline{H^\infty}) \leq \|S\|_{H^p \rightarrow H^p} \leq c_p \text{dist}_{L^\infty}(b, \overline{H^\infty}),$$

where

$$\text{dist}_{L^\infty}(b, \overline{H^\infty}) = \inf\{\|b - f\|_{L^\infty} : f \in \overline{H^\infty}\}, \quad c_p = \frac{1}{\sin \frac{\pi}{\max\{p,q\}}}$$

(see, e.g. [15, p. 32]). But the symbol of the Hankel operator S on H^p does not depend of p (see the proof of the Nehari theorem in [15]). So for $p = 2$, we have

$$\|S\|_{H^2 \rightarrow H^2} = \text{dist}_{L^\infty}(b, \overline{H^\infty}).$$

On the other hand, it is known that the norm and the essential norm of the Hankel operator on the space $H^2(\mathbb{T})$ with Hilbert matrix with respect to the standard basis equal to π (and therefore $\text{dist}_{L^\infty}(b, \overline{H^\infty}) = \pi$; see, e.g. [16, p. 36]), and its spectrum and essential spectrum equal to $[0, \pi]$ (see, e.g. [16, p. 575, Theorem 1.7]).

To prove (1) it remains to show that S is non compact on H^p ($1 < p < \infty$). For this we recall that the symbol of the Hankel operator on the space $H^2(\mathbb{T})$ with Hilbert matrix with respect to the standard basis is $b(e^{it}) = ie^{-it}(\pi - t)$, $0 \leq t < 2\pi$ (see, e.g. [16, p. 6]). According to the Hartman theorem [15, p. 80] if the operator $S = H(b)$ is compact then $b \in C(\mathbb{T}) + \overline{H^\infty}$. But this inclusion contradicts the Lindelöf theorem on one-sided limits of H^∞ -functions (see, e.g. [17, Corollary 5.3.5]). This completes the proof of the first part of the theorem.

(2) As it was shown above $Sf = PM(b)(I - P)Jf$ for $f \in H^p$. Since $J : H^\infty \rightarrow (H^2)^\perp$, it follows that $Sf = PM(b)Jf$ for $f \in H^\infty$. Moreover,

$$J : H^\infty \rightarrow L^\infty, \quad \|J\|_{H^\infty \rightarrow L^\infty} = 1,$$

and

$$M(b) : L^\infty \rightarrow L^\infty, \quad \|M(b)\|_{L^\infty \rightarrow L^\infty} = \|b\|_{L^\infty} = \pi.$$

It is also known (see, e.g. [18, Theorem 8.3.10]) that P is a bounded operator from L^∞ to BMOA. This implies that S is a bounded operator from H^∞ to BMOA and

$$\|S\|_{H^\infty \rightarrow \text{BMOA}} \leq \|P\|_{L^\infty \rightarrow \text{BMOA}} \|M(b)\|_{L^\infty \rightarrow L^\infty} \|J\|_{H^\infty \rightarrow L^\infty} = \pi \|P\|_{L^\infty \rightarrow \text{BMOA}}.$$

Finally, since $b \notin C(\mathbb{T}) + \overline{H^\infty}$, the operator $S : H^\infty(\mathbb{T}) \rightarrow \text{BMOA}(\mathbb{T})$ is non compact by the main result of the paper [19]. ■

Results like the previous theorem **may** have applications to approximation theory. Let \mathbf{r}_n be the set of rational functions of order at most n whose poles lies outside of \mathbb{D} , X a Banach space of functions defined on some set $E \subseteq \mathbb{D}$ and $XR_n(f) = \inf_{r \in \mathbf{r}_n} \|f - r\|_X$ the degree of approximation of a function f from X by rationals from \mathbf{r}_n . By the triangle inequality,

$$|XR_n(f) - XR_n(g)| \leq \|f - g\|_X \quad (f, g \in X). \tag{4.4}$$

Corollary 4.3: *The map $f \mapsto H^p R_n(Sf)$ is continuous on H^p .*

Proof: Indeed, the map $g \mapsto H^p R_n g$ is continuous on H^p by the inequality (4.4). ■

The following corollary is a generalization (for a Hausdorff moment problem) of a result due to Zhu [20, p. 372, Proposition 9].

