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Article *in* Mathematical Methods in the Applied Sciences · May 2023

DOI: 10.1002/mma.9318


Alexey Karapetyants
Southern Federal University


Adolf Mirotin
Francisk Skorina Gomel State University

Received: 20 November 2022 | Revised: 24 March 2023 | Accepted: 12 April 2023
DOI: 10.1002/mma.9318

RESEARCH ARTICLE

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Hausdorff–Berezin operators on weighted spaces

Alexey Karapetyants[1] | **Adolf Mirotin**[2,3]

[1]Institute of Mathematics, Mechanics and Computer Sciences and Regional Mathematical Center, Southern Federal University, Rostov-on-Don, Russia

[2]Department of Mathematics and Programming Technologies, Francisk Skorina Gomel State University, Gomel, Belarus

[3]Regional Mathematical Center, Southern Federal University, Rostov-on-Don, Russia

Correspondence
Alexey Karapetyants, Institute of Mathematics, Mechanics and Computer Sciences and Regional Mathematical Center, Southern Federal University, Rostov-on-Don, 344090, Russia.
Email: karapetyants@gmail.com

Communicated by: V. Didenko

Funding information
Ministry of Education and Science of Russia, Grant/Award Number: 075-02-2023-924; State Program of Scientific Research of Republic of Belarus, Grant/Award Number: 20211776

The research presented in this paper is a continuation of the recent studies of new classes of integral operators on the unit disc $\mathbb{D}$, which are called Hausdorff–Berezin operators. We consider general operators of Hausdorff–Berezin type constructed with an arbitrary positive Radon measure on the unit disc within the framework of weighted spaces with the so-called Möbius weights. This is a fairly wide class of weights in the unit disc which includes classical radial weights with singularities or zeroes at the boundary. Sufficient conditions for the boundedness of the Hausdorff–Berezin operators in these spaces are obtained. In the case of nonnegative kernels, a boundedness criterion is obtained in $L^p(\mathbb{D})$. Approximation properties of operators of Hausdorff–Berezin type are also studied. We also discuss some applications.

KEYWORDS
Bloch space, diagonal Besov spaces, Hausdorff operator, Hausdorff–Berezin operator, integral operator

MSC CLASSIFICATION
47G10, 47B38, 46E30

1 | INTRODUCTION

The research presented in this paper is a continuation of the recent studies of new classes of integral operators on the unit disc, which are called Hausdorff–Berezin operators. This class of Hausdorff–Berezin operators was first introduced in Karapetyants et al. [1], where the operators of the form

$$\mathcal{K}f(z) = \int_{\mathbb{D}} K(w) f(\varphi_z(w))\,\mathrm{dH}(w) \tag{1.1}$$

were studied within the framework of the Lebesgue type space $L^p(\mathrm{dH})$. Here, $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$, $z, w \in \mathbb{D}$ stands for involutive Möbius automorphism of $\mathbb{D}$, and dH is the Möbius invariant Haar measure on the unit disc

$$\mathrm{dH}(z) = \frac{\mathrm{dA}(z)}{(1-|z|^2)^2}.$$

Here, also, $\mathrm{dA}(z) = \frac{1}{\pi}dxdy$, $z = x + iy \in \mathbb{D}$, is area measure, normalized so that $\mathrm{A}(\mathbb{D}) = 1$. Operators of the form (1.1) arose as a natural generalization of the Berezin transform, which was the main motivation for the introduction and study of the class of Hausdorff–Berezin operators. Recall that the Berezin transform is defined by

$$\mathbb{B}f(z) = \int_{\mathbb{D}} f(\varphi_z(w))\,\mathrm{dA}(w) = \int_{\mathbb{D}} f(w)|k_z(w)|^2\,\mathrm{dA}(w). \tag{1.2}$$

Here $k_z(\cdot)$ are the normalized reproducing kernels of the classical Bergman space $\mathcal{A}^2(\mathbb{D}, \mathrm{dA})$; see formula (2.1). As is well-known, the Berezin transform plays an important role in complex analysis and operator theory (see, e.g., earlier studies [2–4]). Also, in the class of Hausdorff–Berezin operators, there are other classical operators of complex analysis, for example, the averaging operator over pseudo-hyperbolic discs which, for instance, was used in Mateljevic and Pavlovic [5] to prove a Forelli–Rudin theorem for $0 < p < 1$. If we extend the notion of Hausdorff–Berezin operator to the multidimensional case, then we will have Green potential for invariant Laplacian as another example of such operator (see Karapetyants et al. [6]). The operators (1.1) have the property that

$$\int_{\mathbb{D}} K(w)f(\varphi_z(w))\,\mathrm{dH}(w) = \int_{\mathbb{D}} K(\varphi_z(w))f(w)\,\mathrm{dH}(w).$$

This invariance property, by its similarity with the one-dimensional Hausdorff operators in the real setting, was the motivation to add the name “Hausdorff” to the class of Hausdorff–Berezin operators. The technique used in the study of Hausdorff operators (see previous studies [7–10]) and the so-called Operators with Homogeneous Kernels (see earlier works [11, 12]; see also Karapetyants and Avetisyan [13]) in real analysis appeared to be useful in the present case of operators in complex analysis, certainly with changes and modifications.

Quite naturally, the question arises of studying operators constructed according to a more general measure, also in weighted spaces with general weights. We will no longer have such a pleasant property of invariance (with Jacobian which is equal to 1). However, this setting will include other more general operators and function spaces. For example, the weighted Berezin transform

$$B_\lambda f(z) = (\lambda+1)\int_{\mathbb{D}} f(\varphi_z(w))(1-|w|^2)^\lambda \mathrm{dA}(w) \tag{1.3}$$

can be considered as an operator with the kernel $(\lambda+1)(1-|w|^2)^\lambda$ with respect to a Lebesgue measure, or we can refer this function as a weight, so that we obtain a more general measure. We recall that the operators of the type (1.3) are also used in the study of various questions in weighted spaces with, generally speaking, different weights. In addition, the change in the weight parameter itself can provide approximation properties—see Theorem 4.21. So whether we are considering more general kernels or measures (weights in the definitions of the spaces), the introduction of such a generalization makes sense for the development of the theory as a whole.

All of the above motivates the study of general operations of the following type, which are the main object of the investigation in the present paper:

$$\mathcal{K}_\mu f(z) = \int_{\mathbb{D}} K(w)f(\varphi_z(w))d\mu(w). \tag{1.4}$$

Here, μ stands for an arbitrary positive Radon measure on $\mathbb{D}$. In particular, for a given sequence (d_n) of complex numbers, we get discrete operators of Hausdorff–Berezin type

$$f \mapsto \sum_{n=0}^{\infty} d_n f(\varphi_z(w_n));$$

however, in this paper, we do not pay attention to the discrete case and focus on the continuous one. Mainly, we study the operators (1.4) in weighted space $L^p(\nu) := L^p(\mathbb{D}, \nu\mathrm{dA})$, $1 \leqslant p < \infty$, where ν is the so-called Möbius weight; see below Section 3.1 for details. Note that the Möbius weight class is a fairly wide class of weights that includes, as a particular case, the classical and most commonly used radial weights on the unit disc of the form $(1-|z|^2)^\alpha$, $\alpha \in \mathbb{R}$.

Sufficient conditions for the boundedness of the Hausdorff–Berezin operators in these spaces are obtained in Theorem 3.8. In the case of nonnegative kernels, the necessary boundedness conditions are also given in Theorem 3.10. Moreover, for nonnegative kernels, a boundedness criterion is obtained in unweighted Lebesgue spaces $L^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{dA})$; see Corollary 3.11 (and also Corollary 3.12). We also focus a special attention on operators constructed from the usual Lebesgue measure dA in unweighted Lebesgue spaces $L^p(\mathbb{D})$, since in this case we can fully apply the technique of operators with homogeneous kernels and obtain another type of boundedness conditions. These boundedness conditions are consistent with those obtained earlier in a more general case; see Remark 1. Approximation properties of the operators of Hausdorff–Berezin type are also studied. The approximation approach is demonstrated on two ideas, when the properties of the Möbius automorphism are used (see Theorem 4.20) and also when the general property of averaging for this type of constructions is used, when the weight parameter (in our case, the kernel parameter) changes to infinity (see Theorem 4.21). In Section 5, we focus on certain particular case of Hausdorff–Berezin operator with factorized kernel and discuss some applications. In particular, in Zhu [4], it is shown that there is a bounded embedding of the Bloch space $\mathcal{B}$ in $L^\infty(\mathbb{D})$ and a bounded embedding of the analytic Besov space B_p in $L^\infty(\mathrm{dH}) = L^\infty(\mathbb{D}, \mathrm{dH})$. We show (Theorem 5.23) that Hausdorff–Berezin operator with factorized kernel implements the isometric embedding of corresponding spaces.

