МАТЕМАТИКА

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КОНЕЧНЫЕ ГРУППЫ С ЗАДАННЫМИ ЛОКАЛЬНЫМИ СЕКЦИЯМИ

Б. Ху[1], Ц. Хуан[1], А.Н. Скиба[2]

[1]*Цзянсунский педагогический университет, Сюйчжоу*
[2]*Гомельский государственный университет им. Ф. Скорины*

FINITE GROUPS WITH GIVEN LOCAL SECTIONS

B. Hu[1], J. Huang[1], A.N. Skiba[2]

[1]*Jiangsu Normal University, Xuzhou*
[2]*F. Scorina Gomel State University*

Группа называется *примарной*, если она является конечной p-группой для некоторого простого числа p. Если $\sigma=\{\sigma_i \mid i\in I\}$ – некоторое разбиение множества $\mathbb{P}$, т. е. $P=\bigcup_{i\in I}\sigma_i$ и $\sigma_i\cap\sigma_j=\varnothing$ для всех $i\neq j$, то мы говорим, что конечная группа G является: σ-*примарной*, если она является σ_i-группой для некоторого i; σ-нильпотентной, если $G=G_1\times\cdots\times G_n$ для некоторых σ-примарных групп $G_1,\ldots,G_n$. Если $N=N_G(A)$ для некоторой примарной неединичной подгруппы A из G, то мы говорим, что N/A_G – *локальная секция группы* G. В данной работе изучается конечная группа G при условии, что все собственные локальные секции из G принадлежат насыщенной наследственной формации $\mathfrak{F}$, также устанавливается нормальная структура G в случае, когда все локальные секции из G являются σ-нильпотентными.

Ключевые слова: *конечная группа, наследственная насыщенная формация, $\mathfrak{F}$-гиперцентр, локальная секция, σ-нильпотентная группа.*

A group is called *primary* if it is a finite p-group for some prime p. If $\sigma=\{\sigma_i \mid i\in I\}$ is some partition of $\mathbb{P}$, that is, $P=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\varnothing$ for all $i\neq j$, then we say that a finite group G is: σ-*primary* if it is a σ_i-group for some i; σ-*nilpotent* if $G=G_1\times\cdots\times G_n$ for some σ-primary groups $G_1,\ldots,G_n$. If $N=N_G(A)$ for some primary non-identity subgroup A of G, then we say that N/A_G is a *local section of* G. In this paper, we study a finite group G under hypothesis that all proper local sections of G belong to a saturated hereditary formation $\mathfrak{F}$, and we determine the normal structure of G in the case when all local sections of G are σ-nilpotent.

Keywords: *finite group, hereditary saturated formation, $\mathfrak{F}$-hypercentre, local section, σ-nilpotent group.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $p\in\pi\subseteq\mathbb{P}$ and $\pi'=\mathbb{P}\setminus\pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G)=\pi(|G|)$, the set of all primes dividing the order of G. If $K\trianglelefteq H\leq G$, then H/K is called a *section* of G.

A group is called *primary* if it is a p-group for some prime p. If $\sigma=\{\sigma_i \mid i\in I\}$ is some partition of $\mathbb{P}$, that is, $P=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\varnothing$ for all $i\neq j$, then we say, following [1], that a group is: σ-*primary* if it is a σ_i-group for some i; σ-*soluble* if every its chief factor is σ-primary. Observe that a group is primary (respectively soluble) if and only if it is σ-primary (respectively σ-soluble) where $\sigma=\{\{2\},\{3\},\ldots\}$.

Definition. (i) If A is a non-identity primary subgroup of G, then $N_G(A)$ is a *local subgroup* of G and $N_G(A)/A_G=N_{G/A_G}(A/A_G)$ is a *local section* of G. Moreover, if $N_G(A)\neq G$, then we say that $N_G(A)/A_G$ is a *proper* local section of G.

(ii) If A is a non-identity σ-primary subgroup of G, then we say that $N_G(A)$ is a σ-*local subgroup* of G and $N_G(A)/A_G$ is a σ-*local section* (a *proper* σ-*local section* if $N_G(A)\neq G$) of G.

Example. (i) The group G is said to be σ-*nilpotent* or σ-*decomposable* (Shemetkov [2]) if $G=$ $=G_1\times\cdots\times G_n$ for some σ-primary groups $G_1,\ldots,G_n$. It can be proved (see Corollary 1.9 below) that G is σ-nilpotent if and only if every σ-local subgroup of G is σ-nilpotent.

