MATH. SCI. RES. J. 17(6) 2013 150-155

Finite groups with all 2-maximal subgroups K-$\mathbb{P}$-subnormal

Viktoria A. Kovaleva
Department of Mathematics, Francisk Skorina Gomel State University,
Sovetskaya str. 104, Gomel 246019, Belarus
E-mail: vika.kovalyova@rambler.ru

Communicated by the Editors

Abstract. A subgroup H of a group G is said to be *K-$\mathbb{P}$-subnormal in G* (A.N. Skiba) if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \dots, n$. In this paper we describe finite groups in which every 2-maximal subgroup is K-$\mathbb{P}$-subnormal.

Keywords: *2-maximal subgroup, strictly 2-maximal subgroup, soluble group, supersoluble group, minimal nonsupersoluble group, K-$\mathbb{P}$-subnormal subgroup.*

Mathematics Subject Classification (2010): 20D10, 20D15, 20D20.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We write $\mathfrak{U}$ to denote the class of all supersoluble groups, $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups N of G with $G/N \in \mathfrak{U}$.

Recall that a subgroup H of G is said to be: (i) *$\mathfrak{U}$-subnormal* in G if there exists a chain of subgroups $H = H_0 \leq H_1 < \cdots \leq H_n = G$ such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$, for $i = 1, \dots, n$; (ii) *$\mathfrak{U}$-subnormal* (in the sense of Kegel [14]) or *K-$\mathfrak{U}$-subnormal* in G (see p. 236 in [3]) in G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \dots, t$.

A subgroup H of G is called a *2-maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define *3-maximal subgroups*, and so on. If H is n-maximal in G but not n-maximal in any proper subgroup of G, then H is said to be a *strictly* n-maximal subgroup of G.

One of the interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its n-maximal subgroups. In partiqular, there are many papers in which the structure of groups with given second maximal subgroups are described. One of the earliest publication in this direction is the article of B. Huppert [12] who established the supersolubility of G whose all second maximal subgroups are normal. This result was developed by many authors. In partiqular, it was proved that G is supersoluble if every 2-maximal subgroup of G is either permutable with every maximal subgroup of G (L.Ja. Poljakov [22]) or S-quasinormal in G (R.K. Agrawal [1], [25, Chapter 1, Theorem 6.5]). In [2], M. Asaad proved that G is supersoluble if every strictly 2-maximal subgroup of G is normal. P. Flavell [6] obtained an upper bound for the number of maximal subgroups containing a strictly 2-maximal subgroup and classify the extremal examples.

Among the recent interesting results on 2-maximal subgroups we can mention the paper of X.Y. Guo and K.P. Shum [8], where the solubility of groups is established in which all 2-maximal subgroups enjoy the cover-avoidance property, and the papers of W. Guo, K.P. Shum, A.N. Skiba and Li Baojun [9, 19, 10], where new characterizations of supersoluble groups in terms of 2-maximal subgroups were obtained. Li Shirong [18] got the classification of nonnilpotent groups whose all 2-maximal subgroups are TI-subgroups. In [4], A. Ballester-Bolinches, L.M. Ezquerro and A.N. Skiba obtained a full classification of groups in which the second maximal subgroups of the Sylow subgroups cover or avoid the chief factors of some of its chief series. Yu.V. Lutsenko and A.N. Skiba [20] got a description of groups whose all 2-maximal subgroups are subnormal. In [17], V.N. Kniahina and V.S. Monakhov studied the groups whose every 2-maximal subgroup permutes with each Schmidt subgroup.

Recall that a subgroup H of G is said to be $\mathbb{P}$-*subnormal in* G [24] if either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for all $i = 1, \ldots , n$.

Another important results on 2-maximal subgroups were obtained by V.A. Kovaleva and A.N. Skiba in [15, 16] and V.S. Monakhov and V.N. Kniahina in [21]. In [15], authors described the groups whose all 2-maximal subgroups are $\mathfrak{U}$-subnormal. In [16], it was obtained a description of groups with all 2-maximal subgroups $\mathfrak{F}$-subnormal for some saturated formation $\mathfrak{F}$. In [21], the groups with all 2-maximal subgroups $\mathbb{P}$-subnormal were studied.

