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ON THE SYMBOL CALCULUS FOR MULTIDIMENSIONAL HAUSDORFF OPERATORS

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Abstract
The aim of this work is to derive a symbol calculus on $L^2(\mathbb{R}^n)$ for multidimensional Hausdorff operators. Two aspects of this activity result in two almost independent parts. While throughout the perturbation matrices are supposed to be self-adjoint and form a commuting family, in the second part they are additionally assumed to be positive definite. What relates these two parts is the powerful method of diagonalization of a normal Hausdorff operator elaborated earlier by the second named author.

Keywords Hausdorff operator · Commuting family · Commutative algebra · Symbol · Matrix symbol · Fourier transform · Convolution · Positive definiteness · Holomorphic function · Fractional power

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Introduction

Not diving too deep for the history of the modern Hausdorff operators on the Euclidean spaces, which usually starts with the reference to Hardy's book [8], where accurate reference to independent pioneer works by Rogosinski can be found (in fact, the same results were independently obtained by Garabedian), we instead refer the reader to the survey papers [10] and [5]. Even more recent and relevant is [9], where an attempt is undertaken to figure out what the notion of Hausdorff operator in Euclidean spaces means. We will use a version of the definition given there.

Definition 1.1 *Let K be a locally Lebesgue integrable function on $\mathbb{R}^n$ and let $(A(u))$, with $u \in \mathbb{R}^n$, be a measurable family of real $(n \times n)$-matrices almost everywhere defined in the support of K and satisfying* $\det A(u) \neq 0$ there. *We define the Hausdorff operator $\mathcal{H}_{K,A}$ with kernel K by*

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$$\mathcal{H}_{K,A}f(x) = \int_{\mathbb{R}^n} K(u)f(A(u)x)du,\ x \in \mathbb{R}^n \text{is a column vector.} \tag{1.1}$$

In the sequel, we shall assume that $|\det A(u)|^{-\frac{1}{2}}K(u) \in L^1(\mathbb{R}^n)$. This guarantees the boundedness of $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}^n)$ and induces the introduction of the set

$$\mathcal{A}_A := \left\{ \mathcal{H}_{K,A} : |\det A(u)|^{-\frac{1}{2}}K(u) \in L^1(\mathbb{R}^n) \right\}$$

of bounded operators in $L^2(\mathbb{R}^n)$.

One more assumption throughout the paper will be that the matrices $A(u)$ *are self-adjoint and form a commuting family*. This implies that there is an orthogonal $n \times n$-matrix C and a family of diagonal non-singular real matrices $A'(u) = \mathrm{diag}(a_1(u), ..., a_n(u))$ such that $A'(u) = C^{-1}A(u)C$ for all $u \in \mathbb{R}^n$, where $A(u)$ is defined. By this, $a(u) := (a_1(u), ..., a_n(u))$ is the family of all the eigenvalues (with their multiplicities) of the matrix $A(u)$.

It is known that in this case the Hausdorff operator $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}^n)$ is normal [17].

The aim of this work is to extend the symbol calculus for one-dimensional Hausdorff operators on $L^2(\mathbb{R})$ elaborated in [11] to the multivariate case. It is not a plain business, and in order to use the approach in [11], various constraints are posed on the matrices $A(u)$. This leaves a room for further research in the cases where both the results and the methods of proof are questionable.

It is worth noting one more peculiarity of this work. In fact, a complete definition of Hausdorff operators in [9] suggests that (1.1) allows $u \in \mathbb{R}^m$, with m not necessarily equal to n. To this end, we mention [12], where in certain cases it is necessary that $m > n$. On the other hand, in some problems, the case $m < n$ may also be meaningful (see, e.g., [1]).

There are three main results in this work: Theorems 2.3, 3.5, and 3.6. We formulate, prove, and discuss them and their consequences in the two following sections. All considerations are based on the diagonalization of a normal Hausdorff operator obtained in [17] and [16].

