МАТЕМАТИКА

УДК 512.542

ОБ ОБОБЩЕННОЙ НОРМЕ КОНЕЧНОЙ ГРУППЫ

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ON THE GENERALIZED NORM OF A FINITE GROUP

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Пусть *G* конечная группа $\pi=\{p_1,\ldots,p_n\}\subseteq\mathbb{P}$. Тогда *G* называется π*-специальной* если $G=O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G)$. Мы используем $\mathfrak{N}_{\pi sp}$ для обозначения класса всех конечных π-специальных групп. Пусть $\mathrm{N}_{\pi sp}(G)$ пересечение нормализаторов π-специальных корадикалов из всех подгрупп *G*, то есть, $\mathrm{N}_{\pi sp}(G)=\bigcap_{H\leq G}N_G(H^{\mathfrak{N}_{\pi sp}})$. Мы говорим, что $\mathrm{N}_{\pi}(G)$ является π*-специальной нормой* группы *G*. Изучены основные свойства π-специальной нормы в *G*. В частности, доказана π-разрешимость группы $\mathrm{N}_{\pi sp}(G)$.

Ключевые слова: *конечная группа*, π*-специальная группа*, π*-разрешимая группа*, π*-специальный корадикал группы*, π*-специальная норма группы*.

Let *G* be a finite group and $\pi=\{p_1,\ldots,p_n\}\subseteq\mathbb{P}$. Then *G* is called π*-special* if $G=O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G)$. We use $\mathfrak{N}_{\pi sp}$ to denote the class of all finite π-special groups. Let $\mathfrak{N}_{\pi sp}(G)$ be the intersection of the normalizers of the π-special residuals of all subgroups of *G*, that is, $\mathrm{N}_{\pi sp}(G)=\bigcap_{H\leq G}N_G(H^{\mathfrak{N}_{\pi sp}})$. We say that $\mathrm{N}_{\pi}(G)$ is the π*-special norm* of *G*. We study the basic properties of the π-special norm of *G*. In particular, we prove that $\mathrm{N}_{\pi sp}(G)$ is π-soluble.

Keywords: *finite group*, π*-special group*, π*-soluble group*, π*-special residual of a group*, π*-special norm of a group*.

1 Introduction

Throughout this paper, all groups are finite and *G* always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi=\{p_1,\ldots,p_n\}\subseteq\mathbb{P}$ and $\pi'=\mathbb{P}\setminus\pi$.

The group G is said to be: π*-special* [1], [2] if $G=O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G)$; *meta*-$\pi$*-special* if *G* is an extension of a π-special group by a π-special group. We use $\mathfrak{N}_{\pi sp}$ to denote the class of all π-special groups.

Various classes of π-special and meta-π-special groups have been studied in many papers and, in particular, in the recent papers [2]–[7]. In this paper, we consider some new properties and applications of such groups.

If $1\in\mathfrak{F}$ is the class of groups, then $G^{\mathfrak{F}}$ is the $\mathfrak{F}$*-residual* of G, that is, the intersection of all normal subgroups N of G with $G/N\in\mathfrak{F}$. In particular, $G^{\mathfrak{N}}$ is the nilpotent residual of *G*; $G^{\mathfrak{N}_{\pi sp}}$ is the π-special residual of G.

Recall that the *norm* $N(G)$ of G is the intersection of the normalizers of all subgroups of G. This concept was introduced by R. Baer [8] (see also [9]) and the norm and the generalized norm of a group have been studied by many authors. In particular, in the recent paper [10] the following analogues of the subgroup $N(G)$ were introduced:

(i) $S(G)=\bigcap_{H\leq G}N_G(H^{\mathfrak{N}})$;

(ii) let $1=S_0(G)\leq S_1(G)\leq\cdots\leq S_n(G)\leq\cdots$, where $S_{i+1}(G)/S_i(G)=S(G/S_{i+1}(G))$ for all $i=0,1,2,\ldots.$ Then $S_\infty(G)=S_n(G)$, where let n is the smallest n such that $S_n(G)=S_{n+1}(G)$.

The basic properties and some applications of the subgroups $S(G)$ and $S_\infty(G)$ were considered in [10]. In this paper we consider the following generalizations of the subgroups $S(G)$ and $S_\infty(G)$.

