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TO THE SPECTRAL THEORY OF DISCRETE HAUSDORFF OPERATORS

Adolf R. Mirotin[1,2]

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Abstract
We show that under an arithmetic condition the spectrum of a bounded multidimensional discrete Hausdorff operator in the Lebesgue space is an annulus (or a disc) centered at the origin, provided the perturbation matrices commute and are either positive or negative definite. Conditions for a point spectrum of such an operator to be empty are given and its norm is computed.

Keywords Hausdorff operator · Discrete Hausdorff operator · Spectrum · Norm of an operator · Lebesgue space

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Introduction

One-dimensional Hausdorff operators were introduced by Rogosinskii and Garabedyan as a continuous analog of the Hausdorff method of generalized summation of series and sequences (see, e.g., [14, Chapter XI]). After the seminal work of Liflyand and Móricz [9] the theory of Hausdorff operators has become an active research area, see, e.g., [6, 10, 11]. The first non-routine results on the boundedness of multidimensional Hausdorff operators were given by Lerner and Liflyand [8].

In [16] and [17], the diagonalization of normal Hausdorff operators over Euclidean spaces was established and as a consequence, their spectra and norms were described in terms of the symbols of these operators. In the case where the kernel is positive these formulas give a direct expression of the norm in terms of the initial data, namely, in terms of the kernel and perturbation matrices.

The main goal of this paper is to compute the spectra of a wide class of discrete Hausdorff operators. More precisely, we show that under an arithmetic condition the spectrum of a bounded multidimensional discrete Hausdorff operator in the Lebesgue space is an annulus (or a disc) centered at the origin, provided the perturbation matrices commute and are either positive or negative definite. As a corollary, we give conditions for the validity of "direct" formulas for the norm of an operator for signed kernels as well. Also, conditions for a point spectrum of such an operator to be empty are given and several examples and counterexamples are considered.

✉ Adolf R. Mirotin
amirotin@yandex.ru

1 Department of Mathematics and Programming Technologies, Francisk Skorina Gomel State University, Sovietskaya, 104, Gomel 246019, Belarus

2 Regional Mathematical Center, Southern Federal University, Rostov-on-Don 344090, Russia

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Preliminaries

The general form of a Hausdorff operator over $\mathbb{R}^d$ is

$$(\mathcal{H}_{K,A}f)(x) = \int_\Omega K(u)f(A(u)x)d\mu(u),$$

where $x \in \mathbb{R}^d$ is a column vector. Here (Ω, μ) is a topological space endowed with positive regular Borel measure μ, K is a locally integrable function on Ω ("kernel"), and $(A(u))_{u\in\Omega}$ is a μ-measurable family of real $d \times d$ matrices ("perturbation matrices") defined almost everywhere in the support of K and satisfying $\det A(u) \neq 0$. This is a particular case of a Hausdorff operator over a locally compact group [18].
Let $a(u) := (a_1(u), \dots, a_d(u))$ be the family of eigenvalues (with their multiplicities) of the matrix $A(u)$ and $|\det A(u)|^{-1/p}K(u) \in L^1(\Omega)$. Then the function

$$\varphi(s) := \int_\Omega K(u)|a(u)|^{-1/p-\iota s}d\mu(u)\ (s = (s_j) \in \mathbb{R}^d)$$

is called *the scalar symbol of the Hausdorff operator* $\mathcal{H}_{K,A}$ in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$ [16, 17]. It is assumed that

$$|a(u)|^{-1/p-\iota s} := \prod_{j=1}^{d} |a_j(u)|^{-1/p-\iota s_j},$$

where $|a_j(u)|^{-1/p-\iota s_j} := \exp((-1/p - \iota s_j)\log|a_j(u)|)$.
We have the following special case of a Hausdorff operator on Euclidean spaces.

Definition 1 . Let $c = (c(k))_{k\in\mathbb{Z}}$ be a sequence of complex numbers, and $A(k) \in \mathrm{GL}(d, \mathbb{R})$ for all $k \in \mathbb{Z}$. A discrete Hausdorff operator acts on a function $f : \mathbb{R}^d \to \mathbb{C}$ by the rule

$$\mathcal{H}_{c,A}f(x) = \sum_{k=-\infty}^{\infty} c(k)f(A(k)x)$$

provided the series converges absolutely.