The algebra $\mathcal{A}_A$

Prior to the formulation and proof of the result in this section, we need more preliminaries. We split $\mathbb{R}^n$ into 2^n hyper-octants $\mathbb{R}^n_i$, fixing also an enumeration of this family. For every pair of the indices (i,j), $i,j = 1, \dots, 2^n$, there is a unique $\varepsilon(i,j) \in \{-1,1\}^n$ such that

$$\varepsilon(i,j)\,\mathbb{R}^n_i := \{(\varepsilon(i,j)_1\,u_1, \dots, \varepsilon(i,j)_n u_n) : u \in \mathbb{R}^n_i\} = \mathbb{R}^n_j.$$

It is worth noting that $\varepsilon(i,j)\,\mathbb{R}^n_j = \mathbb{R}^n_i$ and $\varepsilon(i,j)\,\mathbb{R}^n_l \cap \mathbb{R}^n_i = \emptyset$ whenever $l \neq j$. We put

$$\Omega_{ij} := \left\{ u \in \mathbb{R}^n : \left(\mathrm{sgn}\,(a_1(u)), \dots, \mathrm{sgn}\,(a_n(u))\right) = \varepsilon(i,j) \right\}.$$

For $\mathcal{H}_{K,A} \in \mathcal{A}_A$, assuming that

$$|a(u)|^{-\frac{1}{2}-\imath s} := \prod_{k=1}^{n} |a_k(u)|^{-\frac{1}{2}-\imath s_k},\ s = (s_1, \dots, s_n) \in \mathbb{R}^n,$$

with

$$|a_k(u)|^{-\frac{1}{2}-\imath s_k} := e^{-(\frac{1}{2}+\imath s_k)\log|a_k(u)|},$$

we define

$$\varphi_{ij}(s) := \int_{\Omega_{ij}} K(u)\,|a(u)|^{-\frac{1}{2}-\imath s}\,du.$$

Obviously, each $\varphi_{ij} \in C_b(\mathbb{R}^n)$ and $\varphi_{ij} = \varphi_{ji}$.

Following [17] (see also [16]), we are now in a position to define the *matrix symbol* of the Hausdorff operator $\mathcal{H}_{K,A} \in \mathcal{A}_A$ by

$$\Phi = \left(\varphi_{ij}\right)_{i,j=1}^{2^n}. \tag{2.2}$$

By this, Φ is a symmetric element of the matrix algebra $\mathrm{Mat}_{2^n}\left(C_b(\mathbb{R}^n)\right)$.
We also need a property of the map defined above.

Lemma 2.2 *The map* $\mathrm{Smb} : \mathcal{A}_A \to \mathrm{Mat}_{2^n}(C_0(\mathbb{R}^n))$, *where* $C_0(\mathbb{R}^n)$ *stands for the algebra of continuous functions on* $\mathbb{R}^n$ *vanishing at infinity, is an isometry, if we endow the algebra* $\mathrm{Mat}_{2^n}(C_0(\mathbb{R}^n))$ *with the norm* $\|\Phi\| = \sup_{s\in\mathbb{R}^n} \|\Phi(s)\|_{op}$.

Here, $\|\cdot\|_{op}$ stands for the operator norm of a matrix as the norm of the operator of multiplication by this matrix.

Proof Let M_Φ denote the operator of multiplication by the matrix function $\Phi \in \mathrm{Mat}_{2^n}(C_0(\mathbb{R}^n))$ in the space of vector valued functions $L^2(\mathbb{R}^n, \mathbb{C}^{2^n})$. It is known (see [16] and [17]) that the map $\mathcal{H}_{K,A} \mapsto M_\Phi$ is an isometry (with respect to operator norms) if $\Phi = \mathrm{Smb}(\mathcal{H}_{K,A})$. On the other hand, $\|M_\Phi\| = \|\Phi\|$ by [17, Corollary 3]. □

In this section, we will assume that for each pair of indices i, j, the system of equations

$$|a_k(u)| = e^{t_k}, k = 1, \dots, n, \tag{2.3}$$

has the unique solution $u = (b_1(t), \dots, b_n(t)) \in \Omega_{ij}$, $t = (t_1, ..., t_n)$, for almost every $t_k \in \mathbb{R}$. Hence, we have a measurable map $\mathbb{R}^n \to \Omega_{ij}$, $t \mapsto b(t)$, which is almost bijective.
Finally, we are ready to formulate and prove our first main result.

Theorem 2.3 *Under the above assumptions, the set* $\mathcal{A}_A$ *is a non-closed commutative subalgebra without unit of the Banach algebra* $\mathcal{L}(L^2(\mathbb{R}^n))$ *of bounded operators on* $L^2(\mathbb{R}^n)$.