***Definition* 1.1.** Let $\mathrm{N}_{\pi sp}(G)$ be the intersection of the normalizers of the π-special residuals of all subgroups of G, that is,

$$\mathrm{N}_{\pi sp}(G)=\bigcap_{H\leq G}N_G(H^{\mathfrak{N}_{\pi sp}}).$$

We say that $\mathrm{N}_{\pi sp}(G)$ is the π*-special norm* of *G*.

***Definition* 1.2.** Let

$$1=\mathrm{N}^0_{\pi sp}(G)\leq\mathrm{N}^1_{\pi sp}(G)\leq$$
$$\leq\mathrm{N}^2_{\pi sp}(G)\leq\cdots\leq\mathrm{N}^n_{\pi sp}(G)\leq\cdots,$$

where

$$\mathrm{N}^{i+1}_{\pi sp}(G)/\mathrm{N}^{i}_{\pi sp}(G)=\mathrm{N}_{\pi sp}(G/\mathrm{N}^{i}_{\pi sp}(G))$$

for all $i=0,1,2,\ldots$. And let n be the smallest n such that $\mathrm{N}^{n}_{\pi sp}=\mathrm{N}^{n+1}_{\pi sp}$. Then we write $\mathrm{N}^{\infty}_{\pi sp}(G)=$ $=\mathrm{N}^{n}_{\pi sp}(G)$ and say that $\mathrm{N}^{\infty}_{\pi sp}(G)$ is the π*-special hypernorm* of G.

Obviously, $\mathrm{N}_{\pi sp}(G)$ and $\mathrm{N}^{\infty}_{\pi sp}(G)$ are characteristic subgroups of G.

Before continuing, consider the following example.

***Example* 1.3.** (i) Let $G=P\rtimes(Q\rtimes R)$, where $Q\rtimes R$ is a non-abelian group of order 6 and P is a simple $\mathbb{F}_7(Q\rtimes R)$-module which is faithful for $Q\rtimes R$. Let $\sigma=\{\{2,7\},\{2,7\}'\}$. Then G every proper non-π-special subgroup H of G is either of the form $C_7\rtimes Q^x$, where $|C_7|=7$, or of the form $(Q\rtimes R)^y$ for some $x,y\in G$. In the former case we have $H^{\mathfrak{N}_{\pi sp}}=C_7$ and $N_G(C_7)=PQ^x=PQ$. In the second case we have $((Q\rtimes R)^y)^{\mathfrak{N}_{\pi sp}}=Q^y$ and $N_G(Q^y)=(Q\rtimes R)^y$. Moreover,

$$\bigcap_{y\in G}(Q\rtimes R)^y=(Q\rtimes R)_G\le C_G(P)=P$$

and so $\mathrm{N}_{\pi sp}(G)=1=\mathrm{N}^{\infty}_{\pi sp}(G)$.

(ii) Let G and σ are the same as in (i). Let $A=G\times C_2$. Let $B=(Q\rtimes R)C_2$, where C_2 is a group of order 2. Then $B^{\mathfrak{N}_{\pi sp}}=Q$, $C_2\le\mathrm{N}_{\pi sp}(A)$ and $N_A(Q)=B<A$. Hence $\mathrm{N}_{\pi sp}(A)=C_2=\mathrm{N}^{\infty}_{\pi sp}(G)$.

Our first observation is the following fact.

***Theorem* 1.4.** *For any group* G, *the subgroup* $\mathrm{N}^{\infty}_{\pi sp}(G)$ *is* σ*-soluble.*

***Corollary* 1.5** (Shen, Shi, Qian [10]). *For any group* G, *the subgroup* $S_\infty(G)$ *is soluble.*

Our next result generalizes Theorem 1.4 in [10].

***Theorem* 1.6.** *Let* $G=A\times B$, *where* A *and* B *are subgroups of* G *and* $(|A|,|B|)=1$. *Then*

$$\mathrm{N}_{\pi sp}(G)=\mathrm{N}_{\pi sp}(A)\times\mathrm{N}_{\pi sp}(B).$$

***Corollary* 1.7** (Shen, Shi, Qian [10]). *Let* $G=A\times B$, *where* A *and* B *are subgroups of* G *and* $(|A|,|B|)=1$. *Then* $S(G)=S(A)\times S(B)$.

