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# Some remarks on the solution of the cell growth equation

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A process of growth and division of cells is modelled by an initial boundary value problem that involves a first-order linear functional partial differential equation, the so-called sell growth equation. The analytical solution to this problem was given in the paper Zaidi *et al.* (Zaidi *et al.* 2015 Solutions to an advanced functional partial differential equation of the pantograph type (*Proc. R. Soc. A* **471**, 20140947 (doi:10.1098/rspa.2014.0947)). In this note, we simplify the arguments given in the paper mentioned above by using the theory of operator semigroups. This theory enables us to prove the existence and uniqueness of the solution and to express this solution in terms of Dyson–Phillips series. The asymptotics of the solution is also discussed from the point of view of the theory of operator semigroups.

## 1. Introduction

The analytical solution to the initial-boundary value problem for the cell growth equation was given at the first time by Zaidi *et al.* [1]. They proved also the uniqueness of the solution and find its asymptotics (for the asymptotic result, see also [2]).

A cell growth model under consideration was developed in [3]. See [1] for the history of the issue and a detailed bibliography.

Let $n(x,t)$ denote the number density functions of cells of size $x$ at time $t$. Then

$$\frac{\partial n(x,t)}{\partial t} + g\frac{\partial n(x,t)}{\partial x} = b\alpha^2 n(\alpha x,t) - (b+\mu)n(x,t), \quad (1.1)$$

where $g > 0$ is the rate of growth, $\mu > 0$ is the rate of death and $b > 0$ is the rate at which cells divide into $\alpha > 1$ equally sized daughter cells.

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The above equation is supplemented by a given initial distribution

$$n(x,0)=n_0(x), \tag{1.2}$$

where $n_0$ is a probability distribution function, and the boundary condition,

$$n(0,t)=0. \tag{1.3}$$

The main goal of this note, is to simplify the arguments given in [1] on the existence and uniqueness of the solution of the initial boundary value problem (1.1), (1.2), (1.3) and to discuss its asymptotics from the point of view of the theory of operator semigroups.

The results were announced in [4].

## 2. Existence and uniqueness of the solution

Following [1], we put

$$n(x,t)=\mathrm{e}^{-(b+\mu)t}u(x,t). \tag{2.1}$$

Then,

$$\frac{\partial u(x,t)}{\partial t}=-g\frac{\partial u(x,t)}{\partial x}+b\alpha^2 u(\alpha x,t), \tag{2.2}$$

and conditions (1.2) and (1.3) take the form

$$u(x,0)=n_0(x) \tag{2.3}$$

and

$$u(0,t)=0. \tag{2.4}$$

Note that each solution $u$ to the initial-boundary value problem (2.2)–(2.4) for $x\geq 0$ can be extended to a solution to the initial problem (2.2), (2.3) for $x\in\mathbb{R}$ if we put $u(x,t)=0$ and $n_0(x)=0$ for all $x\leq 0$, $t\geq 0$. Thus, we shall seek the solutions $u(x,t)$ to the problem (2.2), (2.3) for $x\in\mathbb{R}$, $t\geq 0$ putting $n_0(x)=0$ for $x\leq 0$.

We rewrite the Cauchy problem (2.2), (2.3) with $x\in\mathbb{R}$ in an abstract form in the usual manner. Let $X=L^p(\mathbb{R})$, $1\leq p<\infty$ or $X=C_{ub}(\mathbb{R})$ the space of uniformly continuous bounded functions on $\mathbb{R}$ endowed with the sup norm. Consider the vector-valued function $u(t):=u(\cdot,t)$, $u:\mathbb{R}_+\to X$ and the following operator on $X$:

$$\mathcal{H}f(x)=b\alpha^2 f(\alpha x).$$

Then equation (2.2) takes the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=\left(-g\frac{\mathrm{d}}{\mathrm{d}x}+\mathcal{H}\right)u(t).$$