Proof Straightforward calculations yield the commutativity of $\mathcal{A}_A$.
Putting $|a_k(u)| = e^{t_k}$, we get, since

$$|\det A(u)| = \prod_{k=1}^{n} |a_k(u)| = e^{\sum\limits_{k=1}^{n} t_k},$$

that ($s \in \mathbb{R}^n$)

$$\varphi_{ij}(s) = \int_{\mathbb{R}^n} K(b_1(t), \dots, b_n(t)) e^{-\frac{1}{2}\sum\limits_{k=1}^{n} t_k} |J(t)| e^{-\imath s\cdot t}\, dt = \widehat{K_{ij}}(s),$$

where the “hat” stands for the Fourier transform, $t = (t_1, ..., t_n)$,

$$K_{ij}(t) := K(b_1(t), \dots, b_n(t)) e^{-\frac{1}{2}\sum\limits_{l=1}^{n} t_l} |J(t)|,$$

and $J(t) := \frac{\partial(b_1,\dots,b_n)}{\partial(t_1,\dots,t_n)}$ is the Jacobian.
Since the map

$$\mathrm{Smb} : \mathcal{H}_{K,A} \mapsto \Phi, \quad \mathcal{A}_A \to \mathrm{Mat}_{2^n}(C_0(\mathbb{R})) \tag{2.4}$$

is an isometry (and therefore, injective) and multiplicative (see [17] for details), to prove that $\mathcal{A}_A$ is an algebra, it suffices to show that the product of two symbols is also a symbol. More precisely, it suffices to show that if $\mathrm{Smb}(\mathcal{H}_{K,A}) = \Phi$ and $\mathrm{Smb}(\mathcal{H}_{L,A}) = \Psi$, then $\Phi\Psi = \mathrm{Smb}(\mathcal{H}_{Q,A})$ for some $\mathcal{H}_{Q,A} \in \mathcal{A}_A$.

Denoting ψ_{ij} and L_{ij} for Ψ similarly to φ_{ij} and K_{ij} for Φ and replacing the notation $\widehat{}$ for the Fourier transform by $\mathcal{F}$, we have

$$\begin{aligned}\Phi\Psi = (\varphi_{ij})(\psi_{ij}) &= \left(\sum_{k=1}^{2^n} \varphi_{ik}\psi_{kj}\right)_{i,j=1}^{2^n}\\ &= \left(\mathcal{F}\Big(\sum_{k=1}^{2^n} K_{ik} * L_{kj}\Big)\right)_{i,j=1}^{2^n},\end{aligned}$$

where $\psi_{kj} = \widehat{L_{kj}}$, $L_{kj} \in L^1(\mathbb{R}^n)$, and $*$ denotes the convolution in $L^1(\mathbb{R}^n)$.

Defining the functions Q_{ij} on $\mathbb{R}^n$ by

$$Q_{ij} := \sum_{k=1}^{2^n} K_{ik} * L_{kj},$$

we arrive at

$$\Phi\Psi = \left(\widehat{Q_{ij}}\right)_{i,j=1}^{2^n}.$$

Let now Q be a function on $\mathbb{R}^n$ satisfying

$$Q_{ij}(t) = Q(b_1(t), \dots, b_n(t))e^{-\frac{1}{2}\sum_{l=1}^{n} t_l}|J(t)|.$$

By this, $\Phi\Psi = \mathrm{Smb}(\mathcal{H}_{Q,A})$ in accordance with the above arguments. Since $Q_{ij} \in L^1(\mathbb{R}^n)$, we have $|\det A(u)|^{-\frac{1}{2}}Q(u) \in L^1(\mathbb{R}^n)$. Hence, $\mathcal{H}_{Q,A} \in \mathcal{A}_A$.