In conclusion, we note that, along with operators of the form (1.4), it is natural to study their partners of the following form:

$$\mathcal{H}_\mu f(z) = \int_{\mathbb{D}} K(w) f(\varphi_w(z)) d\mu(w).$$

The class of such operators was introduced in Mirotin [14] and studied in earlier studies [15–17] under the name of Hausdorff–Zhu operators.

Despite the apparent similarity, these classes of operators differ significantly from each other. More details, including the historical background, are written in the paper [15]; here, we do not touch on these operators.

2 | PRELIMINARIES

Let $\mathrm{dA}(z) = \frac{1}{\pi} dxdy$ be the normalized Lebesgue measure on the unit disc $\mathbb{D}$, $z = x+iy \in \mathbb{D}$, and dH is the Möbius invariant Haar measure on the unit disc $\mathrm{dH}(z) = \frac{\mathrm{dA}(z)}{(1-|z|^2)^2}$. The symbol $L^p(\mathbb{D})$ stands for the Lebesgues space on the unit disc with the measure dA, that is, $L^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{dA})$. Also, $L^p(\mathrm{dH}) := L^p(\mathbb{D}, \mathrm{dH})$, $L^p(d\mu) := L^p(\mathbb{D}, d\mu)$ for an arbitrary positive Radon measure μ on $\mathbb{D}$, and finally, for a weight ν on $\mathbb{D}$, the symbol $L^p(\nu) := L^p(\mathbb{D}, \nu\mathrm{dA})$ denotes weighted space. Also, for brevity, we will write $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{D})}$. Instead, for the other cases, we will always indicate the space in the index of the norm, that is, $\|\cdot\|_{L^p(\mathrm{dH})}$, and so on. Let $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$, $z, w \in \mathbb{D}$ stand for involutive Möbius automorphism of $\mathbb{D}$. Recall that the function $K(z, w) = (1-\bar{z}w)^{-2}$, $z, w \in \mathbb{D}$, is the Bergman reproducing kernel for the unit disc, and the normalized reproducing kernels $k_z(\cdot)$, $z \in \mathbb{D}$, are given by the formula

$$k_z(w) = \frac{K(z, w)}{\|K(z, \cdot)\|_{L^2(\mathbb{D})}} = \frac{1-|z|^2}{(1-\bar{z}w)^2}. \tag{2.1}$$

We will frequently use the fact that the substitution $z = \varphi_w(\zeta)$ in the formula for $\varphi_z(w)$ gives

$$\varphi_{\varphi_w(\zeta)}(w) = -\zeta \frac{1-w\bar{\zeta}}{1-\bar{w}\zeta}, \quad \zeta, w \in \mathbb{D}. \tag{2.2}$$

The next lemma is well-known; see Zhu [18, p. 381]. We provide its proof for the sake of completeness.

Lemma 2.1. *([18]). For all $w, \zeta \in \mathbb{D}$, the equation $\zeta = \varphi_z(w)$ has the unique solution*

$$z = \psi_w(\zeta) := \frac{(1-|w|^2)\zeta + (1-|\zeta|^2)w}{1-|\zeta w|^2}. \tag{2.3}$$

Moreover, the function ψ_w is a one-to-one map of the disc $\mathbb{D}$ onto itself. The map ψ_w is real differentiable, and its real Jacobian determinant has the form

$$J_w(\zeta) = \left|\frac{\partial \psi_w(\zeta)}{\partial \zeta}\right|^2 - \left|\frac{\partial \psi_w(\zeta)}{\partial \bar{\zeta}}\right|^2 = \frac{(1-|w|^2)^2 |1-w\bar{\zeta}|^2}{(1-|\zeta w|^2)^3}. \tag{2.4}$$

Proof. The equality $\zeta = \varphi_z(w)$ implies $z - w = \zeta(1 - \bar{z}w)$ and $\bar{z} - \overline{w} = \bar{\zeta}(1 - z\overline{w})$ and (2.3) follows. Next, we have that $(1 - |\zeta|)(1 - |w|) > 0$ implies $|\zeta| + |w| < 1 + |\zeta||w|$ and hence

$$(|\zeta| + |w|)(1 - |\zeta||w|) < (1 + |\zeta||w|)(1 - |\zeta||w|) = 1 - |\zeta w|^2.$$

It follows that

$$|\psi_w(\zeta)| \le \frac{(1 - |w|^2)|\zeta| + (1 - |\zeta|^2)|w|}{1 - |\zeta w|^2} = \frac{(|\zeta| + |w|)(1 - |\zeta||w|)}{1 - |\zeta w|^2} < 1.$$

Then $\psi_w(\mathbb{D}) \subseteq \mathbb{D}$. The converse inclusion is obvious since $z = \psi_w(\varphi_z(w))$. Formula (2.4) is a matter of direct calculus. □

Our arguments will essentially be based on the following result.

Lemma 2.2. *([10]). Let $(G; m)$ be a measure space, $\mathcal{F}(G)$ be some Banach space of m-measurable functions on G, (Ω, μ) be a σ-compact quasi-metric space with positive Radon measure μ, and $F(w, z)$ be a function on $\Omega \times G$. Assume that*

a. *the convergence of a sequence in norm in $\mathcal{F}(G)$ yields the convergence of some subsequence to the same function for m-a. e. $z \in G$;*
b. *$F(w, \cdot) \in \mathcal{F}(G)$ for μ-a. e. $w \in \Omega$;*
c. *the map $w \mapsto F(w, \cdot) : \Omega \to \mathcal{F}(G)$ is Bochner integrable with respect to μ.*

Then for the Bochner integral, the following formula

$$\left((B)\int_\Omega F(w, \cdot)d\mu(w)\right)(z) = \int_\Omega F(w, z)d\mu(w)$$

is valid for m - a. e. $z \in G$.

3 | BOUNDEDNESS OF HAUSDORFF–BEREZIN OPERATORS IN LEBESGUE SPACES

3.1 | Boundedness of operators of Hausdorff–Berezin type on weighted Lebesgue spaces

In this subsection, we study general operators of the form

$$\mathcal{K}_\mu f(z) = \int_{\mathbb{D}} K(w) f(\varphi_z(w)) d\mu(w), \tag{3.1}$$

where μ stands for an arbitrary positive Radon measure on $\mathbb{D}$.

Definition 3.3. We call a measurable function $v : \mathbb{D} \to \mathbb{R}_+$ a *Möbius weight*, if

$$v(z) \le c(w)v(\varphi_z(w)), \qquad z, w \in \mathbb{D},$$

for some measurable function $c : \mathbb{D} \to \mathbb{R}_+$.

Definition 3.4. A positive Möbius weight v is called *a strong Möbius weight* if in addition

$$c'(w) \le \frac{v(z)}{v(\varphi_z(w))} \le c(w), \qquad z, w \in \mathbb{D}, \tag{3.2}$$

for some measurable functions $c, c' : \mathbb{D} \to (0, +\infty)$.

Example 3.5. The function $v_1(z) = 1 - |z|^2$ is a positive Möbius weight. Indeed, since

$$1 - |\varphi_z(w)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{z}w|^2}, \tag{3.3}$$

we get

$$v_1(\varphi_z(w)) \geq \frac{1-|w|^2}{(1+|w|)^2}(1-|z|^2) = \frac{1-|w|}{1+|w|}v_1(z).$$

Example 3.6. The function $v_2(z) = v_1(z)^{-1}$ is a positive Möbius weight, too, since

$$\begin{aligned} v_2(\varphi_z(w)) = (1-|\varphi_z(w)|^2)^{-1} &= \frac{|1-\bar{z}w|^2}{(1-|w|^2)(1-|z|^2)} \\ &\geq \frac{(1-|w|)^2}{1-|w|^2}\frac{1}{1-|z|^2} = \frac{1-|w|}{1+|w|}v_2(z). \end{aligned}$$

Example 3.7. Previous examples show that functions v_1^α are positive Möbius weights for every $\alpha \in \mathbb{R}$ and in this case

$$c(w) = c_\alpha(w) = \left(\frac{1+|w|}{1-|w|}\right)^{|\alpha|}. \tag{3.4}$$

Moreover, v_1^α are strong Möbius weights for $\alpha \geq 0$, and in this case,

$$c'(w) = c'_\alpha(w) = \left(\frac{1-|w|}{1+|w|}\right)^{\alpha}.$$

For a Möbius weight v, we consider $L^p(v)$, $1 \leqslant p < \infty$ and denote

$$N_c(K,\mu,p) := \int_{\mathbb{D}} |K(w)|\left(c(w)\frac{1+|w|}{1-|w|}\right)^{1/p} d\mu(w)$$

if this expression makes sense.