Corollary 4.4: Let $1 < p \leq 2$. For $f \in H^p$ we let Tf be the sequence (c_n) defined by

$$c_n = \int_0^1 f(t)t^n dt, \quad n \in \mathbb{Z}_+.$$

Then T is a bounded linear operator from H^p to ℓ^q ($1/p + 1/q = 1$) and $\|T\|_{H^p \rightarrow \ell^q} \leq \pi / \sin(\pi/q)$.

Proof: Since

$$Sf(z) = \int_0^1 f(t) \sum_{n=0}^{\infty} (tz)^n dt = \sum_{n=0}^{\infty} c_n z^n,$$

Theorem 6.1 from [14] and Theorem 4.2 imply that

$$\|Tf\|_{\ell^q} \leq \|Sf\|_{H^p} \leq \frac{\pi}{\sin \frac{\pi}{q}} \|f\|_{H^p}. \quad \blacksquare$$

For the following corollary recalls that the Bergman space L_a^2 consists of such functions $f(z) = \sum_{m=0}^{\infty} f_m z^m$ that are holomorphic in \mathbb{D} and

$$\|f\|_{L_a^2} := \left(\sum_{m=0}^{\infty} \frac{|f_m|^2}{m+1} \right)^{1/2} < \infty,$$

the sequence $\xi_n := \sqrt{n+1} \chi_n$ forms an orthonormal basis for L_a^2 (see, e.g. [18]).

Corollary 4.5: The Markov–Stieltjes transform S is an unbounded densely defined operator on the Bergman space L_a^2 .

Proof: Indeed, by the formula (4.3)

$$S\xi_n(z) = \sum_{m=0}^{\infty} \frac{\sqrt{n+1}}{m+n+1} z^m.$$

So, the matrix (a_{jk}) of S with respect to the basis (ξ_j) is

$$a_{jk} = \frac{\sqrt{k+1}}{\sqrt{j+1}(k+j+1)}.$$

Since $\sum_{k=0}^{\infty} |a_{jk}|^2 = \infty$, the operator S is unbounded. It remains to note that H^2 is dense in L_a^2 . ■

5. The Markov–Stieltjes transform as an operator on Lebesgue spaces

In the following theorem S denotes the Markov–Stieltjes transform on $L^p(0, 1)$ ($1 < p < \infty$). In other words, S is defined by the formula (1.1), where $f \in L^p(0, 1)$, $z \in (0, 1)$.

Theorem 5.1:

- (1) *The Markov–Stieltjes transform is a bounded non compact operator on $L^p(0, 1)$ ($1 < p < \infty$). Moreover, the following estimates hold:*

$$\frac{\pi}{\sin \frac{\pi}{p}} \leq \|S\|_{L^p \rightarrow L^p} \leq \pi \cot \frac{\pi}{2 \max\{p, q\}}.$$

- (2) *The Markov–Stieltjes transform on $L^2(0, 1)$ is unitarily equivalent to the Markov–Stieltjes transform on $H^2(\mathbb{D})$. In particular, the norm and the essential norm of S equal to π , and the spectrum and the essential spectrum of S equal to $[0, \pi]$.*

Proof: We begin with the case $1 < p \leq 2$. As was mentioned in Section 1, $Sf(z) = y\pi Hf_1(y)$ where $z = 1/y$ ($y > 0$), H stands for the Hilbert transform of functions on \mathbb{R} , and the function $f_1(t) := f(t)$ for $t \in (0, 1)$ and $f(t) := 0$ for $t \in \mathbb{R} \setminus (0, 1)$ belongs to $L^p(\mathbb{R})$. Now the M. Riesz inequality for the Hilbert transform implies for $1 < p \leq 2$ that

$$\begin{aligned} \|Sf\|_{L^p(0,1)} &= \left(\int_0^1 \left| \frac{\pi}{z} Hf_1 \left(\frac{1}{z} \right) \right|^p dz \right)^{1/p} = \left(\int_1^\infty \frac{1}{y^2} |y\pi Hf_1(y)|^p dy \right)^{1/p} \\ &= \pi \left(\int_1^\infty \frac{1}{y^{2-p}} |Hf_1(y)|^p dy \right)^{1/p} \leq \pi \left(\int_1^\infty |Hf_1(y)|^p dy \right)^{1/p} \\ &\leq \pi \|Hf_1\|_{L^p(\mathbb{R})} \leq \pi A_p \|f\|_{L^p(0,1)}. \end{aligned}$$