(ii) In the group $A=C_p\rtimes \mathrm{Aut}(C_p)$, where $|C_p|$ is an odd prime, each local section is nilpotent. This group is not σ-nilpotent, where $\sigma=\{\{p\},\{p\}'\}$. The only non-nilpotent local section of $GL(2,3)$ is $GL(2,3)/Z(GL(2,3))$.

The σ-nilpotent groups have been proved to be very useful in the formation theory (see, for example, the papers [3], [4] and the books [2, Chapter IV], [5, Chapter 6]). In the recent years, the σ-nilpotent and σ-soluble groups have been found new and to some extent unexpected applications in the theories of permutable and generalized subnormal subgroups (see, in particular, the recent papers [1], [6]–[9]).

In this paper, we study G under hypothesis that all (proper) local sections of G belong to a hereditary local formation $\mathfrak{F}$ containing all nilpotent groups, and we determine the normal structure of G in the case when all local sections of G are σ-nilpotent.

1 The main results

Recall the following

***Definition* 1.1.** The group G is called $\mathfrak{F}$*-critical* [10, p. 517] provided G does not belong to $\mathfrak{F}$ but every proper subgroup of G belongs to $\mathfrak{F}$. A *Schmidt group* is an $\mathfrak{N}$-critical group, where $\mathfrak{N}$ is the class of all nilpotent groups.

We need also some other concepts of the formation theory.

Let $\mathfrak{F}$ be a class of groups containing all identity groups. Then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$*-residual* of G, that is, the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be: *hereditary* if $H \in \mathfrak{F}$ whenever $H \le G \in \mathfrak{F}$; *saturated* if $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \le \Phi(G)$; a *formation* if every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G.

Note that if $\mathfrak{F}$ is a saturated formation, then $G \in \mathfrak{F}$ if and only if every chief factor H/K of G is $\mathfrak{F}$*-central* in G [11, Theorem 17.14], that is, $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. Moreover, every group G has the largest normal subgroup $Z_{\mathfrak{F}}(G)$, the $\mathfrak{F}$*-hypercentre* of G, with the property that every chief factor of G below $Z_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-central in G. In the case when $\mathfrak{F} = \mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups, we write $Z_\sigma(G)$ instead of $Z_{\mathfrak{N}_\sigma}(G)$.

Our basis result is the following

***Theorem* 1.2** (See Theorem A in [12]). *Suppose that $G \notin \mathfrak{F}$, where $\mathfrak{F}$ is a hereditary saturated formation containing all nilpotent groups.*

(i) *If the $\mathfrak{F}$-residual of every $\mathfrak{F}$-critical section of G is σ-soluble and all proper σ-local sections of G belong to $\mathfrak{F}$, then $G^{\mathfrak{F}}$ is σ-nilpotent.*

(ii) *If the $\mathfrak{F}$-residual of every $\mathfrak{F}$-critical section of G is soluble and all proper local sections of G belong to $\mathfrak{F}$, then $F(G) = G^{\mathfrak{F}} Z_{\mathfrak{F}}(G)$ and every minimal normal subgroup of G is abelian.*

Let ϕ be some linear ordering on $\mathbb{P}$. The record $p\phi q$ means that p precedes q in ϕ and $p \ne q$. Recall that a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ is called ϕ*-dispersed* (Baer [13]) whenever $p_1 \phi p_2 \phi \ldots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$.

It is well-known that if $\mathfrak{F}$ is either the class of all nilpotent groups $\mathfrak{N}$, or the class of all supersoluble groups $\mathfrak{U}$, or the class of all ϕ-dispersed groups, then $\mathfrak{F}$ is a hereditary saturated formation [10, p. 358–359] and every $\mathfrak{F}$-critical group G is soluble [2, Chapter V] with nilpotent $G^{\mathfrak{F}}$ [10, Chapter VII, Theorem 6.18].

Hence we get from Theorem 1.2 the following results.

***Corollary* 1.3.** *Suppose that G is not supersoluble. If all proper local sections of G are supersoluble, then:*

(i) *$G/O_p(G)$ is p-nilpotent for all primes p with* $(p-1, |G|) = 1$.

(ii) $F(G) = G^{\mathfrak{U}} Z_{\mathfrak{U}}(G)$.