In this paper we consider the following generalization of $\mathbb{P}$-subnormality and subnormality.

Definition (A.N. Skiba). A subgroup H of G is said to be K-$\mathbb{P}$-*subnormal in* G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \ldots , n$.

It is easy to see that every $\mathfrak{U}$-subnormal subgroup of G is K-$\mathbb{P}$-subnormal in G. Moreover, if G is soluble, then every K-$\mathbb{P}$-subnormal subgroup of G is $\mathfrak{U}$-subnormal in G.

We prove the following result.

Theorem. *The following statements are equivalent:*

(1) *G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

(2) *Every 2-maximal subgroup of G is K-$\mathbb{P}$-subnormal in G.*

(3) *Every strictly 2-maximal subgroup of G is K-$\mathbb{P}$-subnormal in G.*

Corollary 1 (V.S.Monakhov and V.N. Kniahina [21]). *Every 2-maximal subgroup of G is $\mathbb{P}$-subnormal in G if and only if G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Corollary 1 gives the answer to Problem 1 of paper [24].

Corollary 2 (V.A. Kovaleva and A.N. Skiba [15, Theorem 3.1] or [16, Theorem C]). *Every 2-maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Corollary 3 (R.K. Agrawal [1]). *If every 2-maximal subgroup of G is S-quasinormal in G, then G is supersoluble.*

Corollary 4 (M. Asaad [2]). *If every strictly 2-maximal subgroup of G is normal in G, then G is supersoluble.*

All unexplained notation and terminology are standard. The reader is referred to [3], [5] and [7] if necessary.

2 Proof of Theorem

We need the following lemmas.

Lemma 2.1. *Let H and K be subgroups of G such that H is $\mathfrak{U}$-subnormal in G.*

(1) *If $G^{\mathfrak{U}} \leq K$, then K is $\mathfrak{U}$-subnormal in G* [3, Lemma 6.1.7(1)].

(2) *If K is $\mathfrak{U}$-subnormal in H, then K is $\mathfrak{U}$-subnormal in G* [3, Lemma 6.1.6(1)].

Lemma 2.2. *Let H and K be subgroups of G such that H is K-$\mathbb{P}$-subnormal in G.*

(1) *If N is a normal subgroup of G, then $H \cap N$ is K-$\mathbb{P}$-subnormal in N and HN/N is K-$\mathbb{P}$-subnormal in G/N.*

(2) *If K is K-$\mathbb{P}$-subnormal in H, then K is K-$\mathbb{P}$-subnormal in G.*

(3) *If $G^{\mathfrak{U}} \leq K$, then K is K-$\mathbb{P}$-subnormal in G.*

Proof. Since H is K-$\mathbb{P}$-subnormal in G, there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \ldots, n$.

(1) Consider the chain $H \cap N = H_0 \cap N \leq H_1 \cap N \leq \cdots \leq H_n \cap N = N$. If H_{i-1} is normal in H_i, then it is evident that $H_{i-1} \cap N$ is normal in $H_i \cap N$. Assume that $|H_i : H_{i-1}|$ is a prime. If $H_i \cap N = H_{i-1} \cap N$, then $H_{i-1} \cap N$ is normal in $H_i \cap N$. Suppose that $H_i \cap N \neq H_{i-1} \cap N$. Then

$$|H_i \cap N : H_{i-1} \cap N| = |H_i \cap N : H_i \cap N \cap H_{i-1}| =$$

$$= (|H_i \cap N||H_{i-1}(H_i \cap N)|) : (|H_i \cap N||H_{i-1}|) = |H_{i-1}(H_i \cap N) : H_{i-1}| \neq 1.$$

Hence $H_{i-1}(H_i \cap N) \neq H_{i-1}$. Therefore, since $H_{i-1} \leq H_{i-1}(H_i \cap N) \leq H_i$ and $|H_i : H_{i-1}|$ is a prime, we have $H_{i-1}(H_i \cap N) = H_i$. Hence $|H_i \cap N : H_{i-1} \cap N| = |H_i : H_{i-1}|$ is a prime. Thus $H \cap N$ is K-$\mathbb{P}$-subnormal in N.