Since (see, e.g. [12])

$$A_p = \begin{cases} \tan \frac{\pi}{2p}, & 1 < p \leq 2, \\ \cot \frac{\pi}{2p}, & p > 2, \end{cases}$$

we have

$$\|S\|_{L^p \rightarrow L^p} \leq \pi \cot \frac{\pi}{2 \max\{p, q\}}.$$

In the case $p > 2$ using standard duality arguments, this inequality, and the Hölder inequality, we get (below for $f, g \in L^p(0, 1)$, we put $\langle f, g \rangle := \int_0^1 f\bar{g} dt$, $A_q := A_p$, $1/p + 1/q = 1$)

$$\begin{aligned} \|Sf\|_{L^p} &= \sup\{\langle Sf, g \rangle : g \in L^q, \|g\|_{L^q} \leq 1\} = \sup\{\langle S\bar{g}, \bar{f} \rangle : g \in L^q, \|g\|_{L^q} \leq 1\} \\ &\leq \sup\{\|S\bar{g}\|_{L^q} \|\bar{f}\|_{L^p} : g \in L^q, \|g\|_{L^q} \leq 1\} \\ &\leq \sup\{A_q \pi \|g\|_{L^q} \|f\|_{L^p} : g \in L^q, \|g\|_{L^q} \leq 1\} \leq A_q \pi \|f\|_{L^p}. \end{aligned}$$

This proves the right-hand side of the desired inequality.

To prove the left-hand side of this inequality, consider the function

$$f_\gamma(t) := \left(\frac{t}{1-t} \right)^\gamma, \gamma \in \left(-\frac{1}{p}, \frac{1}{p} \right).$$

Then

$$\|f_\gamma\|_{L^p}^p = \int_0^1 \left| \frac{t}{1-t} \right|^{p\gamma} dt = B(1+p\gamma, 1-p\gamma) = \frac{\pi p\gamma}{\sin \pi p\gamma}.$$

Using [21, Section 2.2.6, formula 5], we have

$$Sf_\gamma(t) = -\frac{\pi}{\sin \pi \gamma} \frac{1}{z} \left(1 - \frac{1}{(1-z)^\gamma} \right).$$

Therefore

$$\|Sf_\gamma\|_{L^p}^p = \left| \frac{\pi}{\sin \pi \gamma} \right|^p \int_0^1 \frac{1}{z^p} \left| 1 - \frac{1}{(1-z)^\gamma} \right|^p dz = \left(\frac{\pi}{\sin \pi \gamma} \right)^p \int_0^1 \left(\frac{1-x^\gamma}{1-x} \right)^p \frac{dx}{x^{p\gamma}}.$$

Fix $0 < \gamma_0 < 1/p$. For every $\varepsilon > 0$ there exists such $\delta > 0$ that $(1-x^{\gamma_0})^p(1-x)^{-p} > 1-\varepsilon$ for all $x \in (0, \delta)$. Then for $\gamma > \gamma_0$ and $x \in (0, \delta)$, we have

$$\left(\frac{1-x^\gamma}{1-x} \right)^p > \left(\frac{1-x^{\gamma_0}}{1-x} \right)^p > 1-\varepsilon.$$

It follows that

$$\begin{aligned} \left(\frac{\|Sf_\gamma\|_{L^p}}{\|f_\gamma\|_{L^p}} \right)^p &= \frac{\sin \pi p\gamma}{\pi p\gamma} \left(\frac{\pi}{\sin \pi \gamma} \right)^p \int_0^1 \left(\frac{1-x^\gamma}{1-x} \right)^p \frac{dx}{x^{p\gamma}} \\ &\geq \frac{\sin \pi p\gamma}{\pi p\gamma} \left(\frac{\pi}{\sin \pi \gamma} \right)^p \int_0^\delta (1-\varepsilon) \frac{dx}{x^{p\gamma}} \\ &= \frac{\sin \pi p\gamma}{\pi p\gamma} \left(\frac{\pi}{\sin \pi \gamma} \right)^p \frac{\delta^{1-p\gamma}}{1-p\gamma} (1-\varepsilon). \end{aligned}$$

Since

$$\lim_{\gamma \rightarrow 1/p} \frac{\sin \pi p\gamma}{\pi(1-p\gamma)} \left(\frac{\pi}{\sin \pi \gamma} \right)^p \frac{\delta^{1-p\gamma}}{p\gamma} = \left(\frac{\pi}{\sin \frac{\pi}{p}} \right)^p,$$

we get

$$\|S\|_{L^p \rightarrow L^p} \geq \frac{\pi}{\sin \frac{\pi}{p}}.$$

To prove that S is non compact, assume the contrary. Then $\lim_{\text{mes}(D) \rightarrow 0} \|P_D S\|_{L^p \rightarrow L^p} = 0$, where $P_D f := \chi_D f$ (χ_D denotes the characteristic function of the subset $D \subset [0, 1]$) [22, Theorem 3.1].

