Journal of Mathematical Sciences
https://doi.org/10.1007/s10958-023-06275-7

ON ALGEBRAS OF HAUSDORFF OPERATORS ON THE REAL LINE

E. Liflyand[1] · **A. Mirotin**[2,3]

Accepted: 14 February 2023

Abstract
The aim of this work is to derive a symbol calculus on $L^2(\mathbb{R})$ for one-dimensional Hausdorff operators in apparently the most general form.

Keywords Hausdorff operator · Fourier transform · Symbol calculus · Commutative Banach algebra · Holomorphic functions · Spectrum

Mathematics Subject Classification Primary 46B25 · Secondary 42A38 · 30A99

Introduction and preliminaries

Modern theory and growing interest in Hausdorff operators can be traced back to [4] and, especially, to [8]. One of the simplest properties of such an operator in one of its traditional forms

$$\mathcal{H}_K f(x) := \int_{\mathbb{R}} K(u) f(\frac{x}{u})\, du \tag{1.1}$$

is the fact that it is bounded on $L^2(\mathbb{R})$ if $K(u)|u|^{\frac{1}{2}} \in L^1(\mathbb{R})$.

However, there is a feeling that the study of this traditional form is somewhat exhausted. A more general form of the Hausdorff operator has been recently suggested in [6] and [7]; we present it in a slightly different manner:

$$\mathcal{H}_{K,a} f(x) := \int_{\mathbb{R}} K(u) f(xa(u))\, du, \tag{1.2}$$

where the given functions K and a are measurable and $a(u) \neq 0$ for a.e. $u \in \mathbb{R}$. Similarly, it is bounded on $L^2(\mathbb{R})$ if

✉ A. Mirotin
amirotin@yandex.by

E. Liflyand
liflyand@gmail.com

1 Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

2 Department of Mathematics and Programming Technologies, Francisk Skorina Gomel State University, Gomel, Belarus

3 Regional Mathematical Center of Southern Federal University, Rostov-on-Don, Russia

Published online: 20 April 2023

$$\frac{K(u)}{|a(u)|^{\frac{1}{2}}} \in L^1(\mathbb{R}). \tag{1.3}$$

This follows by straightforward application of generalized Minkowski's inequality.
To make the setting meaningful, we shall assume that a is an odd function, monotonously decreasing to zero on $\mathbb{R}_+$. It may have a singularity at the origin. In other words, this function mimics the usual $\frac{1}{u}$. Throughout, we shall also assume that the condition (1.3) holds.
The aim of this work is to derive a symbol calculus for one-dimensional Hausdorff operators on $L^2(\mathbb{R})$ of the form (1.2). The notion of a symbol for (generalized) multidimensional Hausdorff operators was introduced in [10] and extended in [11]. In our case, the construction of a symbol induces the map

$$\mathrm{Smb} : \mathcal{H}_{K,a} \mapsto \Phi, \mathcal{A}_a \to \mathrm{Mat}_2(C_0(\mathbb{R})), \tag{1.4}$$

which is injective and multiplicative. Here

$$\mathcal{A}_a := \left\{ \mathcal{H}_{K,a} : \frac{K(u)}{|a(u)|^{\frac{1}{2}}} \in L^1(\mathbb{R}) \right\},$$

$C_0(\mathbb{R})$ stands for the algebra of continuous functions on $\mathbb{R}$ vanishing at infinity, and $\mathrm{Mat}_2(C_0(\mathbb{R})$ denotes the algebra of matrices of order 2 with the entries in $C_0(\mathbb{R})$).
It is noteworthy that in some important cases (see, e.g., (1.1)) the symbol of a one-dimensional Hausdorff operator in a sense of [10] is closely related to the notion of a symbol of an integral operator with homogeneous kernel introduced and studied in [5].
There are two main results in this work, Theorems 2.2 and 3.4. We prove and discuss them in the two following sections, respectively.

The algebra $\mathcal{A}_a$

We begin with a property of the map defined in (1.4).

Lemma 2.1 *The map* $\mathrm{Smb} : \mathcal{A}_a \to \mathrm{Mat}_2(C_0(\mathbb{R}))$ *is an isometry, if we endow the algebra* $\mathrm{Mat}_2(C_0(\mathbb{R}))$ *with the norm* $\|\Phi\| = \sup_{s\in\mathbb{R}} \|\Phi(s)\|_{op}$.