Such operators appeared for the first time in [12, Section 4].
Note that

$$\begin{aligned}|a(u)|^{-1/p-\iota s} &= |\prod_{j=1}^{d} a_j(u)|^{-1/p} \prod_{j=1}^{d} \exp((-\iota s_j)\log|a_j(u)|)\\ &= |\det A(u)|^{-1/p} e^{-\iota s\cdot\log|a(u)|},\end{aligned}$$

where $s \cdot \log|a(u)| := \sum_{j=1}^{d} s_j \log|a_j(u)|$. Therefore, the symbol of a discrete Hausdorff operator is of the form

$$\varphi(s) := \sum_{k=-\infty}^{\infty} c(k)|\det A(k)|^{-1/p} e^{-\iota s\cdot\log|a(k)|}. \tag{1}$$

As we shall see, spectral characteristics of a (nonzero) discrete Hausdorff operator depend on the arithmetic properties of $A(k)$.
Notable examples of discrete Hausdorff operators are given by the q-calculus.
Recall that the q-integral for a function $f : \mathbb{R} \to \mathbb{C}$ is defined as

$$\int_{-\infty}^{\infty} f(t)d_qt := (1-q)\sum_{k=-\infty}^{\infty} (f(q^k) + f(-q^k))q^k,$$

with $q \in \mathbb{R}, 0 < |q| < 1$, provided the series converges absolutely.
If $f : \mathbb{R}_+ \to \mathbb{C}$, the q-integral is defined as

$$\int_0^\infty f(t)d_q t := (1-q) \sum_{k=-\infty}^{\infty} f(q^k)q^k,$$

with $q \in \mathbb{R}, 0 < |q| < 1$, provided the series converges absolutely. For these definitions and properties of q-integral, see, e. g., [4, p. 202] or [5, p. 23]). The first definition is, of course, more general and reduces to the second one for functions vanishing on the negative half-axis.
Thus, according to the general definition, we get the following class of discrete Hausdorff operators.

Definition 2 A q-Hausdorff operator acts on a function $f : \mathbb{R} \to \mathbb{C}$ by the rule

$$\begin{aligned}(\mathcal{H}_{K,A}f)(x) &:= \int_{-\infty}^{\infty} K(u)f(A(u)x)d_q u \\ &= (1-q) \sum_{k=-\infty}^{\infty} (K(q^k)f(A(q^k)x) + K(-q^k)f(A(-q^k)x))q^k,\end{aligned}$$

where K and A are given functions on $\mathbb{R}$, $A(u) \in \mathbb{R} \setminus \{0\}$ for all u, provided the series converges absolutely.

This class of operators will serve as a source of useful examples and counterexamples (see below).

Main results

In order to formulate and prove our main results, we need some preparations. First, we note that the L^p boundedness of a Hausdorff operator readily follows from the Minkowski inequality.

Lemma 1 [2]. *Let* $1 \le p \le \infty$. *If*

$$N_p(c, A) := \sum_{k=-\infty}^{\infty} |c(k)||\det A(k)|^{-1/p} < \infty,$$

then the operator $\mathcal{H}_{c,A}$ *is bounded in* $L^p(\mathbb{R}^d)$ *and its norm does not exceed* $N_p(c, A)$.

It is known [16, 17] that the equality

$$\|\mathcal{H}_{c,A}\|_{L^2(\mathbb{R}^d)} = N_2(c, A)$$

holds if $c(k) \ge 0$ for all k. We shall show that in some cases this equality is possible without this positivity condition, too. This statement is a consequence of the description of spectrum of discrete Hausdorff operators.
In the following, $\mathbb{T}^\infty$ stands for the infinite-dimensional torus endowed with the metric

$$\rho(t, s) = \sqrt{\sum_{k=-\infty}^{\infty} |t_k - s_k|^2}.$$

Lemma 2 *If* $N_2(c, A) < \infty$, *then the function*

$$\xi(t) := \sum_{k=-\infty}^{\infty} \frac{c(k)}{\sqrt{|\det A(k)|}} t_k, \quad t = (t_k)_{k \in \mathbb{Z}} \in \mathbb{T}^\infty,$$

is continuous (and a closed map) on $\mathbb{T}^\infty$.