On the other hand, let $D_a := [a, 1]$ and $x_a := (1/\text{mes}(D_a)^{1/p})\chi_{D_a}$. Then $\|x_a\|_{L^p}^p = 1$ and

$$\begin{aligned} Sx_a &= \frac{1}{\text{mes}(D_a)^{1/p}} \int_{D_a} \frac{dt}{1-tz} = \frac{1}{(1-a)^{1/p}} \int_a^1 \frac{dt}{1-tz} \\ &\geq \frac{1}{(1-a)^{1/p}} \frac{1}{1-az} \int_a^1 dt = (1-a)^{1-1/p} \frac{1}{1-az}. \end{aligned}$$

Therefore

$$\begin{aligned} \|P_{D_a} Sx_a\|_{L^p}^p &= \int_0^1 |\chi_{D_a}(z)|^p |Sx_a(z)|^p dz \geq (1-a)^{p-1} \int_a^1 \frac{dz}{(1-az)^p} \\ &= \frac{1}{p-1} \frac{1}{1-a} \left(1 - \frac{1}{(1+a)^{p-1}} \right). \end{aligned}$$

It follows that

$$\limsup_{a \rightarrow 1} \|P_{D_a} S\|_{L^p \rightarrow L^p}^p \geq \limsup_{a \rightarrow 1} \|P_{D_a} Sx_a\|_{L^p}^p \geq \frac{1}{p-1} \left(1 - \frac{1}{2^{p-1}} \right) > 0,$$

which is a contradiction. This completes the proof on noncompactness of the operator S .

To show that the Markov–Stieltjes transform on $L^2(0, 1)$ is unitarily equivalent to the Markov–Stieltjes transform on $H^2(\mathbb{D})$, consider the restriction operator

$$j : H^2(\mathbb{D}) \rightarrow L^2(0, 1), \quad f \mapsto f|(0, 1).$$

Let S_L denotes the Markov–Stieltjes transform on $L^2(0, 1)$, and S_H the Markov–Stieltjes transform on $H^2(\mathbb{D})$. Note that $jS_H = S_L j$, i.e. j is an intertwining operator for S_L and S_H . The operator j is bounded (see the formula (4.2)) and injective and has a dense range. By the Putnam–Douglas theorem (see, e.g. [23, Theorem IX.6.10(c)]), S_L is unitarily equivalent to S_H . Application of Theorem 4.2 completes the proof. ■

Corollary 5.2: *The Markov–Stieltjes transform S is a bounded operator from H^p to $L^p(0, 1)$ and*

$$\|S\|_{H^p \rightarrow L^p} \leq \pi^{1+1/p} \cot \frac{\pi}{2 \max\{p, q\}}.$$

Proof: By the formula (4.2), if $f \in H^p(\mathbb{D})$ then the restriction $f|(0, 1)$ belongs to $L^p(0, 1)$ and the norm of the restriction operator $j_p : H^p \rightarrow L^p(0, 1) : f \mapsto f|(0, 1)$ does not exceed $\pi^{1/p}$. Let S_H denotes the Markov–Stieltjes operator in H^p . Then $S = j_p S_H : H^p \rightarrow L^p(0, 1)$ and

$$\|S\|_{H^p \rightarrow L^p} \leq \|j_p\|_{H^p \rightarrow L^p} \|S_H\|_{H^p \rightarrow H^p} \leq \pi^{1+1/p} \cot \frac{\pi}{2 \max\{p, q\}}. \quad \blacksquare$$