Now consider the chain $HN/N = H_0N/N \leq H_1N/N \leq \cdots \leq H_nN/N = G/N$. If H_{i-1} is normal in H_i, then $H_{i-1}N/N$ is normal in H_iN/N. Suppose that $|H_i : H_{i-1}|$ is a prime. If $H_{i-1}N/N = H_iN/N$, then $H_{i-1}N/N$ is normal in H_iN/N. Let $H_{i-1}N/N \neq H_iN/N$. Then

$$|H_iN/N : H_{i-1}N/N| = |H_i/H_i \cap N : H_{i-1}/H_{i-1} \cap N| =$$

$$= |H_i : H_{i-1}| : |H_i \cap N : H_{i-1} \cap N| \neq 1,$$

so $|H_i \cap N : H_{i-1} \cap N| = 1$. Hence $|H_iN/N : H_{i-1}N/N| = |H_i : H_{i-1}|$ is a prime. Thus HN/N is K-$\mathbb{P}$-subnormal in G/N.

(2) Since K is K-$\mathbb{P}$-subnormal in H, there exists a chain of subgroups $K = K_0 \leq K_1 \leq \cdots \leq K_t = H$ such that either K_{i-1} is normal in K_i or $|K_i : K_{i-1}|$ is a prime, for $i = 1, \ldots, t$. By considering the chain $K = K_0 \leq K_1 \leq \cdots \leq K_t = H = H_0 \leq H_1 \leq \cdots \leq H_n = G$, we see that K is K-$\mathbb{P}$-subnormal in H.

(3) Since $G^{\mathfrak{U}} \leq K$, K is $\mathfrak{U}$-subnormal in G by Lemma 2.1(1). Hence K is K-$\mathbb{P}$-subnormal in G. The lemma is proved.

The next lemma is evident.

Lemma 2.3. *If G is supersoluble, then every subgroup of G is K-$\mathbb{P}$-subnormal in G.*

Proof of Theorem. (1) $\Rightarrow$ (2). If G is supersoluble, then every subgroup of G is K-$\mathbb{P}$-subnormal in G by Lemma 2.3. Suppose that G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G. Let T be a 2-maximal subgroup of G and M a maximal subgroup of G such that T is a maximal subgroup of M. Since M is supersoluble, T is K-$\mathbb{P}$-subnormal in M by Lemma 2.3. If M is K-$\mathbb{P}$-subnormal in G, then T is K-$\mathbb{P}$-subnormal in G by Lemma 2.2(2). Assume that M is not K-$\mathbb{P}$-subnormal in G. Then $G^{\mathfrak{U}} \not\leq M$ by Lemma 2.2(3). Therefore $G = G^{\mathfrak{U}} \rtimes M$ and $G^{\mathfrak{U}}T$ is a maximal subgroup of G. By Lemma 2.2(3), $G^{\mathfrak{U}}T$ is K-$\mathbb{P}$-subnormal in G. Since G is a minimal nonsupersoluble group, $G^{\mathfrak{U}}T$ is supersoluble and so T is K-$\mathbb{P}$-subnormal in $G^{\mathfrak{U}}T$ by Lemma 2.3. Hence by Lemma 2.2(2), T is K-$\mathbb{P}$-subnormal in G.

(2) ⇒ (3). It is evident.

(3) ⇒ (1). Assume that G is not supersoluble. We prove that G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.

First prove that G is soluble. Assume that this is false and let G be a counterexample of minimal order. Suppose that G is simple. Let p be the largest prime divisor of $|G|$ and P a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $P \leq M$. Since G is not soluble and p is the largest prime divisor of $|G|$, $P \neq M$ by well-known Deskins-Janko-Thompson's Theorem [11, Chapter IV, Satz 7.4]. Hence there is a maximal subgroup T of M such that $P \leq T$. Suppose that T is not a strictly 2-maximal subgroup of G. Then there exists a maximal subgroup M_1 of G such that $M_1 \neq M$ and T is a proper nonmaximal subgroup of M_1. Let T_1 be a maximal subgroup of M_1 such that $T \leq T_1$ and T_1 is a strictly 2-maximal subgroup of G. By hypothesis, T_1 is K-$\mathbb{P}$-subnormal in G. Hence there is a proper subgroup H of G such that $T_1 \leq H$ and either H is normal in G or $|G : H| = q$ is a prime. But in the first case we have that G is not simple, a contradiction. Therefore $|G : H| = q$. In view of inclusion $P \leq T \leq T_1 \leq H$, $q \neq p$. Since G is simple, $H_G = 1$ and by considering the permutation representation of G on the right cosets of H, we see that G is isomorphic to some subgroup of the symmetric group S_q of degree q. Hence $|G| \leq q!$ and so q is the largest prime divisor of $|G|$. It follows that $q = p$. This contradiction shows that T is a strictly 2-maximal subgroup of G, so T is K-$\mathbb{P}$-subnormal in G. But then, by using the same arguments as above, we arrive to contradiction. So G is not a simple group.