Theorem 3.8. *Let v be a positive Möbius weight, $1 \leqslant p < \infty$, and*

$$K(w)c(w)^{1/p}(1-|w|)^{-1/p} \in L^1(d\mu).$$

Then the operator $\mathcal{K}_\mu$ is bounded in $L^p(v)$ and its operator norm does not exceed $N_c(K,\mu,p)$.

Proof. Firstly, we are going to apply Lemma 2.2 to the case where $G = \Omega = \mathbb{D}$, m such that $dm = v\mathrm{dA}$, $\mathcal{F}(G) = L^p(v)$, and $F(w,z) = K(w)f(\varphi_z(w))$. The validity of condition (a) in Lemma 2.2 for the convergence in $L^p(v)$ is well-known.

To verify condition (b) from Lemma 2.2, for every $w \in \mathbb{D}$, we consider the norm

$$\|F(w,\cdot)\|^p_{L^p(v)} = |K(w)|^p \int_{\mathbb{D}} |f(\varphi_z(w))|^p v(z)\mathrm{dA}(z). \tag{3.5}$$

Putting $z = \psi_w(\zeta)$ in the integral, we get in view of Lemma 2.1

$$\int_{\mathbb{D}} |f(\varphi_z(w))|^p v(z)\mathrm{dA}(z) = \int_{\mathbb{D}} |f(\zeta)|^p |J_w(\zeta)| v(\psi_w(\zeta))\mathrm{dA}(\zeta). \tag{3.6}$$

Expression (2.4) implies

$$|J_w(\zeta)| = \frac{(1-|w|^2)^2|1-\overline{w}\zeta|^2}{(1-|\zeta w|^2)^3}.$$

We claim that

$$\|J_w\|_\infty = \frac{1+|w|}{1-|w|}. \tag{3.7}$$

Indeed, one can assume that $w \neq 0$. Then

$$|J_w(\zeta)| \leq \frac{(1-|w|^2)^2(1+|w||\zeta|)^2}{(1-|\zeta w|^2)^3} = \frac{(1-|w|^2)^2}{(1+|w||\zeta|)(1-|w||\zeta|)^3}.$$

The function $g(t) = (1+|w|t)(1-|w|t)^3$ decreases for $t \in [0,1]$. It follows that

$$|J_w(\zeta)| \leq \frac{(1-|w|^2)^2}{(1+|w|)(1-|w|)^3} = \frac{1+|w|}{1-|w|}.$$

On the other hand, let $w = re^{-i(\theta+\pi)}$, and $\zeta_n := r_n e^{-i\theta}$, where $r, r_n \in (0,1)$, $\theta \in \mathbb{R}$, and $r_n \to 1$ $(n \to \infty)$. Then

$$J_w(\zeta_n) = \frac{(1-r^2)^2(1+rr_n)^2}{(1-r^2r_n^2)^3} \to \frac{1+r}{1-r} = \frac{1+|w|}{1-|w|}.$$

Now, since ν is a Möbius weight,

$$\nu(\psi_w(\zeta)) \leqslant c(w)\nu(\varphi_{\psi_w(\zeta)}(w)) = c(w)\nu(\zeta).$$

Therefore, (3.7) yields in view of formulas (3.5) and (3.6) that for all $w \in \mathbb{D}$

$$\|F(w,\cdot)\|_{L^p(\nu)} \leq |K(w)| \left(c(w)\frac{1+|w|}{1-|w|} \right)^{1/p} \|f\|_{L^p(\nu)} < \infty.$$

This verifies (b).

To verify condition (c) of Lemma 2.2, we consider the integral

$$\int_{\mathbb{D}} \|F(w,\cdot)\|_{L^p(\nu)} d\mu(w).$$

Again, by formulas (3.5) and (3.6), we get for all $w \in \mathbb{D}$

$$\|F(w,\cdot)\|_{L^p(\nu)} \leq |K(w)| \left(c(w)\frac{1+|w|}{1-|w|} \right)^{1/p} \|f\|_{L^p(\nu)}.$$

Thus, the condition $N_c(K,\mu,p) < \infty$ implies (c).

Finally, it follows by Lemma 2.2 that

$$\mathcal{K}_\mu f(\cdot) = \int_{\mathbb{D}} K(w) f(\varphi_{(\cdot)}(w)) d\mu(w) = \int_{\mathbb{D}} F(w,\cdot) d\mu(w),$$

(the Bochner integral) and so,

$$\begin{aligned} \|\mathcal{K}_\mu f\|_{L^p(\nu)} &\leq \int_{\mathbb{D}} \|F(w,\cdot)\|_{L^p(\nu)} d\mu(w) \\ &\leq \|f\|_{L^p(\nu)} \int_{\mathbb{D}} |K(w)| \|J_w\|_\infty^{1/p} c(w)^{1/p} d\mu(w) \\ &= \|f\|_{L^p(\nu)} \int_{\mathbb{D}} |K(w)| \left(c(w)\frac{1+|w|}{1-|w|} \right)^{1/p} d\mu(w) \\ &= N_c(K,\mu,p)\|f\|_{L^p(\nu)}. \end{aligned}$$

This finishes the proof. □

Recall that $dH(z) = \nu_1(z)^{-2} dA(z)$. Thus, we have the following result.

Corollary 3.9. *Let $1 \le p < \infty$ and $K(w)(1-|w|)^{-3/p} \in L^1(d\mu)$. Then the operator $\mathcal{K}_\mu$ is bounded in $L^p(dH)$, and its operator norm does not exceed $N_{c_2}(K,\mu,p)$, where c_2 is given by formula (3.4) in which one should take $\alpha = 2$.*

Theorem 3.10. *Let the kernel K be nonnegative. Suppose that the operator $\mathcal{K}_\mu$ is bounded on $L^p(v)$ with $1 \le p < \infty$. Then the following statements hold.*

1. *Let $p = 1$ and $v \in L^1(\mathbb{D})$. Then $K \in L^1(d\mu)$.*
2. *Let $1 < p < \infty$ and v be a strong Möbius weight, that is, (3.2) holds. Then $K(w)c'(w)^{1/p}(1-|w|))^{-\sigma/p} \in L^1(d\mu)$ for each $\sigma > -1$.*

Proof. The statement (1) is trivial: since $v \in L^1(\mathbb{D})$ taking $f \equiv 1$, we have that $f \in L^1(v)$ and $\mathcal{K}_\mu f(z) = \int_{\mathbb{D}} K(w)d\mu(w)$.

Let us prove the statement (2). Let $1/p + 1/q = 1$, and let

$$\phi(z) = \left(\frac{(1-|z|^2)^\sigma}{v(z)}\right)^{\frac{1}{p}},\ \psi(z) = \left(\frac{(1-|z|^2)^\sigma}{v(z)}\right)^{\frac{1}{q}},\ \sigma > -1.$$

Then $\phi \in L^p(v), \psi \in L^q(v)$. Consider the integral

$$I = \int_{\mathbb{D}} (\mathcal{K}_\mu\phi)(z)\psi(z)v(z)\mathrm{d}A(z).$$

This integral exists, since $\mathcal{K}_\mu\phi \in L^p(v)$ and $\psi \in L^q(v)$. Taking into account (3.3), we have

$$\phi(\varphi_z(w)) = \left(\frac{(1-|w|^2)^\sigma(1-|z|^2)^\sigma}{|1-\bar{z}w|^{2\sigma}}\right)^{1/p}\frac{1}{v(\varphi_z(w)^{1/p}}.$$

Thus,

$$\begin{aligned} I &= \int_{\mathbb{D}} (1-|z|^2)^\sigma \int_{\mathbb{D}} K(w)\frac{(1-|w|^2)^{\sigma/p}}{|1-\bar{z}w|^{2\sigma/p}}\left(\frac{v(z)}{v(\varphi_z(w))}\right)^{1/p} d\mu(w)\mathrm{d}A(z) \\ &\ge \int_{\mathbb{D}} (1-|z|^2)^\sigma \int_{\mathbb{D}} K(w)\frac{(1-|w|^2)^{\sigma/p}}{|1-\bar{z}w|^{2\sigma/p}} c'(w)^{1/p} d\mu(w)\mathrm{d}A(z) \\ &\ge \int_{\mathbb{D}} (1-|z|^2)^\sigma \mathrm{d}A(z) \int_{\mathbb{D}} K(w)c'(w)^{1/p}\left(\frac{1+|w|}{1-|w|}\right)^{\sigma/p} d\mu(w). \end{aligned}$$

Hence, the integral

$$\int_{\mathbb{D}} K(w)c'(w)^{1/p}\left(\frac{1+|w|}{1-|w|}\right)^{\sigma/p} d\mu(w)$$

exists for each $\sigma > -1$, and result follows. □

Corollary 3.11. *Let the kernel K be nonnegative and $1 < p < \infty$. Then the operator $\mathcal{K}_\mu$ is bounded on $L^p(\mathbb{D})$ if and only if*

$$K(w)(1-|w|)^{-1/p} \in L^1(d\mu).$$

Proof. Since in our case $c = c' = 1$, this follows from Theorems 3.8 and 3.10 with $\sigma = 1$. □

Finally, we single out the following result that will be used to compare the boundedness conditions in this section with those given in the next one. We have the following characterization of the boundedness of the operator on $L^p(\mathbb{D})$ induced by the positive kernel $K(z) = (1-|z|^2)^\alpha$.