Here $\|\cdot\|_{op}$ stands for the operator norm of a matrix as the norm of the operator of multiplication by this matrix.

Proof Let M_Φ denote the operator of multiplication by the matrix function $\Phi \in \mathrm{Mat}_2(C_0(\mathbb{R}))$ in the space of vector valued functions $L^2(\mathbb{R}, \mathbb{C}^2)$. It is known from [10] and [11] that the map $\mathcal{H}_{K,a} \mapsto M_\Phi$ is an isometry (with respect to operator norms) if $\Phi = \mathrm{Smb}(\mathcal{H}_{K,a})$. On the other hand, $\|M_\Phi\| = \|\Phi\|$ by [11, Corollary 3].

We are now in a position to present our first main result.

Theorem 2.1 *The set $\mathcal{A}_a$ is a non-closed commutative subalgebra of the algebra $\mathcal{L}(L^2(\mathbb{R}))$ of bounded operators on $L^2(\mathbb{R})$ without unit.*

Proof Straightforward calculations yield the commutativity of $\mathcal{A}_a$.
Further, the matrix symbol $\mathrm{Smb}(\mathcal{H}_{K,a}) = \Phi$ of an operator $\mathcal{H}_{K,a}$ can be defined as in [10] by

$$\Phi = \begin{pmatrix} \varphi_+ & \varphi_- \\ \varphi_- & \varphi_+ \end{pmatrix}, \tag{2.1}$$

where in our case

$$\varphi_+(s) = \int_{(0,\infty)} \frac{K(u)|u|^{is}}{|a(u)|^{\frac{1}{2}}}\, du = \widehat{K_+}(s), \tag{2.2}$$

$$\varphi_-(s) = \int_{(-\infty,0)} \frac{K(u)|u|^{is}}{|a(u)|^{\frac{1}{2}}}\, du = \widehat{K_-}(s), \tag{2.3}$$

with $K_\pm(t) := \frac{K(\pm e^{-t})e^{-t}}{|a(e^{-t})|^{\frac{1}{2}}} \in L^1(\mathbb{R})$ (the "hat" stands for the Fourier transform).

Since the map (1.4) is an isometry (and therefore, injective) and multiplicative, to prove that $\mathcal{A}_a$ is an algebra, it suffices to show that the product of two symbols is also a symbol. More precisely, it suffices to show that if $\mathrm{Smb}(\mathcal{H}_{K,a}) = \Phi$ and $\mathrm{Smb}(\mathcal{H}_{L,a}) = \Psi$, then $\Phi\Psi = \mathrm{Smb}(\mathcal{H}_{Q,a})$ for some $\mathcal{H}_{Q,a} \in \mathcal{A}_a$.

But

$$\Phi\Psi = \begin{pmatrix} \varphi_+ & \varphi_- \\ \varphi_- & \varphi_+, \end{pmatrix} \begin{pmatrix} \psi_+ & \psi_- \\ \psi_- & \psi_+ \end{pmatrix} = \begin{pmatrix} \varphi_+\psi_+ + \varphi_-\psi_- & \varphi_+\psi_- + \varphi_-\psi_+ \\ \varphi_+\psi_- + \varphi_-\psi_+ & \varphi_+\psi_+ + \varphi_-\psi_- \end{pmatrix}$$

$$= \begin{pmatrix} (K_+ * L_+ + K_- * L_-)^\wedge & (K_+ * L_- + K_- * L_+)^\wedge \\ (K_+ * L_- + K_- * L_+)^\wedge & (K_+ * L_+ + K_- * L_-)^\wedge \end{pmatrix},$$

where $*$ denotes the convolution in $L^1(\mathbb{R})$.