Let us now proceed to the non-closedness. To this end, we choose a sequence of kernels K_ν satisfying $|\det A(u)|^{-\frac{1}{2}}K_\nu(u) \in L^1(\mathbb{R}^n)$ and such that the sequence of Fourier transforms $\widehat{K_{\nu,11}}$ converges to a function in $C_0(\mathbb{R}^n) \setminus W_0(\mathbb{R}^n)$ uniformly on $\mathbb{R}^n$. Here, $W_0(\mathbb{R}^n)$ denotes the Wiener algebra of Fourier transforms of functions in $L^1(\mathbb{R}^n)$; for a comprehensive survey, see [14]. Assume that the sequence of operators $\mathcal{H}_{K_\nu,A}$ converges to an operator $\mathcal{H}_{L,A}$ in $\mathcal{A}_A$ in the operator norm. Then, by Lemma 2.2, the sequence of symbols $\mathrm{Smb}(\mathcal{H}_{K_\nu,A})$ converges in the norm $\|\cdot\|_{op}$ to $\mathrm{Smb}(\mathcal{H}_{L,A})$ uniformly on $\mathbb{R}^n$. Since the convergence in the norm in a finite-dimensional space implies the coordinate-wise convergence, this implies that $\widehat{K_{\nu,11}}$ converges (at least pointwise) to $\widehat{L_{11}} \in W_0(\mathbb{R}^n)$ on $\mathbb{R}^n$, and we arrive at a contradiction.

Finally, let $\mathcal{H}_{K,A} = I$, the identity operator for some $\mathcal{H}_{K,A} \in \mathcal{A}_A$. Then, $\mathrm{Smb}(\mathcal{H}_{K,A}) = E_{2^n}$ (the unit matrix of order 2^n). Therefore, $\widehat{K_{ii}}(s) = 1$, a contradiction. This completes the proof. □

Corollary 2.4 *The normed algebra $\mathcal{A}_A$ is not Banach.*

The above treatment leads to the open

Problem What is the closure of $\mathcal{A}_A$ in the uniform operator topology?

Functions of a Hausdorff operator

We begin with an assumption used throughout the preceding section. However, it is simplified here because of the positive definiteness condition posed on the matrices.

Let $a(u) := (a_1(u), \dots, a_n(u))$ be the family of eigenvalues (with their multiplicities) of a *positive definite* matrix $A(u)$. We will assume in this section that the system of equations

$$a_k(u) = e^{t_k}, k = 1, \dots, n$$

has the unique solution $u = (b_1(t), \ldots, b_n(t))$, $t = (t_1, \ldots, t_n)$, for almost every $t_k \in \mathbb{R}$. Again, we have a measurable map $b : \mathbb{R}^n \to \mathbb{R}^n$, which is almost bijective.

Holomorphic functions of a Hausdorff operator

The following theorem is a variant of functional calculus for Hausdorff operators.

Theorem 3.5 *Let K and A satisfy the conditions listed above, and let each matrix $A(u)$ be positive definite. If a function F is holomorphic in the neighborhood E of the spectrum $\sigma(\mathcal{H}_{K,A})$ and $F(0) = 0$, then $F(\mathcal{H}_{K,A})$ is also a Hausdorff operator of the form $\mathcal{H}_{K_F,A}$ bounded in $L^2(\mathbb{R}^n)$.*

Proof Let

$$\varphi_{K,A}(s) := \int_{\mathbb{R}^n} K(u)a(u)^{-\frac{1}{2}-\imath s}du, \quad s = (s_1, \ldots, s_n) \in \mathbb{R}^n,$$

stand for the symbol of the Hausdorff operator $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}^n)$ [16, 17].

Recall that, by our definitions, $a(u)^{-\frac{1}{2}-\imath s} := \prod_{k=1}^{n} a_k(u)^{-\frac{1}{2}-\imath s_k}$, where $a_k(u)^{-\frac{1}{2}-\imath s_k} := \exp((-\frac{1}{2} - \imath s_k) \log a_k(u))$.