Let N be a minimal normal subgroup of G and T/N a strictly 2-maximal subgroup of G/N. Then T is a strictly 2-maximal subgroup of G. By hypothesis, T is K-$\mathbb{P}$-subnormal in G. Therefore by Lemma 2.2(1), T/N is K-$\mathbb{P}$-subnormal in G/N. Thus the hypothesis holds for G/N. Hence G/N is soluble by the choice of G and so N is the only minimal normal subgroup of G and N is not soluble. Since G/N is soluble, there exists a normal maximal subgroup M/N of G/N. Then M is a normal maximal subgroup of G. We show that M is supersoluble. Let K be an arbitrary maximal subgroup of M. If K is a strictly 2-maximal subgroup of G, then K is K-$\mathbb{P}$-subnormal in G. Consequently, K is K-$\mathbb{P}$-subnormal in M by Lemma 2.2(1). Therefore either K is normal in M or $|M : K|$ is a prime. But in the first case we also see that $|M : K|$ is a prime in view of maximality of K in M. Suppose that K is not a strictly 2-maximal subgroup of G. Then there is a maximal subgroup M_1 of G such that $M_1 \neq M$ and K is a proper nonmaximal subgroup of M_1. Let K_1 be a maximal subgroup of M_1 such that $K \leq K_1$ and K_1 is a strictly 2-maximal subgroup of G. Since $K \leq K_1 \cap M$ and M is a maximal subgroup of G, $K = K_1 \cap M$. By hypothesis, K_1 is K-$\mathbb{P}$-subnormal in G. Therefore $K = K_1 \cap M$ is K-$\mathbb{P}$-subnormal in M by Lemma 2.2(1). Hence as above we see that $|M : K|$ is a prime. Since K is an arbitrary maximal subgroup of M, it follows that all maximal subgroups of M have prime indices. Therefore M is supersoluble and so N. This contradiction completes the proof of solubility of G.

Since G is soluble, every K-$\mathbb{P}$-subnormal subgroup of G is $\mathfrak{U}$-subnormal in G. We show that every maximal subgroup of G is supersoluble. Let M be a maximal subgroup of G and T any maximal subgroup of M. If T is a strictly 2-maximal subgroup of G, then T is K-$\mathbb{P}$-subnormal in G. Hence T is $\mathfrak{U}$-subnormal in G, which implies that T is $\mathfrak{U}$-subnormal in M by Lemma 2.1(2). Therefore $M/T_M \in \mathfrak{U}$, hence $|M : T|$ is a prime. Suppose that T is not a strictly 2-maximal subgroup of G. Then there is a maximal subgroup M_1 of G such that $M_1 \neq M$ and T is a proper nonmaximal subgroup of M_1. Let T_1 be a maximal subgroup of M_1 such that $T \leq T_1$ and T_1 is a strictly 2-maximal subgroup of G. In view of inclusion $T \leq T_1 \cap M$ and maximality of M in G, we have $T = T_1 \cap M$. Since T_1 is $\mathfrak{U}$-subnormal in G, $T = T_1 \cap M$ is $\mathfrak{U}$-subnormal in M by Lemma 2.1(2). Hence $|M : T|$ is a prime. Since T is an arbitrary maximal subgroup of M, we have that M is supersoluble. Thus all maximal subgroups of G are supersoluble, so G is a minimal nonsupersoluble group.