Defining the functions $Q_\pm$ on $\mathbb{R}$ by

$$Q_+(t) := K_+ * L_+(t) + K_- * L_-(t),$$

$$Q_-(t) := K_+ * L_-(t) + K_- * L_+(t),$$

we obtain

$$\Phi\Psi = \begin{pmatrix} \widehat{Q_+} & \widehat{Q_-} \\ \widehat{Q_-} & \widehat{Q_+} \end{pmatrix}.$$

Let Q be a function on $\mathbb{R}$ satisfying

$$Q_\pm(t) = \frac{Q(\pm e^{-t})e^{-t}}{|a(e^{-t})|^{\frac{1}{2}}}. \tag{2.4}$$

Then $\Phi\Psi = \mathrm{Smb}(\mathcal{H}_{Q,a})$ by the formulas similar to (2.1), (2.2), and (2.3). Since $Q_\pm \in L^1(\mathbb{R})$, we have $\frac{Q(u)}{|a(u)|^{\frac{1}{2}}} \in L^1(\mathbb{R})$. Hence, $\mathcal{H}_{Q,a} \in \mathcal{A}_a$.

Choosing a sequence of kernels K_n satisfying (1.3), we enjoy the property that the sequence of Fourier transforms $\widehat{K_{n+}}$ converges to a function from $C_0(\mathbb{R}) \setminus W_0(\mathbb{R})$ uniformly on $\mathbb{R}$. Here $W_0(\mathbb{R})$ denotes the Wiener algebra of Fourier transforms of functions from $L^1(\mathbb{R})$; for a comprehensive survey, see [9]. Assume that the sequence of operators $\mathcal{H}_{K_n,a}$ converges to an operator $\mathcal{H}_{L,a}$ from $\mathcal{A}_a$ in the operator norm. Then by Lemma 2.1, the sequence of symbols $\mathrm{Sym}(\mathcal{H}_{K_n,a})$ converges in the norm $\|\cdot\|_{op}$ to $\mathrm{Sym}(\mathcal{H}_{L,a})$ uniformly on $\mathbb{R}$. But this implies that $\widehat{K_{n+}}$ converges to $\widehat{L_+} \in W_0(\mathbb{R})$ on $\mathbb{R}$, and we arrive at a contradiction.

Finally, let $\mathcal{H}_{K,a} = I$, the identity operator for some $\mathcal{H}_{K,a} \in \mathcal{A}_a$. Then $\mathrm{Smb}(\mathcal{H}_{K,a}) = E_2$ (the unit matrix of order two) and therefore $\widehat{K_+}(s) = 1$, which leads to a contradiction. This completes the proof.

Corollary 2.1 *The algebra $\mathcal{A}_a$ is not Banach.*

The particular case $a(u) = \frac{1}{u}$ reduces to

Example 1 The set

$$\mathcal{A} := \{\mathcal{H}_K : K(u)|u|^{\frac{1}{2}} \in L^1(\mathbb{R})\}$$

is a non-closed commutative subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ without unit.

Functions of Hausdorff operators

In the sequel, let $\sigma(\mathcal{H}_{K,a})$ denote the spectrum of $\mathcal{H}_{K,a}$ in $L^2(\mathbb{R})$.

Theorem 3.1 *Let $\mathcal{H}_{K,a} \in \mathcal{A}_a$. If a function F is holomorphic in the neighborhood N of the set $\sigma(\mathcal{H}_{K,a}) \cup \{0\}$ and $F(0) = 0$, then $F(\mathcal{H}_{K,a}) \in \mathcal{A}_a$.*

Proof Let $\Phi = \mathrm{Smb}(\mathcal{H}_{K,a})$. Then $\mathcal{H}_{K,a} = \mathcal{U}^{-1} M_\Phi \mathcal{U}$, where $\mathcal{U}$ is a unitary operator taking the space $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$ (which is isomorphic to $L^2(\mathbb{R})$) into $L^2(\mathbb{R}) \times L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C}^2)$ [10]. Moreover, the spectrum of $\mathcal{H}_{K,a}$ equals to the spectrum of Φ in the matrix algebra $\mathrm{Mat}_2(C_0(\mathbb{R}))$ [11]. This implies (see, e.g., [1])

$$\begin{aligned}
F(\mathcal{H}_{K,a}) &= \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - \mathcal{H}_{K,a})^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - \mathcal{U}^{-1} M_\Phi \mathcal{U})^{-1} d\lambda \\
&= \mathcal{U}\left(\frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - M_\Phi)^{-1} d\lambda\right)\mathcal{U}^{-1} \\
&= \mathcal{U} F(M_\Phi)\mathcal{U}^{-1} = \mathcal{U} M_{F(\Phi)} \mathcal{U}^{-1},
\end{aligned}$$