Proof Indeed, by the Schwartz inequality, for any $t, s \in \mathbb{T}^\infty$, we have

$$|\xi(t) - \xi(s)| \leq \sum_{k=-\infty}^{\infty} \frac{|c(k)|}{\sqrt{|\det A(k)|}} |t_k - s_k| \leq \sqrt{\sum_{k=-\infty}^{\infty} \frac{|c(k)|^2}{|\det A(k)|}}\rho(t,s).$$

Thus, ξ is continuous. Since $\mathbb{T}^\infty$ is compact, the fact that ξ is a closed map follows. □

Recall that the real numbers $b_1, \dots, b_m$ are linear independent over $\mathbb{Z}$ if the equality $\sum_{k=1}^m l_k b_k = 0$, where all $l_k \in \mathbb{Z}$, yields $l_k = 0$ for all k. As usual, we say that an infinite family of real numbers is linear independent over $\mathbb{Z}$ if this is the case for any of its finite subfamilies.
We denote by $a(k)$ the family of eigenvalues $(a_1(k), \dots, a_d(k))$ (with their multiplicities) of a real self-adjoint $d \times d$ matrix $A(k)$.

The case of positive definite matrices

Theorem 1 *Let* $(A(k))_{k\in\mathbb{Z}}$ *be a commuting family of positive definite* $d \times d$ *matrices, and let for some* $j \in \{1, \dots, d\}$, *the numbers* $\log a_j(k)$ $(k \in \mathbb{Z})$ *be linear independent over* $\mathbb{Z}$. *If* $N_2(c, A) < \infty$, *then for a discrete Hausdorff operator* $\mathcal{H}_{c,A}$ *in* $L^2(\mathbb{R}^d)$, *the following assertions hold.*

(*i*) *The spectrum* $\sigma(\mathcal{H}_{c,A})$ *is the annulus* $\{r(c, A) \leq |\zeta| \leq N_2(c, A)\}$ *if* $r(c, A) := \min_{t\in\mathbb{T}^\infty} |\xi(t)| > 0$, *and the disc* $\{|\zeta| \leq N_2(c, A)\}$ *otherwise.*

(*ii*)

$$\|\mathcal{H}_{c,A}\|_{L^2\to L^2} = \sum_{k=-\infty}^{\infty} \frac{|c(k)|}{\sqrt{\det A(k)}}. \tag{2}$$

(*iii*) *If the symbol* φ *of* $\mathcal{H}_{c,A}$ *is a nonconstant real analytic function on* $\mathbb{R}^d$, *the point spectrum* $\sigma_p(\mathcal{H}_{c,A})$ *is empty.*

Proof One can assume that $\mathcal{H}_{c,A}$ is nonzero.
(i) We split the proof into several steps.
1. We claim that the spectrum of the truncated discrete Hausdorff operator, i.e., the operator

$$\mathcal{H}_{c,A}^{(n)} f(x) := \sum_{k=-n}^{n} c(k) f(A(k)x), \quad n \in \mathbb{N},$$

in $L^2(\mathbb{R}^d)$ is

$$\sigma(\mathcal{H}_{c,A}^{(n)}) = \left\{\xi_n(t) : t = (t_{-n}, \dots, t_n) \in \mathbb{T}^{2n+1}\right\}, \tag{3}$$

where

$$\xi_n(t) := \sum_{k=-n}^{n} \frac{c(k)}{\sqrt{\det A(k)}} t_k.$$

Indeed, the scalar symbol (1) of $\mathcal{H}_{c,A}^{(n)}$ is a trigonometric polynomial of the form

$$\varphi_n(s) = \sum_{k=-n}^{n} \frac{c(k)}{\sqrt{\det A(k)}} e^{-\imath s\cdot\log a(k)} \quad s \in \mathbb{R}^d, \tag{4}$$

where $s \cdot \log a(k) := s_1 \log a_1(k) + \cdots + s_d \log a_d(k)$. Moreover ([16, 17, Corollary 7]), $\sigma(\mathcal{H}^{(n)}_{c,A})$ equals to the closure of $\varphi_n(\mathbb{R}^d)$. But the corollary of Kronecker's approximation theorem (see, e.g., [19, p. 44]) implies that the set

$$\{(e^{-is_j \log a_j(-n)}, \dots, e^{-is_j \log a_j(n)}) : s_j \in \mathbb{R}\}$$