The operator

$$G:=-g\frac{\mathrm{d}}{\mathrm{d}x}+\mathcal{H}$$

is a generator of a $C_0$-group $S(t)$ on $X$, since the operator $A=-g\mathrm{d}/\mathrm{d}x$ with an appropriate domain $D(A)$[1] is a generator of a $C_0$-group of shifts $S_0(t)f(x)=f(x-gt)$ on $X$, $||S_0(t)||=1$, and $\mathcal{H}$ is bounded on $X$ [6, theorem 13.2.2]. Therefore,

$$u(t)=S(t)u_0 \tag{2.5}$$

is the unique solution to the Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=Gu(t), u(0)=u_0, \tag{2.6}$$

for any $u_0\in D(A)$ (recall that $u_0=n_0$ by (2.3)). The formula (2.5) gives also a so-called mild solution to the Cauchy problem for any $u_0\in X$.

[1]For instance for $X=L^p(\mathbb{R})$, we have $D(A)=\{f\in L^p(\mathbb{R}): f$ absolutely continuous and $f'\in L^p(\mathbb{R})\}$ (e.g. [5, p. 66]).

We shall assume that $u_0(x) = n_0(x)$ for $x \in \mathbb{R}_+$ and $u_0(x) = 0$ for $x < 0$.

The group $S(t)$ can be calculated via the Dyson–Phillips series

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{2.7}$$

where $||S_n(t)|| \leq ||\mathcal{H}||^n t^n/n!$ for $t \geq 0$ (see [6, (13.2.5)]) and

$$S_{n+1}(t) = \int_0^t S_0(t-s)\mathcal{H}S_n(s)\,\mathrm{d}s, \quad n \in \mathbb{Z}_+ \tag{2.8}$$

(see [6, (13.2.4)] or [5, theorem III.1.10]).

Thus,

$$u(x,t) = \sum_{n=0}^{\infty} S_n(t)u_0(x). \tag{2.9}$$

Note that $S_0(t)u_0(x) = u_0(x - gt) \geq 0$ for all $x \in \mathbb{R}$, $t \geq 0$ if $u_0(x) \geq 0$ for all $x \in \mathbb{R}$, and $S_0(t)u_0(x) = 0$ if $x \leq 0$, $t \geq 0$. Now it follows from (2.8) by induction that $u(x,t)$ is non-negative for all $x \in \mathbb{R}$, $t \geq 0$ and equals zero for all $x \leq 0$, $t \geq 0$.

Moreover, since $||S_0(t)||_{X\to X} = 1$, it follows ([6, corollary of the theorem 13.2.1], or [5, theorem III.1.3]), that

$$||S(t)||_{X\to X} \leq e^{t||\mathcal{H}||_{X\to X}} \quad (t \geq 0). \tag{2.10}$$

This yields

$$||u(\cdot,t)||_X \leq e^{t||\mathcal{H}||_{X\to X}}||u_0||_X \quad (t \geq 0). \tag{2.11}$$

In particular, if we assume as in [1] that $u_0 \in L^1(\mathbb{R}_+)$ we get for $X = L^1(\mathbb{R})$ that

$$||u(\cdot,t)||_{L^1} \leq e^{b\alpha t}||u_0||_{L^1} \quad (t \geq 0).$$

This estimate is consistent with the asymptotics for $u(x,t)$ proven in [1].

On the other hand, let $X = C_{\mathrm{ub}}(\mathbb{R})$ and $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$. Then we deduce from (2.11) that

$$|u(x,t)| \leq e^{b\alpha^2 t} \sup_{\mathbb{R}_+} |u_0| \quad \text{for all } x, t \in \mathbb{R}_+.$$

In summary, we have the following result.

**Theorem 2.1.** *Let $X = L^p(\mathbb{R})$, $1 \leq p < \infty$, or $X = C_{\mathrm{ub}}(\mathbb{R})$. Let $n_0 \in X$ be non-negative. Then formula* (2.9) *presents a non-negative solution u to the initial–boundary value problem* (2.2)–(2.4) *such that $u(\cdot,t) \in X$ for $t \geq 0$. Moreover, this solution is unique and the estimate* (2.11) *holds.*

**Remark 2.2.** It follows from (2.9) that

$$u(\cdot,t) \approx \sum_{k=0}^{n} S_k(t)u_0,$$

and by [6, (13.2.6)]

$$||u(\cdot,t) - \sum_{k=0}^{n} S_k(t)u_0||_X \leq ||\mathcal{H}||^{n+1}_{X\to X} t^{n+1} \frac{e^{t||\mathcal{H}||_{X\to X}}}{(n+1)!} \ (t \geq 0).$$

## 3. On the asymptotics of the solution as $t \to \infty$

It was proven in [1] (cf. [2]) for the case $n_0 \in L^1(\mathbb{R}_+)$ that $u(x,t) \sim e^{b\alpha t}y(x)$ pointwise as $t \to \infty$.