***Corollary* 1.4** (Fedri, Tiberio [14]). *If all proper local subgroups of G are supersoluble, then:*

(i) *$G/O_p(G)$ is p-nilpotent, where p is the smallest prime dividing $|G|$.*

(ii) *$G/F(G)$ is supersoluble.*

***Corollary* 1.5** (Beidleman [15]). *If all proper local subgroups of G are ϕ-dispersed, then $G/F(G)$ is ϕ-dispersed.*

The class $\mathfrak{N}_\sigma$ of all σ-nilpotent groups is a hereditary saturated formation [1]. Moreover, $A^{\mathfrak{N}_\sigma}$ is nilpotent for every $\mathfrak{N}_\sigma$-critical group A (see Lemma 3.1 in [12]), so we get from Theorem 1.2 the following

***Corollary* 1.6** (Zhang Chi, Skiba [16]). *If all proper σ-local subgroups of G are σ-nilpotent, then $G/F(G)$ is σ-nilpotent.*

Note that if $G^{\mathfrak{F}}$ is soluble, then $A^{\mathfrak{F}}$ is evidently soluble for every section $A = H/K$ of G. Hence we get from Theorem 1.2 also the following

***Corollary* 1.7** (Gorbachev [17]). *Suppose that $G^{\mathfrak{F}}$ is soluble, where $\mathfrak{F}$ is a hereditary saturated formation containing all nilpotent groups. If $G \notin \mathfrak{F}$ but every proper local subgroup of G belongs to $\mathfrak{F}$, then $F(G) = G^{\mathfrak{F}} Z_{\mathfrak{F}}(G)$.*

Examples shows that a group in which all local sections are σ-nilpotent can be non-nilpotent. Nevertheless, our next result shows that such groups have a structure close to the structure of σ-nilpotent groups.

***Theorem* 1.8** (See Theorem B in [12]). *If each local section of G is σ-nilpotent, then G is either of the following two types*:

(a) *G is σ-nilpotent.*

(b) $G = R \rtimes M$, *where*

(i) $R = C_G(R) = G^{\mathfrak{N}_\sigma}$ *is the Sylow p-subgroup of G for some $p \in \sigma_i$ and M is a σ-nilpotent maximal subgroup of G.*

(ii) $G = H_1 \rtimes H$, *where H_1 is a normal Hall σ_i-subgroup of G, $H \le M$ and either $R = H_1$ or H_1 is a Frobenius group with the kernel R and the complement $M \cap H_1$.*

(iii) *Every non-identity element of H acts irreducibly on R.*

Conversely, *if G is a group of type* (*a*) *or* (*b*), *then every σ-local section of G is σ-nilpotent.*

***Corollary* 1.9.** *G is σ-nilpotent if and only if all σ-local subgroups of G are σ-nilpotent.*

***Corollary* 1.10.** *All local sections of G are nilpotent if and only if G is either of the following two types*:

(i) *G is nilpotent.*

(ii) $G = R \rtimes M$, *where $R = C_G(R) = G^{\mathfrak{N}}$ is the Sylow p-subgroup of G, $p \in \pi(G)$, and M is a nilpotent maximal subgroup of G such that every non-identity element of M acts irreducibly on R.*

A set $\mathcal{H}$ of subgroups of *G* is a *complete Hall σ-set* of *G* [9] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of *G* for some *i* and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of *G* for every $\sigma_i \in \sigma(G)$. If $AB = BA$ for all $A, B \in \mathcal{H}$, then $\mathcal{H}$ is said to be a *σ-basis* [9] of *G*.

We say that *G* is *special* if either *G* is nilpotent or *G* is non-nilpotent and for every Schmidt subgroup A of *G* we have $\Phi(A) = \Phi(F(A))$. Note that the group A in Example, where $p = 13$, is not special but all its subgroups of odd order are special.

Being based on Theorem 1.2, one can prove also the following

***Theorem* 1.11** (See Theorem C in [12]). *Suppose that G is not σ-nilpotent but every proper local section of G is σ-nilpotent. Then the following statements hold*:

(a) $F(G) = G^{\mathfrak{N}_\sigma} Z_\sigma(G)$ *and $F_\sigma(G)$ is a maximal σ-nilpotent subgroup of G. Hence every minimal normal subgroup of G is abelian and G possesses a σ-basis $\mathcal{H} = \{H_0, H_1, \ldots, H_t\}$, where $H_i \trianglelefteq G$ for all $i \le r$ and H_i is not normal in G for all $i > r$.*

(b) *If H_i is special for all $i > r$, then $G / F_\sigma(G)$ is abelian. Moreover*, *if in addition*, *for each such an index i the subgroup $N_G(H_i)$ is σ-nilpotent*, *then $H_{r+1} \cdots H_t$ is σ-nilpotent and*

$$F_\sigma(G) = (H_1 \times \cdots \times H_r) Z_\sigma(G).$$

In this theorem $F_\sigma(G)$ denotes the *σ-Fitting subgroup* of *G* [1], that is, the product of all normal σ-nilpotent subgroups of *G*.