is dense in $\mathbb{T}^{2n+1}$. Therefore, the set

$$\{\Lambda(s) := (e^{-\imath s\cdot\log a(-n)}, \dots, e^{-\imath s\cdot\log a(n)}) : s \in \mathbb{R}^d\} \tag{5}$$

is dense in $\mathbb{T}^{2n+1}$ and (3) follows. Indeed, since ξ_n is a continuous and closed map, we have $\xi_n(\mathrm{Cl}(M)) = \mathrm{Cl}(\xi_n(M))$ for all $M \subset \mathbb{T}^{2n+1}$, where Cl stands for the closure (see, e.g., [1, Chapter 1, §5, Proposition 9]). Therefore,

$$\sigma(\mathcal{H}^{(n)}_{c,A}) = \mathrm{Cl}(\varphi_n(\mathbb{R}^d)) = \mathrm{Cl}(\xi_n(\Lambda(\mathbb{R}^d))) = \xi_n(\mathrm{Cl}(\Lambda(\mathbb{R}^d))) = \xi_n(\mathbb{T}^{2n+1}).$$

2. Now we are going to show that

$$\sigma(\mathcal{H}_{c,A}) = \left\{\xi(t) : t = (\dots, t_{-n}, \dots, t_n, \dots) \in \mathbb{T}^\infty\right\}. \tag{6}$$

Let $\zeta = \xi(t)$ for some $t = (t_k) \in \mathbb{T}^\infty$. Then

$$|\xi(t) - \xi_n(t)| \le r_n := \sum_{|k|>n} \frac{|c(k)|}{\sqrt{\det A(k)}},$$

and $r_n \to 0$ as $n \to \infty$. As has been proven above, $\zeta_n := \xi_n(t) \in \sigma(\mathcal{H}^{(n)}_{c,A})$. If $\mathcal{R}^{(n)} := \mathcal{H}_{c,A} - \mathcal{H}^{(n)}_{c,A}$, then the operators $\mathcal{R}^{(n)}$ and $\mathcal{H}^{(n)}_{c,A}$ commute. It follows [20, Theorem IV.3.6] that $\mathrm{dist}(\zeta_n, \sigma(\mathcal{H}_{c,A})) \le \|\mathcal{R}^{(n)}\| \le r_n$, and hence $\|\zeta_n - \eta_n\| \le r_n$ for some $\eta_n \in \sigma(\mathcal{H}_{c,A})$. Passing, if necessary, to a subsequence, one can assume that $\eta_n \to \eta \in \sigma(\mathcal{H}_{c,A})$. Since $r_n \to 0$, this yields $\zeta_n \to \eta$ as $n \to \infty$. But $\zeta_n \to \zeta$. Thus, $\zeta = \eta \in \sigma(\mathcal{H}_{c,A})$.

Let us now choose an arbitrary $\zeta \in \sigma(\mathcal{H}_{c,A})$. Again by [20, Theorem IV.3.6], we have $\mathrm{dist}(\zeta, \sigma(\mathcal{H}^{(n)}_{c,A})) \le r_n$ for all n. Thus, $|\zeta - \xi_n(t^{(n)})| \le r_n$ for some $t^{(n)} = (t^{(n)}_{-n}), \dots, t^{(n)}_n)) \in \mathbb{T}^{2n+1}$. Consider the point $t^{(n)}_\infty := (\dots, 1, 1, t^{(n)}_{-n}), \dots, t^{(n)}_n, 1, 1, \dots)$ in $\mathbb{T}^\infty$. Since $\mathbb{T}^\infty$ is compact in the (Tychonoff) topology induced by the metric ρ, we can assume without loss of generality that $t^{(n)}_\infty \to t \in \mathbb{T}^\infty$ with respect to ρ as $n \to \infty$. Then

$$\begin{aligned}|\zeta - \xi(t)| \le& |\zeta - \xi_n(t^{(n)})| + |\xi_n(t^{(n)}) - \xi(t^{(n)}_\infty)| + |\xi(t^{(n)}_\infty) - \xi(t)| \\ \le& 2r_n + |\xi(t^{(n)}_\infty) - \xi(t)|.\end{aligned}$$

Since ξ is continuous on $\mathbb{T}^\infty$, it follows that $\zeta = \xi(t)$. This proves (6).