Corollary 3.12. *Let $1 \leqslant p < \infty$ and $\mathcal{K}_\mu$ be the Hausdorff–Berezin operator with the kernel $K(z) = (1-|z|^2)^\alpha$. Then $\mathcal{K}_\mu$ is bounded in $L^p(\mathbb{D})$ if and only if*

$$\alpha > -1 + \frac{1}{p}. \tag{3.8}$$

Proof. Follows from Corollary 3.11. □

3.2 | Boundedness of Hausdorff–Berezin operators on the spaces $L^p(\mathbb{D})$

Here we consider the case in which the operators are given with respect to the normalized Lebesgue measure:

$$\mathcal{K}f(z) = \int_{\mathbb{D}} K(w)f(\varphi_z(w))\mathrm{dA}(w). \tag{3.9}$$

Substitution $w = \varphi_z(\zeta)$ in the above formula gives

$$\mathcal{K}f(z) = \int_{\mathbb{D}} K(\varphi_z(\zeta))|k_z(\zeta)|^2 f(\zeta)\mathrm{dA}(\zeta). \tag{3.10}$$

Let $1 \leqslant p < \infty$, and let

$$\kappa_p = \sup_{\zeta\in\mathbb{D}} \left(\int_{\mathbb{D}} |K(\varphi_z(\zeta))|^p |k_z(\zeta)|^{2p}\mathrm{dA}(z) \right)^{\frac{1}{p}} \tag{3.11}$$

provided this expression exists.

Theorem 3.13. *Let $1 \le p, r < \infty$. If $\kappa_p < \infty$, then the operator $\mathcal{K}$ is bounded as operator between $L^r(\mathbb{D})$ and $L^p(\mathbb{D})$ and $\|\mathcal{K}\|_{L^r(\mathbb{D})\to L^p(\mathbb{D})} \leqslant \kappa_p$.*

Proof. Consider the case $r = 1$. We will employ Lemma 2.2 where $G = \Omega = \mathbb{D}$, $m = \mu = \mathrm{A}$, $\mathcal{F}(G) = L^p(\mathbb{D})$, and $F(\zeta, z) = K(\varphi_z(\zeta))\,|k_z(\zeta)|^2 f(\zeta)$ (we use the representation (3.10)). Condition (a) of Lemma 2.2 is valid. For the proof of the validity of (b) and (c) from Lemma 2.2, we note that in our case

$$\|F(\zeta,\cdot)\|_p^p = \int_{\mathbb{D}} |K(\varphi_z(\zeta))\,|^p|k_z(\zeta)|^{2p}|f(\zeta)|^p\mathrm{dA}(z) \le \kappa_p^p|f(\zeta)|^p < \infty$$

for every fixed $\zeta \in \mathbb{D}$. Thus, as in the proof of Theorem 3.8, one has for $f \in L^1(\mathbb{D})$

$$\|\mathcal{K}f\|_p \le \int_{\mathbb{D}} \|F(\zeta,\cdot)\|_p\mathrm{dA}(\zeta) \le \kappa_p\|f\|_1.$$

The rest follows by the fact that $L^r(\mathbb{D}) \subset L^1(\mathbb{D})$ and $\|f\|_1 \le \|f\|_r$. □

Corollary 3.14. *Under the conditions of the previous theorem,*

$$\|\mathcal{K}\|_{L^r(\mathbb{D})\to L^p(\mathbb{D})} \le \sup_{\zeta\in\mathbb{D}} \left(\int_{\mathbb{D}} |K(\varphi_z(\zeta))|^p \left(\frac{1+|z|^2}{1-|z|^2}\right)^{2p} dA(z) \right)^{\frac{1}{p}}.$$

Proof. Indeed, $|k_z(\zeta)| \le \frac{1+|z|^2}{1-|z|^2}$ for all $z, \zeta \in \mathbb{D}$. □

Corollary 3.15. *Assume that $\kappa_1 < \infty$. Then the operator $\mathcal{K}$ is bounded on $L^1(\mathbb{D})$ and its operator norm on $L^1(\mathbb{D})$ satisfies $\|\mathcal{K}\| \leqslant \kappa_1$.*

Given $\sigma \in \mathbb{R}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, let us write

$$\kappa_1(p,\sigma) = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{p}}|K(\zeta)|\,\mathrm{dA}(\zeta), \tag{3.12}$$

$$\kappa_2(q,\sigma) = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{q}+2} \left| K\left(-\zeta\frac{1-z\bar{\zeta}}{1-\bar{z}\zeta}\right)\right| \mathrm{dA}(\zeta). \tag{3.13}$$

Note that if a kernel K is radial, then the formula in (3.13) reads as

$$\kappa_2(q,\sigma) = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{q}+2}\, |K(\zeta)|\, \mathrm{dA}(\zeta).$$

Theorem 3.16. *Let $1 < p < \infty$. If there exists $\sigma \in \mathbb{R}$ such that*

$$\kappa_1(p,\sigma) < \infty, \qquad \kappa_2(q,\sigma) < \infty, \tag{3.14}$$

then the operator $\mathcal{K}$ is bounded in $L^p(\mathbb{D})$ and its operator norm on $L^p(\mathbb{D})$ satisfies

$$\|\mathcal{K}\| \leqslant \inf\left\{ \kappa_1(p,\sigma)^{\frac{1}{q}} \kappa_2(q,\sigma)^{\frac{1}{p}} \right\},$$

where infimum is taken with respect to all those σ for which (3.14) holds.

Proof. Set $\tau_\sigma(z) = (1-|z|^2)^\sigma$, $\sigma \in \mathbb{R}$. By Hölder's inequality, we have

$$\begin{aligned}
|\mathcal{K}f(z)| &= \left| \int_{\mathbb{D}} \tau_\sigma(w)^{\frac{1}{pq}} \tau_\sigma(w)^{-\frac{1}{pq}} K(\varphi_z(w)) |k_z(w)|^2 f(w)\, \mathrm{dA}(w) \right| \\
&\leqslant \left(\int_{\mathbb{D}} \tau_\sigma(w)^{\frac{1}{p}} |K(\varphi_z(w))| |k_z(w)|^2\, \mathrm{dA}(w) \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_{\mathbb{D}} \tau_\sigma(w)^{-\frac{1}{q}} |K(\varphi_z(w))| |k_z(w)|^2 |f(w)|^p\, \mathrm{dA}(w) \right)^{\frac{1}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\mathbb{D}} \tau_\sigma(w)^{\frac{1}{p}} |K(\varphi_z(w))| |k_z(w)|^2\, \mathrm{dA}(w) &= \int_{\mathbb{D}} \tau_\sigma(\varphi_z(\zeta))^{\frac{1}{p}} |K(\zeta)|\, \mathrm{dA}(\zeta) \\
&= (1-|z|^2)^{\frac{\sigma}{p}} \int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{p}} |K(\zeta)|\, \mathrm{dA}(\zeta) \\
&\leqslant \kappa_1(p,\sigma)(1-|z|^2)^{\frac{\sigma}{p}}.
\end{aligned}$$