***Theorem* 1.8.** *The group* G *is meta-*π*-special if and only if* $G/\mathrm{N}_{\pi sp}(G)$ *is meta-*π*-special.*

***Corollary* 1.9** (Shen, Shi, Qian [10]). *The group* G *is meta-nilpotent if and only if* $G/S(G)$ *is meta-nilpotent*.

2 Proofs of the results

We will use in our proofs the following facts about the subgroups $\mathrm{N}_{\pi sp}(G)$ and $\mathrm{N}^{\infty}_{\pi sp}(G)$.

***Lemma* 2.1.** *If* E *is a subgroup of* G, *then* $\mathrm{N}_{\pi sp}(G)\cap E\le\mathrm{N}_{\pi sp}(E)$.

***Lemma* 2.2.** *If* N *is a normal subgroup of* G, *then* $\mathrm{N}_{\pi sp}(G)N/N\le\mathrm{N}_{\pi sp}(G/N)$.

***Lemma* 2.3.** *If* N *is a normal subgroup of* G *and* $N\le\mathrm{N}^{\infty}_{\pi sp}(G)$, *then* $\mathrm{N}^{\infty}_{\pi sp}(G/N)=\mathrm{N}^{\infty}_{\pi sp}(G)/N$.

Proof of Theorem 1.4. It is enough to show that $\mathrm{N}_{\pi sp}(G)$ is π-soluble. Let $X=\mathrm{N}_{\pi sp}(G)$. Then the group X has the following property: the π-special residual of every subgroup of X is normal in X. We show that every group with such a property is π-soluble. Assume that this is false and let X be a counterexample of minimal order. Let M be a maximal subgroup of X and let $N=M^{\mathfrak{N}_{\pi sp}}$ be the π-special residual of M. Then N is normal in G. If $N\ne 1$, then X/N and N are π-soluble since the hypothesis evidently holds for X/N and N and so in this case X π-soluble by the choice of X. Therefore every maximal subgroup M of X is π-special and so G is minimal non-π-special group. Then G is a Schmidt group and so soluble. This contradiction completes the proof of the result.

***Lemma* 2.4.** *Let* $G=A\times B$, *where* A *and* B *are subgroups of* G. *Then* $G^{\mathfrak{N}_{\pi sp}}=A^{\mathfrak{N}_{\pi sp}}\times B^{\mathfrak{N}_{\pi sp}}$.

Proof. First observe that

$$AB/A^{\mathfrak{N}_{\pi sp}}B\simeq A/(A\cap A^{\mathfrak{N}_{\pi sp}}B)=$$
$$=A/A^{\mathfrak{N}_{\pi sp}}(A\cap B)=A/A^{\mathfrak{N}_{\pi sp}}\in\mathfrak{N}_{\pi sp},$$

$B/B^{\mathfrak{N}_{sp}}A\in\mathfrak{N}_{\pi sp}$ and

$$A^{\mathfrak{N}_{\pi sp}}B\cap B^{\mathfrak{N}_{\pi sp}}A=A^{\mathfrak{N}_{\pi sp}}(B\cap B^{\mathfrak{N}_{\pi sp}}A)=$$
$$=A^{\mathfrak{N}_{\pi sp}}B^{\mathfrak{N}_{\pi sp}}(A\cap B)=A^{\mathfrak{N}_{\pi sp}}B^{\mathfrak{N}_{\pi sp}},$$

so $G^{\mathfrak{N}_{\pi sp}}\le A^{\mathfrak{N}_{\pi sp}}B^{\mathfrak{N}_{\pi sp}}$. On the other hand,

$$A/(A\cap G^{\mathfrak{N}_{\pi sp}})\simeq AG^{\mathfrak{N}_{\pi sp}}/G^{\mathfrak{N}_{\pi sp}}\in\mathfrak{N}_{\pi sp}$$

since the formation $\mathfrak{N}_{\pi sp}$ is hereditary. Hence $A^{\mathfrak{N}_{\pi sp}}\le G^{\mathfrak{N}_{\pi sp}}$. Similarly, $B^{\mathfrak{N}_{\pi sp}}\le G^{\mathfrak{N}_{\pi sp}}$. Therefore

$$G^{\mathfrak{N}_{\pi sp}}=A^{\mathfrak{N}_{\pi sp}}\times B^{\mathfrak{N}_{\pi sp}}.$$ □