3. Further, the continuity of the map $\xi : \mathbb{T}^\infty \to \mathbb{C}$ and (6) imply that the set $\sigma(\mathcal{H}_{c,A})$ is connected. The observation that this set is rotational invariant shows that the spectrum is either an annulus $\{r(c,A) \le |\zeta| \le R(c,A)\}$ if $r(c,A) > 0$ or a disc $\{|\zeta| \le R(c,A)\}$ if $r(c,A) = 0$, where $r(c,A) := \min_{t\in\mathbb{T}^\infty} |\xi(t)|$ and $R(c, A)$ is a spectral radius of $\mathcal{H}_{c,A}$. Since this operator is normal [17], $R(c, A)$ equals to its norm. So, to complete the proof of (i) it suffices to prove (ii).

(ii) It was shown in [16, 17] that

$$\|\mathcal{H}^{(n)}_{c,A}\|_{L^2\to L^2} = \sup_{\mathbb{R}^d} |\varphi_n|.$$

Since (by the corollary of Kronecker's theorem) the set (5) is dense in $\mathbb{T}^{2n+1}$, formula (4) yields

$$\|\mathcal{H}^{(n)}_{c,A}\|_{L^2\to L^2} = \sum_{k=-n}^{n} \frac{|c(k)|}{\sqrt{\det A(k)}}.$$

Taking into account that

$$\|\mathcal{H}_{c,A}^{(n)}f - \mathcal{H}_{c,A}f\|_{L^2(\mathbb{R}^d)} \le \sum_{|k|>n} |c_k| \|f(A(k)\cdot)\|_{L^2(\mathbb{R}^d)} = \sum_{|k|>n} \frac{|c(k)|}{\sqrt{\det A(k)}} \|f\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$, we get

$$\|\mathcal{H}_{c,A}\|_{L^2\to L^2} = \lim_{n\to\infty} \|\mathcal{H}_{c,A}^{(n)}\|_{L^2\to L^2} = \sum_{k=-\infty}^{\infty} \frac{|c(k)|}{\sqrt{\det A(k)}} = N_2(c, A).$$

(iii) It is known [16, Theorem 1 (iii)] that

$$\sigma_p(\mathcal{H}_{c,A}) = \{\lambda \in \mathbb{C} : \text{mes}(\varphi^{-1}(\{\lambda\}) > 0\},$$

where mes denotes the Lebesgue measure in $\mathbb{R}^d$. If $\lambda \in \sigma_p(\mathcal{H}_{c,A})$, it follows that $\text{mes}(\Pi \cap \varphi^{-1}(\{\lambda\}) > 0$ for some parallelepiped $\Pi = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$. It follows, in turn, that each orthogonal projection of $\Pi \cap \varphi^{-1}(\{\lambda\})$ on $[a_j, b_j]$ is of positive Lebesgue measure. Since all of these projections have a cluster point, $\varphi(s) \equiv \lambda$ by a version of a uniqueness theorem for real analytic functions [15] (see also [7, p. 83]). A contradiction. □

Remark. Since $\mathcal{H}_{c,A}$ is normal, the residual spectrum of $\mathcal{H}_{c,A}$ is empty.

Corollary 1 *Let $(A(k))_{k\in\mathbb{Z}}$ be a commuting family of positive definite $d \times d$ matrices, and let for some $j \in \{1, \dots, d\}$, the numbers $\log a_j(k)$ $(k \in \mathbb{Z})$ be linear independent over $\mathbb{Z}$. If $N_2(c, A) < \infty$, then the discrete Hausdorff operator $\mathcal{H}_{c,A}$ in $L^2(\mathbb{R}^d)$ is invertible if and only if $r(c, A) > 0$.*

Example 1. Let θ be a real transcendental number. Consider the following one-dimensional nonzero discrete Hausdorff operator:

$$\mathcal{H}f(x) = \sum_{k\in\mathbb{Z}} c(k) f(e^{\theta^k} x), \quad x \in \mathbb{R}.$$