Therefore, by Fubini's theorem and (2.2), we have

$$\begin{aligned}
\kappa_1(p,\sigma)^{-\frac{p}{q}} \|\mathcal{K}f\|_p^p &\leq \int_{\mathbb{D}} (1-|z|^2)^{\frac{\sigma}{q}}\, \mathrm{dA}(z) \int_{\mathbb{D}} \tau_\sigma(w)^{-\frac{1}{q}} |K(\varphi_z(w))| |k_z(w)|^2 |f(w)|^p\, \mathrm{dA}(w) \\
&= \int_{\mathbb{D}} |f(w)|^p\, \mathrm{dA}(w) \int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{\sigma}{q}+2}}{(1-|w|^2)^{\frac{\sigma}{q}+2}} |K(\varphi_z(w))| |k_w(z)|^2\, \mathrm{dA}(z) \\
&= \int_{\mathbb{D}} |f(w)|^p\, \mathrm{dA}(w) \int_{\mathbb{D}} \frac{(1-|\varphi_w(\zeta)|^2)^{\frac{\sigma}{q}+2}}{(1-|w|^2)^{\frac{\sigma}{q}+2}} \left| K\left(-\zeta \frac{1-w\overline{\zeta}}{1-\overline{w}\zeta} \right) \right| \mathrm{dA}(\zeta) \\
&= \int_{\mathbb{D}} |f(w)|^p\, \mathrm{dA}(w) \int_{\mathbb{D}} |k_\zeta(w)|^{\frac{\sigma}{q}+2} \left| K\left(-\zeta \frac{1-w\overline{\zeta}}{1-\overline{w}\zeta} \right) \right| \mathrm{dA}(\zeta) \\
&\leqslant \kappa_2(q,\sigma) \|f\|_p^p.
\end{aligned}$$

Therefore,

$$\|\mathcal{K}f\|_p \leq \kappa_1(p,\sigma)^{\frac{1}{q}} \kappa_2(q,\sigma)^{\frac{1}{p}} \|f\|_p.$$

This finishes the proof. □

Let us calculate the adjoint operator. We have

$$\begin{aligned}\langle \mathcal{K}f, g\rangle &= \int_{\mathbb{D}} \mathcal{K}f(z)\overline{g(z)}\mathrm{dA}(z) = \int_{\mathbb{D}} \overline{g(z)}\mathrm{dA}(z)\int_{\mathbb{D}} K(w)f(\varphi_z(w))\mathrm{dA}(w)\\ &= \int_{\mathbb{D}} \overline{g(z)}\mathrm{dA}(z)\int_{\mathbb{D}} K(\varphi_z(\zeta))f(\zeta)|k_z(\zeta)|^2\mathrm{dA}(\zeta)\\ &= \int_{\mathbb{D}} f(\zeta)\mathrm{dA}(\zeta)\int_{\mathbb{D}} \overline{g(z)}K(\varphi_z(\zeta))|k_z(\zeta)|^2\mathrm{dA}(z)\\ &= \int_{\mathbb{D}} f(\zeta)\mathrm{dA}(\zeta)\int_{\mathbb{D}} \overline{g(\varphi_\zeta(w))}K(\varphi_{\varphi_\zeta(w)}(\zeta))|k_w(\zeta)|^2\mathrm{dA}(w)\\ &= \int_{\mathbb{D}} f(\zeta)\overline{\mathcal{K}^*g(\zeta)}\,\mathrm{dA}(\zeta).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{K}^*g(\zeta) &= \int_{\mathbb{D}} g(\varphi_\zeta(w))\overline{K(\varphi_{\varphi_\zeta(w)}(\zeta))}\,|k_w(\zeta)|^2\mathrm{dA}(w)\\ &= \int_{\mathbb{D}} g(\varphi_\zeta(w))\overline{K\left(-w\frac{1-\zeta\overline{w}}{1-\overline{\zeta}w}\right)}\,|k_w(\zeta)|^2\mathrm{dA}(w).\end{aligned}$$

If the kernel K is radial, then

$$\mathcal{K}^*g(\zeta) = \int_{\mathbb{D}} g(\varphi_\zeta(w))\overline{K(w)}\,|k_w(\zeta)|^2\mathrm{dA}(w).$$

The adjoint operator $\mathcal{K}^*$ is not necessarily a Hausdorff–Berezin operator, so it is not included in the above considerations. In view of this observation, we formulate below the corresponding boundedness result for the adjoint operator which follows from general functional analysis.

Corollary 3.17. *Let $1 < p < \infty$. If there exist $\sigma \in \mathbb{R}$ such that (3.14) holds, then the operator $\mathcal{K}^*$ is bounded on $L^q(\mathbb{D})$ and its operator norm on $L^q(\mathbb{D})$ satisfies*

$$\|\mathcal{K}^*\| \leqslant \inf\{\kappa_1(p,\sigma)^{\frac{1}{q}}\kappa_2(q,\sigma)^{\frac{1}{p}}\},$$

where infimum is taken with respect to all those σ for which (3.14) holds.

For nonnegative K, let us write

$$\begin{aligned}\varkappa_0 &= \int_{\mathbb{D}} K(z)\,\mathrm{dA}(z),\\ \varkappa &= \inf_{\zeta\in\mathbb{D}}\int_{\mathbb{D}} K\,(\varphi_z(\zeta))\;|k_z(\zeta)|^2\mathrm{dA}(z),\\ \varkappa_1(p,\sigma) &= \inf_{z\in\mathbb{D}}\int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{p}}K(\zeta)\,\mathrm{dA}(\zeta),\\ \varkappa_2(q,\sigma) &= \inf_{z\in\mathbb{D}}\int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{q}+2}K\left(-\zeta\frac{1-z\overline{\zeta}}{1-\overline{z}\zeta}\right)\mathrm{dA}(\zeta).\end{aligned}$$

If K is radial, then

$$\varkappa_2(q,\sigma) = \inf_{z\in\mathbb{D}}\int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{q}}K(\zeta)\,\mathrm{dA}(\zeta).$$

Theorem 3.18. *Let the kernel K be nonnegative. Suppose that the operator $\mathcal{K}$ is bounded on $L^p(\mathbb{D})$ with $1 \leqslant p < \infty$. Then the following statements hold.*

1. *If $p = 1$, then $\varkappa_0 < \infty$ and $\varkappa < \infty$.*
2. *If $1 < p < \infty$, then $\varkappa_1(p, \sigma) < \infty$ and $\varkappa_2(q, \sigma) < \infty$ for any $\sigma > -1$.*

Proof. First, suppose that the operator $\mathcal{K}$ is bounded in $L^1(\mathbb{D})$. Then for $\phi(z) = 1$, we have

$$\begin{aligned}\|\mathcal{K}\phi\|_1 &= \int_{\mathbb{D}} \left| \int_{\mathbb{D}} K(w)\,\mathrm{dA}(w) \right| \mathrm{dA}(z) \\ &= \int_{\mathbb{D}} \mathrm{dA}(z) \int_{\mathbb{D}} K(w)\,\mathrm{dA}(w) = \varkappa_0.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathcal{K}\phi\|_1 &= \int_{\mathbb{D}} \left| \int_{\mathbb{D}} K(\varphi_z(w))\,|k_z(w)|^2\,\mathrm{dA}(w) \right| \mathrm{dA}(z) \\ &= \int_{\mathbb{D}} \mathrm{dA}(w) \int_{\mathbb{D}} K(\varphi_z(w))\,|k_z(w)|^2\,\mathrm{dA}(z) \geqslant \varkappa.\end{aligned}$$

Further, suppose that the operator $\mathcal{K}$ is bounded in $L^p(\mathbb{D})$ for some $1 < p < \infty$. Then for all $\phi \in L^p(\mathbb{D})$ and $\psi \in L^q(\mathbb{D})$, we have

$$\left| \int_{\mathbb{D}} (\mathcal{K}\phi)(z)\,\psi(z)\,\mathrm{dA}(z) \right| \leqslant \|\mathcal{K}\|\,\|\phi\|_p \|\psi\|_q,$$

where $\|\mathcal{K}\|$ denotes the operator norm in $L^p(\mathbb{D})$. Let

$$\phi(z) = (1 - |z|^2)^{\frac{\sigma}{p}},\ \ \psi(z) = (1 - |z|^2)^{\frac{\sigma}{q}},\ \ \sigma > -1.$$

We obtain

$$\begin{aligned}\int_{\mathbb{D}} (\mathcal{K}\phi)(z)\,\psi(z)\,\mathrm{dA}(z) &= \int_{\mathbb{D}} (1 - |z|^2)^{\sigma}\,\mathrm{dA}(z) \int_{\mathbb{D}} |k_w(z)|^{\frac{\sigma}{p}} K(w)\,\mathrm{dA}(w) \\ &\geqslant \varkappa_1(p, \sigma) \int_{\mathbb{D}} (1 - |z|^2)^{\sigma}\,\mathrm{dA}(z).\end{aligned}$$

This implies that $\varkappa_1(p, \sigma) < \infty$. By the same arguments applied to the adjoint operator $\mathcal{K}^*$, we obtain

$$\begin{aligned}&\int_{\mathbb{D}} (\mathcal{K}^*\psi)(z)\,\phi(z)\,\mathrm{dA}(z) \\ &= \int_{\mathbb{D}} (1 - |z|^2)^{\frac{\sigma}{p}}\,\mathrm{dA}(z) \int_{\mathbb{D}} (1 - |w|^2)^{\frac{\sigma}{q}}\,\overline{K(\varphi_w(z))}\,|k_w(z)|^2 \mathrm{dA}(w) \\ &= \int_{\mathbb{D}} (1 - |z|^2)^{\sigma}\,\mathrm{dA}(z) \int_{\mathbb{D}} |k_\zeta(z)|^{\frac{\sigma}{q}+2}\,\overline{K\left(-\zeta \frac{1 - z\overline{\zeta}}{1 - \overline{z}\zeta}\right)}\mathrm{dA}(\zeta) \\ &\geqslant \varkappa_2(q, \sigma) \int_{\mathbb{D}} (1 - |z|^2)^{\sigma}\,\mathrm{dA}(z).\end{aligned}$$

This implies that $\varkappa_2(q, \sigma) < \infty$. □

Let us consider again the important example of the kernel K given by the formula $K(z) = (1 - |z|^2)^{\alpha}$.