By [7, Chapter 3, Theorem 3.11.9] or [23, Chapter VI, Theorem 26.5], $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G for some prime p. Moreover, by [7, Chapter 3, Theorem 3.4.2] or [23, Chapter VI, Theorem 24.2], $G_p/\Phi(G_p)$ is a chief factor of G. Suppose that $\Phi(G_p) \neq 1$. Since G is not supersoluble, there exists an $\mathfrak{U}$-abnormal maximal subgroup L of G. By Lemma 2.1(1), $G_p \nleq L$. It follows that $G = G_pL$ and $L = (G_p \cap L)G_{p'} = \Phi(G_p)G_{p'}$, where $G_{p'}$ is a Hall p'-subgroup of G. Since $\Phi(G_p) \nleq \Phi(L)$, there is a maximal subgroup T of L such that $\Phi(G_p) \nleq T$. Then $L = \Phi(G_p)T$, hence $G = G_pL = G_p\Phi(G_p)T = G_pT$. If T is not a strictly 2-maximal subgroup of G, then there exists a maximal subgroup V of G such that $V \neq L$ and T is a proper nonmaximal subgroup of V. Let W be a maximal subgroup of V such that $T \leq W$ and W is a strictly 2-maximal subgroup of G. Then W is $\mathfrak{U}$-subnormal in G. Hence there is a proper subgroup H of G such that $W \leq H$ and $G/H_G \in \mathfrak{U}$. Therefore we have $G_p = G^{\mathfrak{U}} \leq H_G$ and so $G = G_pT \leq H$, a contradiction. Consequently, T is a strictly 2-maximal subgroup of G. But then arguing as above we get contradiction. Thus $\Phi(G_p) = 1$ and hence G_p is a minimal normal subgroup of G. So (3) ⇒ (1). The theorem is proved.

Proof of Corollary 3. Suppose that this corollary is false and let G be a counterexample of minimal order. Since every S-quasinormal subgroup is subnormal by [13], all 2-maximal subgroups of G are subnormal in G. Hence all 2-maximal subgroups of G are K-$\mathbb{P}$-subnormal in G. Therefore, by Theorem, G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G. By [7, Chapter 3, Theorem 3.11.9] or [23, Chapter VI, Theorem 26.5], $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $G = G_p \rtimes M$. Let T be a maximal subgroup of M. Then T is subnormal in G, so $T \leq M_G$. If $M_G = 1$, then $T = 1$ and hence $|M| = q$ is a prime. Thus M is a Sylow subgroup of G. Let K be a maximal subgroup of G_p. Since G is not supersoluble, $K \neq 1$. Moreover, K is a 2-maximal subgroup of G and so $MK = KM$, which contradicts the maximality of M. Thus $M_G \neq 1$. Hence $T = M_G$ is normal in G. In view of [13, Lemma 1], the hypothesis holds for G/T. Therefore G/T is supersoluble by the choice of G. But then $G \simeq G/T \cap G_p$ is supersoluble. This contradiction completes the proof of Corollary 3.

Proof of Corollary 4. Suppose that this corollary is false and let G be a counterexample of minimal order. Since every strictly 2-maximal subgroup of G is normal in G, G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G by Theorem. By [7, Chapter 3, Theorem 3.11.9] or [23, Chapter VI, Theorem 26.5], $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $G = G_p \rtimes M$. Let T be a maximal subgroup of M. If $T = 1$, then $|M| = q$ is a prime. Let P be a maximal subgroup of G_p. Since by hypothesis P is normal in G and G_p is a minimal normal subgroup of G, $P = 1$. It follows that G is supersoluble, a contradiction. Therefore $T \neq 1$. If T is a strictly 2-maximal subgroup of G, then T is normal in G. It is evident that the hypothesis holds for G/T. Hence G/T is supersoluble by the choice of G, which implies that $G \simeq G/T \cap G_p$ is supersoluble, a contradiction. Consequently, T is not a strictly 2-maximal subgroup of G. This implies that there exists a maximal subgroup M_1 of G such that $M_1 \neq M$ and T is a proper nonmaximal subgroup of M_1. Let T_1 be a maximal subgroup of M_1 such that $T \leq T_1$ and T_1 is a strictly 2-maximal subgroup of G. By hypothesis, T_1 is normal in G. Therefore $T_1 \cap C_p$ is normal in G. But G_p is a minimal normal subgroup of G. Thus $T_1 \cap G_p = 1$, which as above leads us to a contradiction.