Proof of Theorem 1.6. Let H be any subgroups of G. Then $H=(H\cap A)\times(H\cap B)$ since $(|A|,|B|)=1$. Therefore $H^{\mathfrak{N}_{\pi sp}}=(H\cap A)^{\mathfrak{N}_{\pi sp}}\times(H\cap B)^{\mathfrak{N}_{\pi sp}}$ by Lemma 2.4. Hence

$$N_G(H^{\mathfrak{N}_{\pi sp}})=N_G((H\cap A)^{\mathfrak{N}_{\pi sp}})\cap N_G((H\cap B)^{\mathfrak{N}_{\pi sp}})=$$
$$=N_A((H\cap A)^{\mathfrak{N}_{\pi sp}})B\cap N_B((H\cap A)^{\mathfrak{N}_{\sigma}})A=$$
$$=N_A((H\cap A)^{\mathfrak{N}_{\sigma}})\times N_B((H\cap A)^{\mathfrak{N}_{\pi sp}}).$$

Therefore we have $\mathrm{N}_{\pi sp}(G)=\mathrm{N}_{\pi sp}(A)\times\mathrm{N}_{\pi sp}(B)$. □

Proof of Theorem 1.8. It is enough to show that if $G/\mathrm{N}_{\pi sp}(G)$ is meta-π-special, then also G is meta-π-special. Assume that this is false and let G be a counterexample of minimal order. Then $\mathrm{N}_{\pi sp}(G)\ne 1$.

Let R be a minimal normal subgroup of G. Then $R\mathrm{N}_{\pi sp}(G)/R\le\mathrm{N}_{\pi sp}(G/R)$ by Lemma 2.2. Moreover,

$$G/R\mathrm{N}_{\pi sp}(G)\simeq$$
$$\simeq (G/\mathrm{N}_{\pi sp}(G)/(R\mathrm{N}_{\sigma}(G)/\mathrm{N}_{\pi sp}(G))\in\mathfrak{N}_{\pi sp}$$

since the class of all meta-π-special groups is a homomorph. Therefore the hypothesis holds for G/R, so the choice of G implies that G/R is meta-π-special. Hence

$$(G/R)^{\mathfrak{N}_{\pi sp}}=G^{\mathfrak{N}_{\pi sp}}R/R\simeq G^{\mathfrak{N}_{\pi sp}}/(G^{\mathfrak{N}_{\pi sp}}\cap R)$$

is π-special. Therefore $R\leq G^{\mathfrak{N}_{\pi sp}}$ and $G^{\mathfrak{N}_{\pi sp}}/R$ is π-special. If G has a minimal normal subgroup $N\neq R$, then $G^{\mathfrak{N}_{\pi sp}}/L$ is also π-special and so $G^{\mathfrak{N}_{\pi sp}}\simeq G^{\mathfrak{N}_{\pi sp}}/(R\cap L)$ is π-special and so G is meta-π-special, contrary to the choice of G. Therefore R is the unique minimal normal subgroup of G, so $R\leq \mathrm{N}_{\pi sp}(G)$ since $\mathrm{N}_{\pi sp}(G)\neq 1$. It is clear also that $R\Phi(G)$ and so for some maximal subgroup M of G we have $G=RM$ and $M_G=1$. Moreover, $M^{\mathfrak{N}_{\pi sp}}\neq 1$ since G is not meta-π-special and R is π-special in view of Theorem 1.4 and the inclusion $R\leq \mathrm{N}_{\pi sp}(G)$. Now observe that $R,\leq N_G(M^{\mathfrak{N}_{\pi sp}})$ and so $M^{\mathfrak{N}_{\pi sp}}$ is normal in G. Hence $M_G\neq 1$. This contradiction completes the proof of the result.

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Поступила в редакцию 13.11.18.