where Γ is the boundary of any open neighborhood U of the set $\sigma(\mathcal{H}_{K,a}) \cup \{0\}$ such that N contains its closure. To finish the proof, it remains to show that $F(\Phi)$ is the symbol of an operator in $\mathcal{A}_a$. For all regular λ, we have

$$(\lambda - \Phi)^{-1} = \frac{1}{\Delta}\begin{pmatrix} \lambda - \varphi_+ & -\varphi_- \\ -\varphi_- & \lambda - \varphi_+ \end{pmatrix},$$

where $\Delta := (\lambda - \varphi_+(s))^2 - \varphi_-(s)^2 \neq 0$ for all $s \in \mathbb{R}$. Then

$$\begin{aligned}
F(\Phi) &= \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - \Phi)^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma F(\lambda)\begin{pmatrix} \frac{\lambda-\varphi_+}{\Delta} & \frac{-\varphi_-}{\Delta} \\ \frac{-\varphi_-}{\Delta} & \frac{\lambda-\varphi_+}{\Delta} \end{pmatrix} d\lambda \\
&= \begin{pmatrix} \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{\lambda-\varphi_+}{\Delta} d\lambda & \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{-\varphi_-}{\Delta} d\lambda \\ \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{-\varphi_-}{\Delta} d\lambda & \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{\lambda-\varphi_+}{\Delta} d\lambda \end{pmatrix} \\
&= \begin{pmatrix} F_1(\varphi_-, \varphi_+) & F_2(\varphi_-, \varphi_+) \\ F_2(\varphi_-, \varphi_+) & F_1(\varphi_-, \varphi_+) \end{pmatrix},
\end{aligned}$$

where

$$F_1(z_1,z_2) := \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{\lambda - z_2}{(\lambda - z_2)^2 - z_1^2}d\lambda,$$

and

$$F_2(z_1,z_2) := \frac{-z_1}{2\pi i}\int_\Gamma \frac{F(\lambda)}{(\lambda - z_2)^2 - z_1^2}d\lambda.$$

Functions $\varphi_\pm = \widehat{K_\pm}$ belong to $W_0(\mathbb{R})$. We are going to prove that $F_1(\varphi_-,\varphi_+)$ and $F_2(\varphi_-,\varphi_+)$ belong to $W_0(\mathbb{R})$, too. To this end, we employ the functional calculus of several elements of a commutative Banach algebra with unit (see [3, §13] or, e.g., [2, Ch. III, §4]). However, the convolution algebra $L^1(\mathbb{R})$ is a commutative Banach algebra without unit. Let

$$\mathcal{V} := \{\mathfrak{z} = \lambda e + f : \lambda \in \mathbb{C}, f \in L^1(\mathbb{R})\}$$

be a Banach algebra obtained by the formal adjunction of a unit element e to $L^1(\mathbb{R})$ (see [3, §16]). Each non-zero complex homomorphism of $\mathcal{V}$ is $\psi_s(\lambda e + f) = \widehat{f}(s)$, where $s \in \mathbb{R}$ or $\psi_\infty(\lambda e + f) = \lambda$ [3, §17]. In particular, $\mathfrak{z} \in L^1(\mathbb{R})$ if and only if $\psi_\infty(\mathfrak{z}) = 0$. We denote by $\mathrm{Spec}(\mathcal{V})$ the Gelfand spectrum (the space of all non-zero complex homomorphisms) of the algebra $\mathcal{V}$. The map $\mathcal{G}\mathfrak{z}(\psi) := \psi(\mathfrak{z})$ $(\psi \in \mathrm{Spec}(\mathcal{V}))$ is called the Gelfand transform. Then

$$\mathcal{G}(\lambda e + f) = \lambda + \widehat{f}.$$

The joint spectrum of elements $K_\pm$ of the commutative Banach algebra $\mathcal{V}$ is

$$\begin{aligned}\sigma_\mathcal{V}(K_-,K_+) &:= \{(\mathcal{G}K_-)(\psi),(\mathcal{G}K_+)(\psi) : \psi \in \mathrm{Spec}(\mathcal{V})\}\\ &= \{(\widehat{K_-}(s),\widehat{K_+}(s)) : s \in \mathbb{R}\} \cup \{(0,0)\}\\ &= \{(\varphi_-(s),\varphi_+(s)) : s \in \mathbb{R}\} \cup \{(0,0)\}.\end{aligned}$$