Lemma 3.19. *Suppose* $K(z) = (1 - |z|^2)^\alpha$, $\alpha \in \mathbb{R}$, *and* $1 < p < \infty$ *with* $1/p + 1/q = 1$. *Then*

(1) $\kappa_1 < \infty$ *if and only if* $\alpha > 0$.

(2) $\kappa_1(p, \sigma) < \infty$ *if and only if*

$$\alpha > \max\left\{-2 + \frac{\sigma}{p}, -1 - \frac{\sigma}{p}\right\}.$$

(3) $\kappa_2(q, \sigma) < \infty$ *if and only if*

$$\alpha > \max\left\{\frac{\sigma}{q}, -3 - \frac{\sigma}{q}\right\} = \max\left\{\sigma - \frac{\sigma}{p}, -3 - \sigma + \frac{\sigma}{p}\right\}.$$

Proof. The proof follows by putting in the kernel $K(z) = (1 - |z|^2)^\alpha$ into (3.11), (3.12), and (3.13) and then applying Lemma 3.10 from Zhu [4]. □

Remark 1. If we apply the conditions of Theorems 3.16 and 3.18 and Lemma 3.19 to the model example, when the kernel is given by the formula $K(z) = (1 - |z|^2)^\alpha$, $\alpha \in \mathbb{R}$, then we will have $\alpha > -1 + \frac{1}{p}$ for the sufficiency of boundedness and $\alpha \geqslant -1 + \frac{1}{p}$ for the necessity of boundedness of the corresponding operator in $L^p(\mathbb{D})$. Indeed, this can be shown by choosing a minimizing function as in the proof of Theorem 13 from Karapetyants et al. [1]. This observation is consistent to a certain extent with the result in Corollary 3.12 (in that corollary, we have the sharper condition $\alpha > -1 + \frac{1}{p}$ for the necessity of boundedness of the corresponding operator).

4 | IDENTITY APPROXIMATION BY HAUSDORFF–BEREZIN OPERATORS

For the goals of approximation, we consider Hausdorff–Berezin construction of the form

$$\mathcal{K}_\varepsilon f(z) = \int_{\mathbb{D}} K(w) f(\varphi_z(\varepsilon w))\, d\mu(w), \tag{4.1}$$

where $0 < \varepsilon < 1$. For the case of Haar measure $d\mu = \mathrm{dH}$, the construction (4.1) may be written in the form

$$\mathcal{K}_\varepsilon f(z) = \int_{\mathbb{D}} K_\varepsilon(\varphi_z(w)) f(w)\, \mathrm{dH}(w),$$

where the kernel K_ε is given by the formula

$$K_\varepsilon(z) = \begin{cases} \varepsilon^2 K\left(\frac{z}{\varepsilon}\right) \frac{(1-|z|^2)^2}{(\varepsilon^2 - |z|^2)^2}, & |z| < \varepsilon; \\ 0, & \varepsilon \leqslant |z| < 1. \end{cases}$$

Since $\varphi_z(\varepsilon w) \to z$ as $\varepsilon \to 0$, for appropriate functions f and kernels K, there exists a pointwise limit

$$\lim_{\varepsilon \to 0} \mathcal{K}_\varepsilon f(z) = f(z) \int_{\mathbb{D}} K(w)\, d\mu(w),\ z \in \mathbb{D},$$

so in the sequel, we naturally assume that

$$K \in L^1(d\mu) \quad \text{and} \quad \int_{\mathbb{D}} K(w)\, d\mu(w) = 1. \tag{4.2}$$

Theorem 4.20. *Let* $1 \leqslant p < \infty$. *Let* v *be a Möbius weight such that infinitely differentiable functions with compact support in* $\mathbb{D}$ *form a dense set in* $L^p(v)$. *Let also* v *be such that* $c(w)$ *is a bounded function on some subdisc* $\mathbb{D}_\delta = \{z \in \mathbb{D} : |z| < 1 - \delta\}$, $\delta \in (0, 1)$. *Under assumption (4.2), the operators (4.1) are identity approximations in* $L^p(v)$, *namely,*

$$\lim_{\varepsilon \to 0} \|\mathcal{K}_\varepsilon f - f\|_{L^p(v)} = 0. \tag{4.3}$$

Proof. Let us show that the operators $\mathcal{K}_\varepsilon$ are uniformly bounded when ε stays away from 1:

$$\sup_{0<\varepsilon<\gamma} \|\mathcal{K}_\varepsilon f\|_{L^p(\nu)} \leqslant C_\gamma \|f\|_{L^p(\nu)} \tag{4.4}$$

for any fixed $\gamma \in (0,1)$. By Minkowski inequality, we have

$$\|\mathcal{K}_\varepsilon f\|_{L^p(\nu)} \leqslant \int_{\mathbb{D}} |K(w)|\, d\mu(w) \left(\int_{\mathbb{D}} |f(\varphi_z(\varepsilon w))|^p \nu(z) \mathrm{dA}(z) \right)^{\frac{1}{p}}.$$

The change of variables $z \to \zeta$ in the inner integral by the rule $\varphi_z(\varepsilon w) = \zeta$ yields

$$\begin{aligned}
\|\mathcal{K}_\varepsilon f\|_{L^p(\nu)} &\leqslant \int_{\mathbb{D}} |K(w)|\, d\mu(w) \left(\int_{\mathbb{D}} |f(\zeta)|^p |J_{\varepsilon w}(\zeta)| \nu(\psi_{\varepsilon w}(\zeta)) \mathrm{dA}(\zeta) \right)^{\frac{1}{p}} \\
&\leqslant \tilde{C}_\gamma \|f\|_{L^p(\nu)} \int_{\mathbb{D}} |K(w)| \left(c(\varepsilon w) \frac{1+\varepsilon|w|}{1-\varepsilon|w|} \right)^{1/p} d\mu(w) \\
&\leqslant C_\gamma \|f\|_{L^p(\nu)} \int_{\mathbb{D}} |K(w)|\, d\mu(w),
\end{aligned}$$

whenever $\gamma < 1$, which proves (4.4). Here we used estimates for the Jacobian and reasonings as those given in the proof of Theorem 3.8 with w replaced by εw in the corresponding formulas. Also, we used that the function $w \to c(\varepsilon w)$ is bounded on $w \in \mathbb{D}$ $(0 < \varepsilon < 1)$.

In view of the Banach–Steinhaus theorem, it remains to check (4.8) on a dense set in $L^p(\nu)$ of infinitely differentiable functions with compact support in $\mathbb{D}$. This can be done by analogy with the corresponding proof in Theorem 14 from Karapetyants et al. [1]. For the sake of completeness, here we present the details.