References

[1] R.K. Agrawal, *Generalized center and hypercenter of a finite group*, Proc. Amer. Math. Soc. 54 (1976), 13–21.

[2] M. Asaad, *Finite groups some whose n-maximal subgroups are normal*, Acta Math. Hung. 54 (1–2) (1989), 9–27.

[3] A. Ballester-Bolinches, L.M. Ezquerro, *Classes of Finite Groups*, Springer-Verlag, Dordrecht, 2006.

[4] A. Ballester-Bolinches, L.M. Ezquerro, A.N. Skiba, *On second maximal subgroups of Sylow subgroups of finite groups*, J. Pure Appl. Algebra 215 (4) (2011), 705-714.

[5] K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

[6] P. Flavell, *Overgroups of second maximal subgroups*, Arch. Math. 64 (1995), 277–282.

[7] W. Guo, *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

[8] X.Y. Guo, K.P. Shum, *Cover-avoidance properties and the structure of finite groups*, J. Pure Appl. Algebra 181 (2003), 297–308.

[9] W. Guo, K.P. Shum, A.N. Skiba, *X-Semipermutable subgroups of finite groups*, J. Algebra 315 (2007), 31–41.

[10] W. Guo, A.N. Skiba, *Finite groups with given s-embedded and n-embedded subgroups*, J. Algebra 321 (2009), 2843–2860.

[11] B. Huppert, *Endliche Gruppen*, Vol. 1, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[12] B. Huppert, *Normalteiler and Maximal Untergruppen Endlicher Gruppen*, Math. Z. 60 (1954), 409–434.

[13] O.H. Kegel, *Sylow-Gruppen and Subnormalteiler endlicher Gruppen*, Math. Z. 78 (1962), 205-221.

[14] O.H. Kegel, *Zur Struktur mehrfach faktorisierbarer endlicher Gruppen*, Math. Z. 87 (1965), 409–434.

[15] V.A. Kovaleva, A.N. Skiba, *Finite solvable groups with all n-maximal subgroups $\mathfrak{U}$-subnormal*, Sib. Math. J. 54 (1) (2013), 65–73.

[16] V.A. Kovaleva, A.N. Skiba, *Finite soluble groups with all n-maximal subgroups $\mathfrak{F}$-subnormal*, J. Group Theory (2013), DOI 10.1515/jgt-2013-0047.

[17] V.N. Kniahina, V.S. Monakhov, *On the permutability of n-maximal subgroups with Schmidt subgroups*, Trudy Inst. Mat. i Mekh. UrO RAN 18 (3) (2012), 125–130.

[18] Shirong Li, *Finite non-nilpotent groups all of whose second maximal subgroups are TI-groups*, Math. Proc. of the Royal Irish Academy 100A (1) (2000), 65–71.

[19] Baojun Li, A.N. Skiba, *New characterizations of finite supersoluble groups*, Sci. China Ser. A: Math. 50 (1) (2008), 827–841.

[20] Yu.V. Lutsenko, A.N. Skiba, *Finite groups with subnormal second and third maximal subgroups*, Math. Notes 91 (5) (2012), 680–688.

[21] V.S. Monakhov, V.N. Kniahina, *Finite groups with $\mathbb{P}$-subnormal subgroups*, Ricerche di Matematica 62 (2) (2013), 307–322.

[22] L.Ja. Poljakov, *Finite groups with permutable subgroups*, Proc. Gomel Sem. Finite groups, 75–88, Nauka i Tekhnika, Minsk, 1966.

[23] L.A. Shemetkov, *Formations of Finite Groups*, Nauka, Minsk, 1978.

[24] A.F. Vasilyev, T.I. Vasilyeva, V.N. Tyutyanov, *On finite groups similar to supersoluble groups*, Problems of physics, mathematics and technics 2 (3) (2010), 21–27.

[25] M. Weinstein, *Between Nilpotent and Solvable*, Polygonal Publishing House, Passaic N.J., 1982.

Received January 2014