If $N_2(c, A) = \sum_{k\in\mathbb{Z}} |c(k)| e^{-\theta^k/2} < \infty$, this operator is bounded in $L^2(\mathbb{R})$. Moreover, if in addition, $\sum_{k\in\mathbb{Z}} c(k) e^{-\theta^k/2} t_k^0 = 0$ for some $t^0 \in \mathbb{T}^\infty$, Theorem 1 implies that $\sigma(\mathcal{H}) = \{|\zeta| \le N_2(c, A)\}$ for this space.
On the other hand, if, e.g., $|c(0)| e^{-1/2} > \sum_{k\neq 0} |c(k)| e^{-\theta^k/2}$, we have

$$\left| \sum_{k\in\mathbb{Z}} c(k) e^{-\theta^k/2} t_k \right| \ge |c(0)| e^{-1/2} - \sum_{k\neq 0} |c(k)| e^{-\theta^k/2} > 0,$$

for all $t \in \mathbb{T}^\infty$. Thus, in this case $r(c, A) > 0$, and $\sigma(\mathcal{H})$ in $L^2(\mathbb{R})$ is an annulus.
The following example shows that the conditions of Lemma 1 and Theorem 1 are essential.
Example 2. Consider the q-calculus version of a Cesàro operator introduced in [17]:

$$(C_q f)(x) := \frac{1}{x} \int_0^x f(t) d_q t := (1 - q) \sum_{k=0}^{\infty} f(q^k x) q^k. \tag{7}$$

This is a q-Hausdorff operator in the sense of Definition 2, with $K = \chi_{[0,1]}$, the indicator function of [0, 1], and $A(u) = u$ (and a q-analog of the classical Cesàro operator [13]). Since in this case $N_p(c, A) < \infty$ for all $p > 1$, this operator is bounded in $L^p(\mathbb{R})$ for such p.
On the other hand, $N_1(c, A) = \infty$ and C_q does not act in $L^1(\mathbb{R})$. Indeed, let $f(x) = e^{-x} \chi_{\mathbb{R}_+}(x)$. Then for $q > 0$,

$$(C_q f)(x) = (1 - q) \sum_{k=0}^{\infty} e^{-q^k x} q^k \chi_{\mathbb{R}_+}(x),$$

and by the B. Levy theorem,

$$\int_{\mathbb{R}} (C_q f)(x)dx = (1-q)\sum_{k=0}^{\infty} q^k \int_{\mathbb{R}_+} e^{-q^k x}dx = \infty.$$

This example shows that the arithmetic condition in Theorem 1 is essential. Indeed, in this case the numbers $\log a(k) = k\log q$ $(k = 0, 1, \dots)$ are linear dependent over $\mathbb{Z}$ and, as mentioned in [17, Example 3], if $q > 0$, then the spectrum of C_q in $L^2(\mathbb{R})$ is a circle $\{\lambda \in \mathbb{C} : |\lambda - 1| = \sqrt{q}\}$. This implies $\|C_q\| = 1 + \sqrt{q}$, which is consistent with formula (8)[1]. For a detailed study of spectral properties of Cesàro operators in several settings, see, e.g., [3] and references therein.

The case of negative definite matrices

The case of negative definite matrices can be reduced to the previous one.

Theorem 2 *Let $(A(k))_{k\in\mathbb{Z}}$ be a commuting family of negative definite $d \times d$ matrices, and let for some $j \in \{1, \dots, d\}$, the numbers $\log(-a_j(k))$ $(k \in \mathbb{Z})$ be linear independent over $\mathbb{Z}$. If $N_2(c, A) < \infty$, then for a discrete Hausdorff operator $\mathcal{H}_{c,A}$ in $L^2(\mathbb{R}^d)$, the following statements hold.*

(i) The spectrum $\sigma(\mathcal{H}_{c,A})$ is the annulus $\{r(c, -A) \le |\zeta| \le N_2(c, A)\}$ if $r(c, -A) > 0$, and the disc $\{|\zeta| \le N_2(c, A)\}$ otherwise.

(ii)

$$\|\mathcal{H}_{c,A}\|_{L^2\to L^2} = \sum_{k=-\infty}^{\infty} \frac{|c(k)|}{\sqrt{(-1)^d \det A(k)}}. \tag{8}$$

(iii) If the symbol φ of $\mathcal{H}_{c,A}$ is a nonconstant real analytic function on $\mathbb{R}^d$, the point spectrum $\sigma_p(\mathcal{H}_{c,A})$ is empty.