Note that if all non-normal Sylow subgroups of any Schmidt subgroup of *G* have prime order, then *G* is special by Lemma 2.7 in [12]. Hence we get from Theorem 1.11 the following

***Corollary* 1.12** (Zhang Chi, Skiba [16]). *Suppose that G is not σ-nilpotent but every proper local subgroup of G is σ-nilpotent. If non-normal Sylow subgroups of any Schmidt subgroup A contained in a non-normal Hall σ_i-subgroup of G*, *$i = i(A)$, have prime order*, *then $G / F_\sigma(G)$ is abelian.*

The group *G* is called *semi-nilpotent* [18] if all proper local subgroups of *G* are nilpotent.

Note that in the case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, $F_\sigma(G) = F(G)$ is the Fitting subgroup and $Z_\sigma(G) = Z_\infty(G)$ is the hypercentre of G. Therefore we get in this case from Theorem 1.11 the following known result.

***Corollary* 1.13** (See [18] or Theorem 7.6 in [19, Chapter 4]). *If G is semi-nilpotent and $F_0(G)$ denotes the product of its normal Sylow subgroups*, *then $G / F_0(G)$ is nilpotent and $G / F(G)$ is abelian.*

2 Final remarks

1. If $A \le G$, then A / A_G is called the *cofactor* of *A* in *G*. In this paper, in fact, we follow the general idea of studying groups with restrictions on the cofactors of their subgroups (see, for example, the recent papers [20], [21]).

2. Theorem 1.11 allows to prove the following fact which covers one of the main results in [18], [16].

***Theorem* 2.1** (See Theorem 5.1 in [12]). *Suppose that G is not σ-nilpotent but every proper σ-local subgroup of G is σ-nilpotent. Then G has σ-basis $\{H_1, \ldots, H_t\}$ such that for some $1 \le r \le t$ the subgroups $H_1, \ldots, H_r$ are normal in G and H_i is not normal in G for all $i > r$. Moreover*, *if H_i is special for all $i > r$, then $G / F_\sigma(G)$ is cyclic.*

3. Theorems 1.2, 1.8 and 1.11 remain to be new for each specific partition σ of $\mathbb{P}$.

4. In the mathematical practice we often deal with the following three classical partitions of $\mathbb{P}$: $\sigma^1 = \{\{2\}, \{3\}, \ldots\}$, $\sigma^\pi = \{\pi, \pi'\}$ and

$$\sigma^{1\pi} = \{\{p_1\}, \ldots \{p_n\}, \pi'\},$$

where $\pi = \{p_1, \ldots, p_n\}$ (we use here the notations in [7]).

Note that *G* is: σ^π-soluble if and only if *G* is *π-separable*, that is, every chief factor of *G* is either a π-group or a π'-group; σ^π-nilpotent if and only if G is π-*decomposable*, that is, $G = O_{\pi}(G) \times O_{\pi'}(G)$. Therefore in the case when $\sigma = \sigma^{\pi}$ we get from Theorem 1.11 the following result.

***Corollary* 2.2.** *Suppose that G is not* π-*decomposable but every proper local section of G is* π-*decomposable. Then the following statements hold*:

(i) G/V *is* π-*decomposable*, *where*

$$V = O_{\pi}(G) \times O_{\pi'}(G)$$

is a maximal π-*decomposable subgroup of G. Hence G is* π-*separable, so G has a Hall* π-*subgroup* H_1 *and a Hall* π'-*subgroup* H_2.

(ii) $F(G) = DZ$, *where D is the* π-*decomposable residual of G and Z is a normal subgroup of G such that* $(H/K) \rtimes (G/C_G(H/K))$ *is either a* π-*group or a* π'-*group for every chief factor* H/K *of G below* Z.

(iii) *At least one of the normalizers* $N_G(H_1)$ *or* $N_G(H_2)$, $N_G(H_1)$ *say*, *is not* π-*decomposable. Moreover*, *if* $N_G(H_i)$ *is* π-*decomposable for any i such that* $N_G(H_i) \neq G$, *then* H_1 *is normal in G and* $N_G(H_2)$ *is* π-*decomposable. In this case every element of G induces a* π'-*automorphism on every chief factor of G below* $O_{\pi'}(G)$.

A special case of Corollary 2.2 was obtained in the paper [22].

5. In the case when $\sigma = \sigma^{1\pi}$, Theorem 1.11 covers the main result in [23].

REFERENCES

1. *Skiba*, *A.N.* On σ-subnormal and σ-permutable subgroups of finite groups / A.N. Skiba // J. Algebra. – 2015. – Vol. 436. – P. 1–16.