On can derive several complements to this result from a general theory of operator semigroups, as well. Recall that a function $f$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ converges to an element $y \in X$ ($X$ is a Banach space) in a sense of Cesàro as $t \to \infty$ if

$$C\text{-}\lim_{t\to\infty} f(t) := \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s = y,$$

(convergence in the norm of $X$; e.g. [7]). Let $X = L^1(\mathbb{R})$. Note that $T(t) := \mathrm{e}^{-b\alpha t}S(t)$ is a bounded $C_0$-semigroup in $L^1(\mathbb{R}_+)$ by (2.10) with the generator $B := G - b\alpha I$ that satisfies the condition (4.5) from [7, p. 261]. Let $u_0 \in \mathrm{Ker}B + \overline{\mathrm{Ran}B}$ with $\mathrm{Ker}B$ and $\overline{\mathrm{Ran}B}$ denoting the kernel and the closure of the range of $B$. Since $u(\cdot, t) = S(t)u_0$, we have by [7, proposition 4.3.1] that $u(\cdot, t) \sim e^{b\alpha t}y$ for some $y \in L^1(\mathbb{R})$ as $t \to \infty$ in a sense that

$$C\text{-}\lim_{t\to\infty} \mathrm{e}^{-b\alpha t}u(\cdot, t) = y,$$

(convergence in $L^1(\mathbb{R})$). Moreover, by this Proposition

$$y = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{e}^{-b\alpha s}S_0(s)u_0\,\mathrm{d}s,$$

and $y$ is non-negative if $n_0$ is non-negative. Next, this Proposition states that $y \in \mathrm{Ker}B$. In other words, $y$ is a non-negative solution of the classical pantograph equation

$$y'(x) = py(x) + ay(\alpha x), \tag{3.1}$$

with $p = -b\alpha/g$, $a = b\alpha^2/g$, which agrees with a result by Hall & Wake [3]. Well-posedness, stability, oscillation, continuity and asymptotic boundedness of the solution of the Cauchi problem as well as analytical and numerical methods for solving this equation have been studied extensively (see [8–11] and the bibliography therein). It is important for us that if $p < 0$, then the equation (3.1) has a unique (up to a constant factor) solution in $L^1(\mathbb{R}_+)$, non-negative on $\mathbb{R}_+$ and this solution $y$ is equal to the solution obtained by Kato & McLeod [10] (see [11]). More precisely, $y = Ly^*$, where $L$ is a positive constant and

$$y^*(x) = \exp(px) + \sum_{n=1}^{\infty} \frac{a^n \exp(p\alpha^n x)}{(-p)^n \prod_{m=1}^{n}(1-\alpha^m)}.$$

This is consistent with the result obtained in [1].

Finally, by Arendt [7, proposition 4.3.1]

$$y = \lim_{\lambda\downarrow 0} \lambda R(\lambda, G - b\alpha I)u_0 = \lim_{\lambda\downarrow 0} \lambda R(\lambda + b\alpha, G)u_0.$$

Thus, the problem of the asymptotics of the solution of our equation is closely related to the asymptotics of the resolvent of the operator $G$.

The results detailed above follow mutatis mutandis for several other choices of $X$, e.g. for $C_0(\mathbb{R}), L^p(\mathbb{R}_+), C_{\mathrm{ub}}(\mathbb{R}_+)$ (in the last two cases we consider spaces of functions on $\mathbb{R}$ that vanish outside $\mathbb{R}_+$) and corresponding semigroup generators and for a more general choice of constants (e.g. $b \in \mathbb{R}$).

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