Proof The scalar symbol of the operator $\mathcal{H}_{c,A}$ is

$$\begin{aligned}\varphi^-(s) &:= \sum_{k=-\infty}^{\infty} \frac{c(k)}{\sqrt{|\det A(k)|}} e^{-\imath s\cdot\log(-a(k))}\\ &= \sum_{k=-\infty}^{\infty} \frac{c(k)}{\sqrt{\det(-A(k))}} e^{-\imath s\cdot\log(-a(k))}.\end{aligned}$$

Thus, φ^- coincides with the scalar symbol φ of the operator $\mathcal{H}_{c,(-A)}$ where all the matrices $(-A(k))$ are positive definite.

According to [17, Corollary 8], $\sigma(\mathcal{H}_{c,A})$ equals to the set

$$-\mathrm{Cl}(\varphi^-(\mathbb{R}^d)) \cup \mathrm{Cl}(\varphi^-(\mathbb{R}^d)) = -\mathrm{Cl}(\varphi(\mathbb{R}^d)) \cup \mathrm{Cl}(\varphi(\mathbb{R}^d)),$$

where Cl denotes the closure in $\mathbb{C}$. But as shown in Theorem 1, $\mathrm{Cl}(\varphi(\mathbb{R}^d))$ is either the annulus $\{r(c, -A) \le |\zeta| \le N_2(c, A)\}$ or the disc $\{|\zeta| \le N_2(c, A)\}$, and so $-\mathrm{Cl}(\varphi(\mathbb{R}^d)) = \mathrm{Cl}(\varphi(\mathbb{R}^d))$. In view of Theorem 1, it follows that $\sigma(\mathcal{H}_{c,A}) = \sigma(\mathcal{H}_{c,(-A)})$. Since $r(c, -A) = r(c, A)$, this proves (i).

The statement (ii) follows from the fact that $\mathcal{H}_{c,A} = J\mathcal{H}_{c,(-A)}$, where $Jg(x) := g(-x)$ is a unitary operator in $L^2(\mathbb{R}^d)$.

(iii) It is known [17, Theorem 1] that the point spectrum $\sigma_p(\mathcal{H}_{c,A})$ consists of such complex numbers λ for which the set $E(\lambda) := \{s \in \mathbb{R}^d : \det(\lambda - \Phi(s)) = 0\}$ has positive Lebesgue measure. Here Φ stands for the matrix symbol of $\mathcal{H}_{c,A}$. The proof of Corollary 8 in [17] shows that in our case $\det(\lambda - \Phi(s)) = (\lambda^2 - \varphi^2(s))^{2^{d-1}}$. Since the function $\lambda^2 - \varphi^2$ is real analytic and nonconstant, $\mathrm{mes}(E(\lambda)) = 0$ by the uniqueness theorem as in the proof of Theorem 1. □

[1] Here we fix a typo in [17, Example 3].

Corollary 2 *Let* $(A(k))_{k\in\mathbb{Z}}$ *be a commuting family of negative definite* $d\times d$ *matrices, and let for some* $j\in\{1,\dots,d\}$, *the numbers* $\log(-a_j(k))$ $(k\in\mathbb{Z})$ *be linear independent over* $\mathbb{Z}$. *If* $N_2(c,A)<\infty$, *then the discrete Hausdorff operator* $\mathcal{H}_{c,A}$ *in* $L^2(\mathbb{R}^d)$ *is invertible if and only if* $r(c,-A)>0$.

Concluding remarks

In this section, we collect two more counterexamples that the q-calculus version of a Cesàro operator gives.

Example 3. We are going to compute the spectrum of the Cesàro operator (7) with $-1<q<0$ by making use of [16, Theorem 2]. (In this case Theorems 1 and 2 do not work.) The scalar symbol (1) for this operator is

$$\begin{aligned}\varphi(s)=&(1-q)\sum_{k\in\mathbb{Z}_+}q^k|q^k|^{-1/2}|q^k|^{-\imath s}=(1-q)\sum_{k=0}^{\infty}(-(-q)^{1/2-\imath s})^k\\ =&\frac{1-q}{1+(-q)^{1/2-\imath s}}=\frac{1-q}{1+\sqrt{-q}(-q)^{-\imath s}}.\end{aligned}$$