2. *Shemetkov*, *L.A*. Formations of finite groups / L.A. Shemetkov. – Moscow: Nauka, 1978. – 272 p.

3. *Ballester-Bolinches*, *A*. On the lattice of $\mathfrak{F}$-subnormal subgroups / A. Ballester-Bolinches, K. Doerk, M.D. Pèrez-Ramos // J. Algebra. – 1992. – Vol. 148. – P. 42–52.

4. *Vasil'ev*, *A.F.* On lattices of subgroups of finite groups / A.F. Vasil'ev, A.F. Kamornikov, V.N. Semenchuk // Infinite groups and related algebraic structures. – Kiev: Institut Matematiki AN Ukrainy, 1993. – P. 27–54.

5. *Ballester-Bolinches*, *A*. Classes of Finite Groups / A. Ballester-Bolinches, L.M. Ezquerro. – Dordrecht: Springer-Verlag, 2006. – 385 p.

6. *Kovaleva*, *V.A*. A criterion for a finite group to be σ-soluble / V.A. Kovaleva // Commun. Algebra. – 2019. – Vol. 46. – P. 5410–5415.

7. Skiba, A.N. Some characterizations of finite σ-soluble $P\sigma T$-groups / A.N. Skiba // J. Algebra. – 2018. – Vol. 495. – P. 114–129.

8. *Beidleman*, *J.C.* On τ_{σ}-quasinormal subgroups of finite groups / J.C. Beidleman, A.N. Skiba // J. Group Theory. – 2017. – Vol. 20, № 5. – P. 955–964.

9. *Skiba*, *A.N.* A generalization of a Hall theorem / A.N. Skiba // J. Algebra Appl. – 2015. – Vol. 15, № 4. – P. 21–36.

10. *Doerk*, *K.* Finite Soluble Groups / K. Doerk, T. Hawkes. – Berlin – New York: Walter de Gruyter, 1992. – 891 p.

11. *Shemetkov*, *L.A.* Formations of Algebraic Systems / L.A. Shemetkov, A.N. Skiba. – Moscow: Nauka, 1989. – 254 p.

12. *Skiba*, *A.N.* On generalized local subgroups of finite groups / A.N. Skiba, B. Hu, J. Huang // Preprint. – 2019. – 18 p.

13. *Baer*, *R*. The influence on a finite groups of certain types of its proper subgroups / R. Baer // Illinois J. Math. – 1957. – Vol. 1. – P. 115–187.

14. *Fedri*, *V.* Sui gruppi finiti i sui sottogruppi locali propri sono superssolubili / V. Fedri, U. Tiberio // Boll. Unione Mat. Ital. – 1980. – Vol. 17, № 1. – P. 73–78.

15. *Beidleman*, *J.C*. The influence ons a finite groups of certain types of its proper subgroups / J.C. Beidleman // Riv. Mat. Univ. Parma. – 1968. – Vol. 9, № 2. – P. 247–260.

16. *Chi*, *Z.* On semi-σ-nilpotent finite groups / Z. Chi, A.N. Skiba // Journal of Algebra and Its Applications – 2019. – DOI: 10.1142/S021949881950 2001.

17. *Gorbachev*, *V.I.* Local F-subgroups of finite groups / V.I. Gorbachev // Problems in Algebra. – 1986. – № 2. – P. 62–72.

18. *Sah*, *Chih-Han.* On a generalization of finite nilpotent groups / Chih-Han Sah // Math. Z. – 1957. – Vol. 68, № 1. – P. 189–204.

19. *Weinstein*, *M.* Between Nilpotent and Solvable / M. Weinstein. – Passaic: Polygonal Publishing House, 1982. – 232 p.

20. *Monakhov*, *V.S.* On cofactors of subnormal subgroups / V.S. Monakhov, I.L. Sokhor // Journal of Algebra and Its Applications. – 2016. – Vol. 15, № 9. – DOI: 10.1142/S0219498816501693.

21. *Monakhov*, *V.S.* On Groups with Formational Subnormal Sylow Subgroups / V.S. Monakhov, I.L. Sokhor // J. Group Theory. – 2018. – Vol. 21, № 2. – P. 273–287.

22. *Adarchenko*, *N.M.* On finite semi-*p*-decomposable groups / N.M. Adarchenko, I.G. Blistets, V.N. Rizhik // Problems of Physics, Mathematics and Technics. – 2018. – Vol. 34, № 1. – P. 41–44.

23. *On finite semi-*π*-special groups* / N.S. Kosenok, V.M. Selkin, V.N. Rizhik, V.N. Mitsik // Problems of Physics, Mathematics and Technics. – 2019. – Vol. 39, № 2. – P. 88–91.

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