Note that

$$\begin{aligned}\varphi_{K,A}(s) &= \int_{\mathbb{R}^n} K(u) \prod_{k=1}^{n} a_k(u)^{-\frac{1}{2}} e^{-\imath s_k \log a_k(u)} du \\ &= \int_{\mathbb{R}^n} K(u)(\det A(u))^{-\frac{1}{2}} e^{-\imath s \cdot \log a(u)} du,\end{aligned}$$

where

$$s \cdot \log a(u) := \sum_{k=1}^{n} s_k \log a_k(u).$$

Putting $a_k(u) = e^{t_k}$, with $t = (t_1, ..., t_n) \in \mathbb{R}^n$, we get, since $\det A(u) = \prod_{k=1}^{n} a_k(u) = e^{\sum_{k=1}^{n} t_k}$, that

$$\varphi_{K,A}(s) = \int_{\mathbb{R}^n} K(b_1(t), \ldots, b_n(t))e^{-\frac{1}{2}\sum_{k=1}^{n} t_k}|J(t)|e^{-\imath s \cdot t}dt = \widehat{L}(s),$$

where

$$L(t) := K(b_1(t), \ldots, b_n(t))e^{-\frac{1}{2}\sum_{k=1}^{n} t_k}|J(t)|,$$

and $J(t) := \frac{\partial(b_1,\ldots,b_n)}{\partial(t_1,\ldots,t_n)}$ is the Jacobian.

According to [16, Theorem 1], [17], the Hausdorff operator $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}^n)$ is unitary equivalent to the operator $M_{\varphi_{K,A}}$ of coordinate-wise multiplication by $\varphi_{K,A}$ in the space $L^2(\mathbb{R}^n, \mathbb{C}^{2^n})$ of $\mathbb{C}^{2^n}$-valued functions. More precisely, $\mathcal{H}_{K,A} = \mathcal{V}^{-1}M_{\varphi_{K,A}}\mathcal{V}$, where $\mathcal{V}$ is a unitary operator between $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n, \mathbb{C}^{2^n})$ independent of K. Moreover, the spectrum $\sigma(\mathcal{H}_{K,A})$ is equal to the closure of the range of the symbol $\varphi_{K,A}$.

Then, as in [11, Theorem 2], we have

$$
\begin{aligned}
F(\mathcal{H}_{K,A}) =& \frac{1}{2\pi\imath}\int_{\Gamma} F(\lambda)(\lambda-\mathcal{H}_{K,A})^{-1}d\lambda \\
=& \frac{1}{2\pi\imath}\int_{\Gamma} F(\lambda)(\lambda-\mathcal{V}^{-1}M_{\varphi_{K,A}}\mathcal{V})^{-1}d\lambda \\
=& \mathcal{V}^{-1}\left(\frac{1}{2\pi\imath}\int_{\Gamma} F(\lambda)(\lambda-M_{\varphi_{K,A}})^{-1}d\lambda\right)\mathcal{V} \\
=& \mathcal{V}^{-1}F(M_{\varphi_{K,A}})\mathcal{V} = \mathcal{V}^{-1}M_{F(\varphi_{K,A})}\mathcal{V},
\end{aligned}
\tag{3.5}
$$

where Γ is the boundary of any open neighborhood of the set $\sigma(\mathcal{H}_{K,A})$ such that E contains its closure.

Here, the identity

$$
(\lambda-\mathcal{V}^{-1}M_{\varphi_{K,A}}\mathcal{V})^{-1} = (\mathcal{V}^{-1}\lambda\mathcal{V}-\mathcal{V}^{-1}M_{\varphi_{K,A}}\mathcal{V})^{-1} = \mathcal{V}^{-1}(\lambda-M_{\varphi_{K,A}})^{-1}\mathcal{V}
$$

and properties of the Bochner integral are used.

To finish the proof, it remains to show that $F(\varphi_{K,A})$ is the symbol of some Hausdorff operator $\mathcal{H}_{K_F,A}$ (this operator is bounded in $L^2(\mathbb{R}^n)$ by the holomorphic functional calculus [6]).

To this end, observe that $F(\varphi_{K,A}) \equiv F(\widehat{L}) = \widehat{Q_F}$ for some $Q_F \in L^1(\mathbb{R}^n)$ (see, e. g., [18, Theorem 6.2.4]; this theorem is applicable, since F is holomorphic on the open set E, which contains the closure of the range of $\varphi_{K,A}$). Thus, if we denote $\log a(u) := (\log a_1(u), \dots, \log a_n(u))$ and put

$$
K_F(u) := (\det A(u))^{\frac{1}{2}}\frac{Q_F(\log a(u))}{|J(\log a(u))|},
\tag{3.6}
$$

then

$$
K_F(b_1(t), \dots, b_n(t)) = e^{\frac{1}{2}\sum\limits_{k=1}^{n} t_k}\frac{Q_F(t)}{|J(t)|},
$$