Corollary 5.3: *The map $f \mapsto L^p R_n(Sf)$ is continuous on L^p .*

Proof: The proof is similar to the case of H^p . ■

Remark 5.1: The Markov–Stieltjes transform is an unbounded operator on $L^1(0, 1)$ and $L^\infty(0, 1)$, but it is bounded as an operator from $L^p(0, 1)$ ($1 < p < \infty$) to $L^1(0, 1)$ [8, p. 187]. It is also a continuous map from $L^1(0, 1)$ to $L^p(0, 1)$ for $0 < p < 1$ because if $f, g \in L^p(0, 1)$ then

$$\begin{aligned} \|S(f - g)\|_{L^p(0,1)}^p &= \int_0^1 \left| \int_0^1 \frac{f(t) - g(t)}{1 - tz} dt \right|^p dz \leq \int_0^1 \left(\int_0^1 \frac{|f(t) - g(t)|}{1 - tz} dt \right)^p dz \\ &\leq \int_0^1 \left(\int_0^1 \frac{|f(t) - g(t)|}{1 - z} dt \right)^p dz = \int_0^1 \frac{dz}{(1 - z)^p} \|f - g\|_{L^1(0,1)}^p \\ &= \frac{1}{1 - p} \|f - g\|_{L^1(0,1)}^p. \end{aligned}$$

Recall that the Banach space ℓ_A^p ($1 < p \leq \infty$) consists of functions f that are holomorphic on the unit disc, $f(z) = \sum_{n=0}^\infty f_n z^n$ ($z \in \mathbb{D}$), and such that $\|f\|_{\ell_A^p}^p := \sum_{n=0}^\infty |f_n|^p < \infty$. Obviously, the space ℓ_A^p can be identified with ℓ^p . According to Böttcher and Silbermann [15, p. 53], a Hankel operator $H(a)$ on ℓ_A^p associated with a sequence $a = (a_n)$ is defined by

$$H(a)f(z) = \sum_{j=0}^\infty b_j z^j \quad (z \in \mathbb{D}),$$

where

$$b_j = \sum_{k=0}^\infty a_{k+j+1} f_k \quad \left(f(z) = \sum_{n=0}^\infty f_n z^n \right).$$

In the following theorem, we consider S as an operator on ℓ_A^p (this means that S is defined by the formula (1.1), where $f \in \ell_A^p, z \in \mathbb{D}$).

Theorem 5.4: *The Markov–Stieltjes transform S is a bounded Hankel operator on ℓ_A^p ($1 < p < \infty$) and has Hilbert matrix with respect to the standard basis. Moreover,*

$$\|S\|_{\ell_A^p \rightarrow \ell_A^p} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

Proof: First note, that the Markov–Stieltjes transform exists for $f \in \ell_A^p$, since $f|(0, 1) \in L^1(0, 1)$. In fact, if $f(t) = \sum_{n=0}^\infty f_n t^n$ then

$$\int_0^1 |f(t)| dt \leq \int_0^1 \sum_{n=0}^\infty |f_n| t^n dt = \sum_{n=0}^\infty \frac{|f_n|}{n+1},$$

the series converges by the Hölder inequality.

Arguing as in the proof of Theorem 4.2, we get that the Markov–Stieltjes transform S is a Hankel operator on ℓ_A^p and has Hilbert matrix with respect to the standard basis $\{e_n : n \in \mathbb{Z}_+\}$, $e_n(z) := z^n$ of ℓ_A^p .

To compute the norm of S , note that the Hardy–Littlewood–Polya–Shur inequality [24, Theorem 318] applied to the function $K(x, y) = 1/(x + y)$ implies, that

$$\|Sf\|_{\ell_A^p}^p = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} \frac{f_m}{n + m + 1} \right|^p \leq k^p \|f\|_{\ell_A^p}^p,$$

where the best possible constant k is [24, p. 229]

$$k = \int_0^{\infty} \frac{dx}{x^{1/p}(1+x)} = \frac{\pi}{\sin \frac{\pi}{p}}$$

(for the last equality see, e.g. [21, Section 2.2.4, formula 25]). ■

Corollary 5.5: *Let $1 < p \leq 2$. The Markov–Stieltjes transform S is a bounded operator from ℓ_A^p to H^q ($1/p + 1/q = 1$) and*

$$\|S\|_{\ell_A^p \rightarrow H^q} \leq \frac{\pi}{\sin \frac{\pi}{p}}.$$

Proof: It follows from the above theorem, because by [14, Theorem 6.1, p. 94] $\ell_A^p \subset H^q(\mathbb{D})$ and the norm of the natural embedding of ℓ_A^p into $H^q(\mathbb{D})$ does not exceed 1. ■

Corollary 5.6: *The Markov–Stieltjes transform S is a bounded operator from ℓ_A^p to $L^q(0, 1)$ and*

$$\|S\|_{\ell_A^p \rightarrow L^q} \leq \frac{\pi^{1+1/q}}{\sin \frac{\pi}{q}}.$$

Proof: The proof follows from formula (4.2) and Corollary 5.7. ■

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