Let f be an infinitely differentiable function supported on the disc $\mathbb{D}_\delta = \{z \in \mathbb{D} : |z| < 1-\delta\}$ for some $\delta \in (0,1)$. Suppose that $\varepsilon < \delta/(2-\delta)$. We have

$$|\varphi_z(\varepsilon w)| = \left| \frac{z - \varepsilon w}{1 - \varepsilon \bar{z} w} \right| \geqslant \frac{|z| - \varepsilon}{1+\varepsilon}.$$

We claim that $f(\varphi_z(\varepsilon w)) = 0$ for all $w \in \mathbb{D}$ and z close to the boundary. This is true when $\frac{|z|-\varepsilon}{1+\varepsilon} > 1-\delta$, or $|z| > \varepsilon(2-\delta) + 1 - \delta$. Denote $\nu = \varepsilon(2-\delta) - \delta$. By Minkowski inequality, we obtain

$$\begin{aligned}
\|\mathcal{K}_\varepsilon f - f\|_{L^p(\nu)} &\le \int_{\mathbb{D}} |K(w)|\, d\mu(w) \left[\int_{\mathbb{D}} |f(\varphi_z(\varepsilon w)) - f(z)|^p \nu(z) \mathrm{dA}(z) \right]^{\frac{1}{p}} \\
&\le C_\nu \int_{\mathbb{D}} |K(w)| d\mu(w) \left[\int_{\mathbb{D}_\nu} |f(\varphi_z(\varepsilon w)) - f(z)|^p \mathrm{dA}(z) \right]^{\frac{1}{p}} \\
&\le C_\nu \|K\|_{L^1(d\mu)} \sup_{z\in\mathbb{D}_\nu, w\in\mathbb{D}} |f(\varphi_z(\varepsilon w)) - f(z)| \\
&\le C_f C_\nu \|K\|_{L^1(d\mu)} \sup_{z\in\mathbb{D}_\nu, w\in\mathbb{D}} |\varphi_z(\varepsilon w) - z| \to 0,\ \varepsilon \to 0,
\end{aligned}$$

where the constants C_f and C_ν depend only on f and ν, respectively. This finishes the proof. □

Let $\lambda \in [0,\infty)$ and let K_λ be parameterized family of nonnegative kernels such that

$$K_\lambda \in L^1(\mathbb{D}, d\mu) \qquad \text{and} \qquad \int_{\mathbb{D}} K_\lambda(w)\, d\mu(w) = 1,\ \lambda \in [0,\infty), \tag{4.5}$$

and for arbitrary fixed $\delta \in (0,1)$

$$\lim_{\lambda\to\infty} \int_{\delta<|w|<1} K_\lambda(w)\, d\mu(w) = 0. \tag{4.6}$$

As an example, one can take $K_\lambda(z) := (\lambda+1)(1-|z|^2)^\lambda$, for the case $d\mu = \mathrm{d}A$. In such a case, the corresponding operator is nothing but the well-known weighted Berezin transform (1.3). The result proved in Theorem 4.21 below for the case of weighted Berezin transform (1.3) can be found in Theorem 61.9 from Zhu [4]. Theorem 4.21 extends this result for the case of the operator

$$\mathcal{K}_\lambda f(z) = \int_{\mathbb{D}} K_\lambda(w) f(\varphi_z(w))\, d\mu(w). \tag{4.7}$$

Theorem 4.21. *Let $1 \leqslant p < \infty$. Let v be a Möbius weight such that infinitely differentiable functions with compact support in $\mathbb{D}$ form a dense set in $L^p(v)$. Let $\sup_{\lambda\geq\lambda_0} N_c(K_\lambda,\mu,p) < \infty$ for some $\lambda_0 > 0$. Under the assumptions (4.5) and (4.6), the operators (4.7) are identity approximations in $L^p(v)$, namely,*

$$\lim_{\lambda\to+\infty} \|\mathcal{K}_\lambda f - f\|_{L^p(v)} = 0. \tag{4.8}$$

Proof. Let $f \in C(\overline{\mathbb{D}})$. We have

$$\begin{aligned}
|\mathcal{K}_\lambda f(z) - f(z)| &= \left| \int_{\mathbb{D}} K_\lambda(w)\, (f(\varphi_z(w)) - f(z))\, d\mu(w) \right| \\
&\leqslant \int_{|w|<\delta} |K_\lambda(w)|\, |f(\varphi_z(w)) - f(z)|\, d\mu(w) \\
&\quad + \int_{\delta<|w|<1} |K_\lambda(w)|\, |f(\varphi_z(w)) - f(z)|\, d\mu(w) \\
&\leqslant \left(\sup_{|w|<\delta} |f(\varphi_z(w)) - f(z)| \right) \int_{|w|<\delta} |K_\lambda(w)|\, d\mu(w) \\
&\quad + 2\|f\|_\infty \int_{\delta<|w|<1} |K_\lambda(w)|\, d\mu(w).
\end{aligned}$$

Given $\varepsilon \in (0,1)$, the first term can be made less than $\varepsilon/2$ by choosing δ small enough, uniformly in $z \in \mathbb{D}$ due to the uniform continuity of the function f. In turn, uniform continuity is applicable, since for $|w| < \delta$, we have for all $z \in \mathbb{D}$ that $|\varphi_z(w) - z| < \delta\frac{1+\delta}{1-\delta}$. Then for such fixed δ, the second term, which does not depend on z, goes to 0 as $\lambda \to \infty$ by assumption (4.6). Hence, it can be made less then $\varepsilon/2$ by choosing λ.

Let now $f \in L^p(v)$. Fix the sequence $\{f_n\}$ of $C(\overline{\mathbb{D}})$ - functions that converge in $L^p(v)$ to f when $n \to +\infty$. We have

$$\mathcal{K}_\lambda f(z) - f(z) = \mathcal{K}_\lambda(f - f_n)(z) + (\mathcal{K}_\lambda f_n(z) - f_n(z)) + (f_n(z) - f(z))$$

It remains to note that the condition $\sup_{\lambda\geq 0} N_c(K_\lambda,\mu,p) < \infty$ implies the uniform estimate for the norm of the operator $\mathcal{K}_\lambda$ as operator acting in $L^p(v)$. □

We note that for the case $K_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda$ and $v(z) = v_\alpha(z) = (1-|z|^2)^\alpha$, the condition $\sup_{\lambda\geq\lambda_0} N_c(K_\lambda,\mu,p) < \infty$ is satisfied automatically. Indeed, in such a case, we have

$$\begin{aligned}
N_c(K_\lambda,\mu,p) &= (\lambda+1)\int_{\mathbb{D}} (1-|w|^2)^\lambda \left(\left(\frac{1+|w|}{1-|w|}\right)^{|\alpha|} \frac{1+|w|}{1-|w|} \right)^{1/p} d\mu(w) \\
&\leqslant 2^{|\alpha+1|/p}(\lambda+1)\int_{\mathbb{D}} (1-|w|^2)^{\lambda-|\alpha+1|/p}\, d\mu(w) \\
&= \frac{2^{|\alpha+1|/p}(\lambda+1)}{\lambda-|\alpha+1|/p}, \quad (\lambda_0 > |\alpha+1|/p),
\end{aligned}$$

in view of condition (4.5).

Finally, let us mention some concluding remarks. Note that we are considering operators of Hausdorff type, so the question of approximation by constructions of this type is extremely important, since, for example, in the paper [9], the approximation property was designated as one of the main defining properties of Hausdorff operators. The fact that

Berezin's transformation also has, in a certain sense, an approximating property reinforces the importance of the study in this section.

The literature on this topic, even in the case of general Hausdorff operators, is not so extensive. An appeal for the study of approximation problems for Hausdorff operators was posed in Liflyand [19, 20]. Let us also mention two more publications [21, 22] devoted to approximation for Hausdorff operators.

5 | HAUSDORFF–BEREZIN OPERATORS WITH FACTORIZED KERNEL $K(w) = k_1(|w|^2)\overline{w}^m$. SOME EXAMPLES AND APPLICATION

Let us first prove the following technical result.

Lemma 5.22. *Let $K(w) = k_1(|w|^2)\overline{w}^m$, $m \in \mathbb{N}$, and let $k_1 \in L^1([0,1], \rho d\rho)$. Then for every $n \in \mathbb{N}$, we have*

$$\mathcal{K}z^n = \left(\int_0^1 k_1(\rho)\rho^m d\rho\right) n\overline{z}^{m-n} \times \sum_{k\geq 0, k\geq n-m}^{n-1} \binom{n-1}{k} \frac{(m-1)!}{(n-k)!(m-n+k)!}(|z|^2-1)^{n-k}|z|^{2k}.$$

In particular, for $m = 1$, we have

$$\mathcal{K}z^n = \left(\int_0^1 k_1(\rho)\rho d\rho\right) n(|z|^2-1)z^{n-1}. \tag{5.1}$$

Proof. Direct substitution gives

$$\begin{aligned}
\mathcal{K}z^n &= \int_{\mathbb{D}} k_1(|w|^2)\overline{w}^m \left(\frac{z-w}{1-\overline{z}w}\right)^n \mathrm{dA}(w) \\
&= \frac{1}{\pi}\int_0^1 k_1(\rho^2)\rho^{m+1}d\rho \int_0^{2\pi} \frac{(z-\rho e^{i\theta})^n}{(1-\rho e^{i\theta}\overline{z})^n} e^{-im\theta}d\theta \\
&= \frac{1}{\pi}\int_0^1 k_1(\rho^2)\rho^{m+1}d\rho \int_0^{2\pi} \frac{(e^{i\theta}z-\rho)^n}{(e^{i\theta}-\rho\overline{z})^n} e^{im\theta}d\theta \\
&= \int_0^1 k_1(\rho^2)\rho^{m+1}d\rho \frac{1}{\pi i}\int_{\mathbb{T}} \frac{(\xi z-\rho)^n}{(\xi-\rho\overline{z})^n}\xi^{m-1}d\theta.
\end{aligned}$$