We claim that the functions $F_1(z_1,z_2)$ and $F_2((z_1,z_2))$ are holomorphic in a neighborhood of $\sigma_\mathcal{V}(K_-,K_+)$. Indeed, it is known [10, Theorem 2] that

$$\sigma(\mathcal{H}_{K,a}) = \mathrm{cl}(\varphi(\mathbb{R}) \cup \varphi^*(\mathbb{R})),$$

where $\varphi = \varphi_+ + \varphi_-$, $\varphi^* = \varphi_+ - \varphi_-$. It follows that for all $\lambda \in \Gamma$ and $z_1 \in \mathrm{cl}(\varphi_-(\mathbb{R}))$, $z_2 \in \mathrm{cl}(\varphi_+(\mathbb{R}))$, we have $(\lambda - z_2)^2 - z_1^2 \neq 0$ (since $\lambda \neq z_2 \pm z_1$). Therefore,

$$\min\{|(\lambda - z_2)^2 - z_1^2| : \lambda \in \Gamma, (z_1,z_2) \in \sigma_\mathcal{V}(K_-,K_+)\} > 0,$$

and thus both functions F_1 and F_2 are holomorphic on some neighborhood of the joint spectrum. The functional calculus in commutative Banach algebras implies that there are $Q_\pm \in \mathcal{V}$ such that $F_{1,2}(\widehat{K_-},\widehat{K_+}) = \mathcal{G}Q_\pm$, respectively.

Observing that $F_1(0,0) = F_2(0,0) = 0$, we conclude (see, e.g., [2, p. 78, Theorem 4.5]) that $\psi_\infty(Q_\pm) = F_{1,2}(\psi_\infty(K_-),\psi_\infty(K_+)) = 0$, and so $Q_\pm \in L^1(\mathbb{R})$. It follows that $F_1(\varphi_-,\varphi_+) = \widehat{Q_-} \in W_0(\mathbb{R})$ and $F_2(\varphi_-,\varphi_+) = \widehat{Q_+} \in W_0(\mathbb{R})$. If the function Q on $\mathbb{R}$ is given by (2.4), then $F(\Phi) = \mathrm{Smb}(\mathcal{H}_{Q,a})$, as desired.

Acknowledgements The authors thank the anonymous referees for their very useful comments and suggestions that improve the presentation.

Data availability This manuscript has no associated data.

Declarations

Competing interests The authors declare no competing interests.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators. Part 1. General theory*, Interscience Publishers, N.Y.-London, 1988.
2. T. Gamelin, *Uniform Algebras*, Chelsea, N.Y., 1969.
3. I. Gelfand, D. Raikov, and G. Shilov, *Commutative normed rings*, Chelsea, N.Y., 1964.
4. C. Georgakis, *The Hausdorff mean of a Fourier–Stieltjes transform*, Proc. Amer. Math. Soc. **116** (1992), 465–471.
5. N. Karapetiants and S. Samko, *Equations with involutive operators*, Birkhäuser, Boston, 2001.
6. J.C. Kuang, *Generalized Hausdorff operators on weighted Morrey-Herz spaces*, Acta Math. Sinica (Chin. Ser.) **55** (2012), 895–902 (Chinese; Chinese, English summaries).
7. J.C. Kuang, *Generalized Hausdorff operators on weighted Herz spaces*, Mat. Vesnik **66** (2014), 19–32.
8. E. Liflyand and F. Móricz, *The Hausdorff operator is bounded on the real Hardy space*$H^1(\mathbb{R})$, Proc. Am. Math. Soc., **128**:5 (2000), 1391 – 1396.
9. E. Liflyand, S. Samko and R. Trigub, *The Wiener algebra of absolutely convergent Fourier integrals: an overview*, Anal. Math. Physics **2** (2012), 1–68.
10. A. R. Mirotin, *On the Structure of Normal Hausdorff Operators on Lebesgue Spaces*, Functional Analysis and Its Applications. **53** (2019), 261–269.
11. A. R. Mirotin, *On the description of multidimensional normal Hausdorff operators on Lebesgue spaces*, Forum Math. **32** (2020), 111–119