and

$$
\begin{aligned}
\varphi_{K_F,A}(s) =& \int_{\mathbb{R}^n} K_F(u)a(u)^{-\frac{1}{2}-\imath s}du \\
=& \int_{\mathbb{R}^n} K_F(b_1(t), \dots, b_n(t))e^{(-\frac{1}{2})\sum_{k=1}^{n} t_k}|J(t)|e^{-\imath s\cdot t}dt \\
=& \widehat{Q}_F(s) = F(\varphi_{K,A})(s).
\end{aligned}
\tag{3.7}
$$

Therefore, by (3.5), we have $F(\mathcal{H}_{K,A}) = \mathcal{H}_{K_F,A}$, as desired. □

Example 1 Let $F(z) = z^l, l \in \mathbb{N}, l \geq 2$. Then, $\mathcal{H}^l_{K,A}$ equals to some Hausdorff operator $\mathcal{H}_{K_l,A}$ with a scalar symbol $\widehat{Q}_l = \varphi^l_{K,A}$, by (3.7).

We now consider the averaging operator of Boyd [3, 4]

$$
P_\alpha f(x) = x^{\alpha-1}\int_0^x t^{-\alpha}f(t)dt = \int_0^1 u^{-\alpha}f(ux)\,du,
$$

where $P_0 = \mathcal{C}$ is the continuous Cesáro operator. For $\alpha < \frac{1}{2}$, this is a bounded Hausdorff operator in $L^2(\mathbb{R})$ with the kernel $K(u) = \chi_{(0,1)}(u)u^{-\alpha}$, where χ stands for the indicator function of the set indicated as a subscript, and $A(u) = a(u) = u$. Its symbol is

$$\varphi_{P_\alpha}(s) = \int_{\mathbb{R}} \chi_{(0,1)}(u) u^{-\alpha-\frac{1}{2}-\imath s} du = \frac{1}{(\frac{1}{2}-\alpha)-\imath s}.$$

As mentioned above,

$$\widehat{Q}_l(s) = \varphi^l_{P_\alpha}(s) = \frac{1}{((\frac{1}{2}-\alpha)-\imath s)^l}.$$

Formula (3) in [2, Ch. III, §3.2] yields

$$Q_l(t) = \frac{(-1)^{l-1}}{(l-1)!} t^{l-1} e^{(\frac{1}{2}-\alpha)t} \chi_{(-\infty,0)}(t).$$

Since in our case $J(t) = e^t$, formula (3.6) implies

$$\begin{aligned} K_l(u) =& u^{\frac{1}{2}} \frac{Q_l(\log u)}{J(\log u)} \\ =& \frac{u^{\frac{1}{2}}}{u} \frac{(-1)^l}{(l-1)!} (\log u)^{l-1} e^{(\frac{1}{2}-\alpha)\log u} \chi_{(-\infty,0)}(\log u) \\ =& \frac{1}{(l-1)!} \left(\log \frac{1}{u}\right)^{l-1} u^{-\alpha} \chi_{(0,1)}(u). \end{aligned}$$

Thus,

$$P^l_\alpha f(x) = \frac{1}{(l-1)!} \int_0^1 u^{-\alpha} \left(\log \frac{1}{u}\right)^{l-1} f(ux) du.$$

For $x > 0$, this is a formula of Boyd [4, Lemma 2].[1] As follows from our considerations, it is valid for all $f \in L^2(\mathbb{R})$ if $\alpha < \frac{1}{2}$. Since for $\alpha < 1 - \frac{1}{p}$, the operator P_α is bounded in $L^p(\mathbb{R})$ with $p \in (1, \infty)$, by the Minkowski inequality, for such α Boyd's formula is valid for all $f \in L^p(\mathbb{R})$, $p \in (1, \infty)$, as well.