Here $\mathbb{T}$ stands for the unit circle. Let us calculate the integral

$$I_{m,n}(z) \equiv \frac{1}{\pi i}\int_{\mathbb{T}} \frac{(\xi z-\rho)^n}{(\xi-\rho\overline{z})^n}\xi^{m-1}d\theta. \tag{5.2}$$

Let $n = 1$, $m = 1, 2, 3 \ldots$. Then

$$I_{m,1}(z) = 2(\xi z-\rho)\xi^{m-1}|_{\xi=\rho\overline{z}} = 2(|z|^2-1)\rho^m\overline{z}^{m-1}. \tag{5.3}$$

Let $n = 2, 3, \ldots, m = 1$. Then

$$\begin{aligned}
I_{m,n}(z) &= \frac{2}{(n-1)!}\frac{d^{n-1}}{d\xi^{n-1}}(\xi z-\rho)^n\,|_{\xi=\rho\overline{z}} \\
&= \frac{2}{(n-1)!}\,(n(n-1)\,\ldots\,2)\,(\xi z-\rho)z^{n-1}\Big|_{\xi=\rho\overline{z}} \\
&= 2n\rho(|z|^2-1)z^{n-1}.
\end{aligned}$$

Let $n = 1, 2, 3, \dots, m = 1, 2, 3, \dots$. Then

$$I_{m,n}(z) = \frac{2}{(n-1)!}\frac{d^{n-1}}{d\xi^{n-1}}(\xi z - \rho)^n \xi^{m-1}|_{\xi=\rho\bar{z}}.$$

We note that

$$\frac{d^{n-1}}{d\xi^{n-1}}(\xi z - \rho)^n \xi^{m-1} = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k}{d\xi^k}(\xi z-\rho)^n \frac{d^{n-1-k}}{d\xi^{n-1-k}}\xi^{m-1}$$
$$= \sum_{k\geq 0, k\geq n-m}^{n-1}\binom{n-1}{k}(n(n-1)\dots(n-k+1))(\xi z-\rho)^{n-k}z^k$$
$$\times((m-1)(m-2)\dots(m-n+k+1))\,\xi^{m-n+k}$$
$$= \sum_{k\geq 0, k\geq n-m}^{n-1}\binom{n-1}{k}\frac{n!}{(n-k)!}(\xi z-\rho)^{n-k}z^k\frac{(m-1)!}{(m-n+k)!}\xi^{m-n+k}.$$

Therefore,

$$I_{m,n}(z) = \frac{2}{(n-1)!}\sum_{k\geq 0, k\geq n-m}^{n-1}\binom{n-1}{k}\frac{n!}{(n-k)!}\rho^{n-k}(|z|^2-1)^{n-k}|z|^{2k}$$
$$\times\frac{(m-1)!}{(m-n+k)!}\rho^{m-n+k}\bar{z}^{m-n}$$
$$= 2n\bar{z}^{m-n}\rho^m\sum_{k\geq 0, k\geq n-m}^{n-1}\binom{n-1}{k}\frac{(m-1)!}{(n-k)!(m-n+k)!}(|z|^2-1)^{n-k}|z|^{2k}.$$

The results now follow by the combining all the above calculations:

$$\mathcal{K}z^n = \left(\int_0^1 k_1(\rho^2)\rho^{2m+1}2d\rho\right)n\bar{z}^{m-n}$$
$$\times\sum_{k\geq 0, k\geq n-m}^{n-1}\binom{n-1}{k}\frac{(m-1)!}{(n-k)!(m-n+k)!}(|z|^2-1)^{n-k}|z|^{2k}.$$

This finishes the proof. □

The following theorem represents some examples in which, in some special cases, our operators implement isomorphisms of some classical spaces of complex analysis. For the definitions and properties of Bloch space $\mathcal{B}$, and diagonal Besov spaces B_p, see, for example, Wulan and Zhu [23]. In particular, the Bloch space $\mathcal{B}$ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The closure $\mathcal{B}_0$ in $\mathcal{B}$ of the set of polynomials is called the little Bloch space. Given a function $k \in L^1([0,1], \rho d\rho)$, let us denote

$$I(k) := \left|\int_0^1 k(\rho)\rho d\rho\right|.$$

Theorem 5.23. *(cf. Zhu [4, Lemma 5.14. Lemma 5.22]) Let $K(w) = k_1(|w|^2)\overline{w}$, where $k_1 \in L^1([0,1], \rho d\rho)$, and the normalization condition $I(k_1) = 1$ holds. Then the following statements hold true.*

(i) The operator $\mathcal{K}$ is an isometric isomorphism of the little Bloch space $\mathcal{B}_0$ onto a subspace of $L^\infty(\mathbb{D})$.
(ii) Let $1 < p < \infty$. The operator $\mathcal{K}$ is an isometric isomorphism of the diagonal Besov space B_p onto a subspace of $L^p(\mathrm{dH})$.

Proof. Note first that in the case $K(w) = k_1(|w|^2)\overline{w}$, formula (5.1) implies that

$$\mathcal{K}f(z) = \left(\int_0^1 k_1(\rho)\rho d\rho\right)(|z|^2 - 1)f'(z) \tag{5.4}$$

for any polynomial f.

Since polynomials are dense in the little Bloch space, it follows that for all $f \in \mathcal{B}_0$,

$$\|\mathcal{K}f\|_\infty = I(k_1)\|f\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}.$$

This proves the statement (i).

To prove (ii), we notice that again formula (5.4) implies that

$$\begin{aligned}\|\mathcal{K}f\|^p_{L^p(\mathrm{dH})} &= \int_{\mathbb{D}}|\mathcal{K}f(z)|^p\mathrm{dH}(z)\\ &= \left|\int_0^1 k_1(\rho)\rho d\rho\right|^p \int_{\mathbb{D}}(1 - |z|^2)^p|f'(z)|^p\mathrm{dH}(z) = \|f\|^p_{B_p}\end{aligned}$$

for any polynomial $f \in B_p$. On the other hand, polynomials are dense in B_p for all $p > 1$ (see, e.g., Zhu [24]). This finishes the proof. □

Finally, in this context, it is interesting to compare the above formulas with the similar formulas for the Hausdorff–Zhu operators (see Karapetyants & Mirotin [15]). Indeed, if the kernel of a Hausdorff–Zhu operator $\mathcal{H}$ constructed with respect to the normalized Lebesgue measure on the unit disc is $K(w) = k_1(|w|^2)w^m$, where $k_1 \in L^1(0,1)$, and $m \in \mathbb{Z}_+$, then for each monomial $e_n(z) = z^n$ ($n \in \mathbb{N}$), we have

$$\mathcal{H}z^n = \lambda(m,n)z^{m+n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{N},$$

where $\lambda(m,n) = \int_0^1 k_1(t)Q_{m,n}(t)dt$ and

$$Q_{m,n}(t) = \frac{n}{m+n}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n-1}{k}\binom{m+n}{n-k}t^{m+k}(1-t)^{n-k}.$$

6 | CONCLUSION

The article deals with classes of Hausdorff–Berezin operators that have recently appeared in analysis as a natural generalization of the Berezin transform (introduced in Karapetyants et al. [1]). The motivation for such a generalization is, in particular, that the weighted Berezin transform is actually a Hausdorff–Berezin operator with a special radial kernel. The Berezin transform plays an important role in complex analysis, quantization theory, and in general in mathematical physics. A number of properties of operators and operator algebras are described by the behavior of the Berezin transformation of the operator, which is essentially a covariant symbol of the operator. More generally, this is closely related to the symbolic Wick calculus (the Berezin symbol is a restriction of the Wick symbol to a diagonal). In the Hausdorff–Berezin operators, a kernel appears, the choice of which undoubtedly changes the properties of the operator and provides new opportunities for the development of the theory as a whole. The present article is in line with the exploitation of this idea.

ACKNOWLEDGEMENTS

Alexey Karapetyants and Adolf Mirotin acknowledge the support of the Ministry of Education and Science of Russia, agreement no. 075-02-2023-924. Adolf Mirotin is also partially supported by the State Program of Scientific Research of Republic of Belarus, project 20211776.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflict of interest.

DATA AVAILABILITY STATEMENT

The authors confirm that all data generated or analyzed during this study are included in this article.

ORCID

Alexey Karapetyants https://orcid.org/0000-0001-6205-3624

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How to cite this article: A. Karapetyants and A. Mirotin, *Hausdorff–Berezin operators on weighted spaces*. Math. Meth. Appl. Sci. (2023), 1–18. DOI 10.1002/mma.9318