Further, the conjugate scalar symbol is $\varphi^* := \varphi_+ - \varphi_-$, where

$$\begin{aligned}\varphi_+(s)=&(1-q)\sum_{k\in2\mathbb{Z}_+}q^k|q^k|^{-1/2}|q^k|^{-\imath s}=(1-q)\sum_{l=0}^{\infty}(-(-q)^{1/2-\imath s})^{2l}\\ =&\frac{1-q}{1-((-q)^{1/2-\imath s})^2}=\frac{1-q}{1+q(-q)^{-2\imath s}},\end{aligned}$$

and

$$\begin{aligned}\varphi_-(s)=&(1-q)\sum_{k\in2\mathbb{Z}_++1}q^k|q^k|^{-1/2}|q^k|^{-\imath s}\\ =&(1-q)\sum_{l=0}^{\infty}(q(-q)^{-1/2-\imath s})^{2l+1}=q(-q)^{-1/2-\imath s}\varphi_+(s).\end{aligned}$$

Thus,

$$\begin{aligned}\varphi^*(s)=&(1-q(-q)^{-1/2-\imath s})\varphi_+(s)\\ =&\frac{(1-q)(1+(-q)^{1/2-\imath s})}{1-(-q)^{1-2\imath s}}=\frac{1-q}{1-\sqrt{-q}(-q)^{-\imath s}}.\end{aligned}$$

It follows that

$$\varphi(\mathbb{R})=\left\{\frac{1-q}{1+\sqrt{-q}z}\,:\,z\in\mathbb{T}\right\}=\{\lambda\in\mathbb{C}\,:\,|\lambda-1|=\sqrt{-q}\},$$

and

$$\varphi^*(\mathbb{R})=\left\{\frac{1-q}{1-\sqrt{-q}z}\,:\,z\in\mathbb{T}\right\}=\{\lambda\in\mathbb{C}\,:\,|\lambda-1|=\sqrt{-q}\}.$$

Finally, by Theorem 2 in [16], we obtain[2]

$$\sigma(C_q) = \varphi(\mathbb{R}) \cup \varphi^*(\mathbb{R}) = \{\lambda \in \mathbb{C} : |\lambda - 1| = \sqrt{-q}\}.$$

Example 4. Recall that a measurable function a on $\mathbb{R}$ is called an $(1, r)$-*atom* ($r \in (1, \infty]$) if
(i) the support of a is contained in an interval (b, c);
(ii) $\|a\|_\infty \le \frac{1}{b-c}$ if $r = \infty$, and $\|a\|_r \le (b-c)^{\frac{1}{r}-1}$ if $r \in (1, \infty)$[3];
(iii) $\int_{\mathbb{R}} a(x)dx = 0$.
By atom we mean an $(1, r)$-atom on $\mathbb{R}$.
The *Hardy space* $H^1(\mathbb{R}) = H^{1,r}(\mathbb{R})$ is a space of such functions f on $\mathbb{R}$ that admit an atomic decomposition of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where a_j are $(1, r)$-atoms on $\mathbb{R}$ and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.
Theorem 4.1 from [12] shows that the condition $N_1(c, A) < \infty$ is sufficient for the operator $\mathcal{H}_{c,A}$ to be bounded in $H^{1,r}(\mathbb{R})$ ($r \in (1, \infty]$). The operator C_q considered in Example 2 satisfies $N_1(c, A) = \infty$ and does not act in $H^{1,r}(\mathbb{R})$ as well. Indeed, the function

$$a(x) := \frac{1}{2}(\chi_{[0,1]}(x) - \chi_{[0,1]}(x + 1))$$

is an $(1, r)$-atom. If we assume that $C_q a \in H^{1,r}(\mathbb{R})$, then the restriction $(C_q a)|\mathbb{R}_+$ belongs to $L^1(\mathbb{R}_+)$. On the other hand, e.g., for $q > 0$, we have by the B. Levy theorem that

$$\begin{aligned}\int_{\mathbb{R}_+} |(C_q a)(x)|dx =&(1 - q) \sum_{k=0}^{\infty} \int_{\mathbb{R}_+} a(q^k x)q^k dx \\ =&(1 - q) \sum_{k=0}^{\infty} \int_{\mathbb{R}_+} a(t)dt = \infty,\end{aligned}$$

a contradiction.

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[2] Here we correct the mistake in the part (2) of Example 3 in [17].
[3] As usual, $\|\cdot\|_r$ denotes the L^r norm.

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