Fractional powers of a Hausdorff operator

Let Re $\alpha > 0$. Since the function z^α is not holomorphic in any neighborhood of zero, the approach of the previous subsection is not applicable to the case where $0 \in \sigma(\mathcal{H}_{K,A})$ and needs a special treatment.
For fractional power of a non-negative bounded operator B in the Hilbert space $\mathfrak{H}$, we will make use of the following formula [15, Ch. 3, Proposition 3.1.1; Ch. 5, Definition 5.1.1]). For every positive integer $m >$ Re α,

$$B^\alpha f = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1} (B(t+B)^{-1})^m f dt, \quad f \in \mathfrak{H}.$$

Theorem 3.6 *Let* Re $\alpha > 0$, $A(u)$ *satisfy the above conditions and let, in addition, each matrix* $A(u)$ *be positive definite for a. e. u. Let the scalar symbol* $\varphi_{K,A} \geq 0$ *and the fractional power* $\varphi^\alpha_{K,A}$ *be the Fourier transform of a function* $Q_\alpha \in L^1(\mathbb{R}^n)$. *Then, the fractional power* $\mathcal{H}^\alpha_{K,A}$ *is also a Hausdorff operator of the form* $\mathcal{H}_{K_\alpha,A}$ *bounded in* $L^2(\mathbb{R}^n)$.

Proof We first note that the Hausdorff operator $\mathcal{H}_{K,A}$ is normal and its spectrum $\sigma(\mathcal{H}_{K,A})$ equals to the closure of the range of $\varphi_{K,A}$ [16, 17]. Since $\varphi_{K,A} \geq 0$, we conclude that the operator $\mathcal{H}_{K,A}$ is non-negative in $L^2(\mathbb{R}^n)$. Thus, for $f \in L^2(\mathbb{R}^n)$ and every positive integer $m >$ Re α, we have

[1] Boyd's formula is important for the study of the dynamics of $\mathcal{C}$, see, e. g., [7] and the bibliography therein.

$$\mathcal{H}_{K,A}^{\alpha} f = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1} (\mathcal{H}_{K,A}(t+\mathcal{H}_{K,A})^{-1})^m f dt.$$

As in the proof of the previous theorem, $\mathcal{H}_{K,A} = \mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V}$, where $\mathcal{V}$ is a unitary operator taking $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n, \mathbb{C}^{2^n})$ independent of K. Then,

$$\begin{aligned}(\mathcal{H}_{K,A}(t+\mathcal{H}_{K,A})^{-1})^m &= (\mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V}(t+\mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V})^{-1})^m \\ &= (\mathcal{V}^{-1} M_{\varphi_{K,A}} (t+M_{\varphi_{K,A}})^{-1} \mathcal{V})^m \\ &= \mathcal{V}^{-1} (M_{\varphi_{K,A}} (t+M_{\varphi_{K,A}})^{-1})^m \mathcal{V}.\end{aligned}$$

Here, the identity

$$\begin{aligned}\mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V}(t+\mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V})^{-1} =& \mathcal{V}^{-1} M_{\varphi_{K,A}} \mathcal{V}(\mathcal{V}^{-1}(t+M_{\varphi_{K,A}})\mathcal{V})^{-1} \\ =& \mathcal{V}^{-1} M_{\varphi_{K,A}} (t+M_{\varphi_{K,A}})^{-1} \mathcal{V}\end{aligned}$$

is used. Therefore, by [15, Example 3.3.1],

$$\begin{aligned}\mathcal{H}_{K,A}^{\alpha} f =& \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{V}^{-1} (M_{\varphi_{K,A}} (t+M_{\varphi_{K,A}})^{-1})^m \mathcal{V} f\, dt \\ =& \mathcal{V}^{-1} \left(\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1} (M_{\varphi_{K,A}} (t+M_{\varphi_{K,A}})^{-1})^m \mathcal{V} f\, dt \right) \\ =& \mathcal{V}^{-1} M_{\varphi_{K,A}}^{\alpha} \mathcal{V} f = \mathcal{V}^{-1} M_{\varphi_{K,A}^{\alpha}} \mathcal{V} f.\end{aligned} \tag{3.8}$$

Since $\varphi_{K,A}^{\alpha} = \widehat{Q_\alpha}$ for some $Q_\alpha \in L^1(\mathbb{R}^n)$, we can proceed as in the proof of the previous theorem. Indeed, let

$$K_\alpha(u) := (\det A(u))^{\frac{1}{2}} \frac{Q_\alpha(\log a(u))}{|J(\log a(u))|}. \tag{3.9}$$

Then,

$$K_\alpha(b_1(t), \ldots, b_n(t)) = e^{\frac{1}{2}\sum_{k=1}^n t_k} \frac{Q_\alpha(t)}{|J(t)|},$$

and

$$\begin{aligned}\varphi_{K_\alpha,A}(s) &= \int_{\mathbb{R}^n} K_\alpha(u) a(u)^{-\frac{1}{2}-\imath s} du \\ &= \int_{\mathbb{R}^n} K_\alpha(b_1(t), \ldots, b_n(t)) e^{-\frac{1}{2}\sum_{k=1}^n t_k} |J(t)| e^{-\imath s \cdot t} dt \\ &= \widehat{Q}_\alpha(s) = \varphi_{K,A}^{\alpha}(s).\end{aligned}$$

By virtue of (3.8), this yields $\mathcal{H}_{K,A}^{\alpha} = \mathcal{H}_{K_\alpha,A}$, as desired. □

Remark 1 As mentioned in [16, Corollary 4], under the assumptions of Theorem 3.6, the operator $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}_+^n)$ is unitary equivalent to the operator of multiplication by $\varphi_{K,A}$ in $L^2(\mathbb{R}_+^n)$. It follows that Theorem 3.6 is valid for the space $L^2(\mathbb{R}_+^n)$ as well.

Example 2 Consider the Calderón operator

$$(\mathcal{K}f)(x) = \frac{1}{x}\int_0^x f(u)du + \int_x^\infty \frac{f(u)}{u}du$$

in the space $L^2(\mathbb{R}_+)$. This is a Hausdorff operator with

$$K(u) = \frac{1}{u\max(1,u)}\chi_{(0,\infty)}(u), \quad A(u) = \frac{1}{u}.$$

It follows that the symbol of $\mathcal{K}$ is $\varphi(s) = \frac{1}{s^2+\frac{1}{4}}$. Therefore, $\sigma(\mathcal{K}) = [0,4]$. Furthermore, let $\operatorname{Re}\alpha > 0$. It is known (see, e. g., [2, Ch. 1, §1.12 (40)]) that $\varphi^\alpha(s) = \frac{1}{(s^2+\frac{1}{4})^\alpha}$ is the Fourier transform of the function

$$Q_\alpha(t) = \pi^{-\frac{1}{2}}\Gamma(\alpha)^{-1}|t|^{\alpha-\frac{1}{2}}\mathbf{K}_{\alpha-\frac{1}{2}}\left(\frac{|t|}{2}\right),$$

where $\mathbf{K}_\nu$ is the function of Macdonald. Thus,

$$(\mathcal{K}^\alpha f)(x) = \int_0^\infty K_\alpha(u)f\left(\frac{x}{u}\right)du,$$

where the kernel K_α is given by formula (3.9), with $\det A(u) = a(u) = \frac{1}{u}$, $J(t) = -e^{-t}$, and the Q_α mentioned above. In other words, $K_\alpha(u) = u^{-\frac{3}{2}}Q_\alpha(\log u)$ for $u > 0$. In particular,

$$K_{\frac{1}{2}+\iota\tau}(u) = \frac{1}{\sqrt{\pi}\Gamma(\frac{1}{2}+\iota\tau)}u^{-\frac{3}{2}}|\log u|^{\iota\tau}\mathbf{K}_{\iota\tau}\left(\frac{|\log u|}{2}\right)$$

and

$$(\mathcal{K}^{\frac{1}{2}+\iota\tau}f)(x) = \frac{1}{\sqrt{\pi}\Gamma(\frac{1}{2}+\iota\tau)}\int_0^\infty u^{-\frac{3}{2}}|\log u|^{\iota\tau}\mathbf{K}_{\iota\tau}\left(\frac{|\log u|}{2}\right)f\left(\frac{x}{u}\right)du.$$

Putting here $v = \frac{1}{u}$ and $x = 1$, we arrive at the following index transform (for this class of integral transforms, see, e. g., [19])

$$f^*(\tau) := \frac{1}{\sqrt{\pi}\Gamma(\frac{1}{2}+\iota\tau)}\int_0^\infty v^{-\frac{1}{2}}|\log v|^{\iota\tau}\mathbf{K}_{\iota\tau}\left(\frac{|\log v|}{2}\right)f(v)dv.$$

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