

# On $\sigma$ -semipermutable subgroups of finite groups\*

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## Abstract

Let  $\sigma = \{\sigma_i | i \in I\}$  be some partition of the set of all primes  $\mathbb{P}$ ,  $G$  a finite group and  $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$ . A set  $\mathcal{H}$  of subgroups of  $G$  is said to be a *complete Hall  $\sigma$ -set* of  $G$  if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \sigma$  and  $\mathcal{H}$  contains exact one Hall  $\sigma_i$ -subgroup of  $G$  for every  $\sigma_i \in \sigma(G)$ . A subgroup  $H$  of  $G$  is said to be:  *$\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$*  if  $HH_i^x = H_i^xH$  for all  $x \in G$  and all  $H_i \in \mathcal{H}$  such that  $(|H|, |H_i|) = 1$ ;  *$\sigma$ -semipermutable in  $G$*  if  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to some complete Hall  $\sigma$ -set of  $G$ .

We study the structure of  $G$  being based on the assumption that some subgroups of  $G$  are  $\sigma$ -semipermutable in  $G$ .

## 1 Introduction

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. Moreover,  $\mathbb{P}$  is the set of all primes,  $p \in \pi \subseteq \mathbb{P}$  and  $\pi' = \mathbb{P} \setminus \pi$ . If  $n$  is an integer, the symbol  $\pi(n)$  denotes the set of all primes dividing  $n$ ; as usual,  $\pi(G) = \pi(|G|)$ , the set of all primes dividing the order of  $G$ .

In what follows,  $\sigma = \{\sigma_i | i \in I \subseteq \mathbb{N}\}$  is some partition of  $\mathbb{P}$ , that is,  $\mathbb{P} = \cup_{i \in I} \sigma_i$  and  $\sigma_i \cap \sigma_j = \emptyset$  for all  $i \neq j$ . Let  $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$ .

In the mathematical practice, we often deal with the following two special partitions of  $\mathbb{P}$ :  $\sigma = \{\{2\}, \{3\}, \dots\}$  and  $\sigma = \{\pi, \pi'\}$  (in particular,  $\sigma = \{\{p\}, \{p'\}\}$ , where  $p$  is a prime).

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A set  $\mathcal{H}$  of subgroups of  $G$  is a *complete Hall  $\sigma$ -set* of  $G$  [1, 2] if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \sigma$  and  $\mathcal{H}$  contains exact one Hall  $\sigma_i$ -subgroup of  $G$  for every  $\sigma_i \in \sigma(G)$ .

Subgroups  $A$  and  $B$  of  $G$  are called *permutable* if  $AB = BA$ . In this case they also say that  $A$  *permutes* with  $B$ .

**Definition 1.1.** Suppose that  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$ . A subgroup  $H$  of  $G$  is said to be:  *$\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$*  if  $HH_i^x = H_i^xH$  for all  $x \in G$  and all  $i$  such that  $(|H|, |H_i|) = 1$ ;  *$\sigma$ -semipermutable in  $G$*  if  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to some complete Hall  $\sigma$ -set of  $G$ .

Many known results deal with two special cases of the  $\sigma$ -semipermutability condition: when  $\sigma = \{\{2\}, \{3\}, \dots\}$  and  $\sigma = \{\pi, \pi'\}$ .

Consider some typical examples.

**Example 1.2.** A subgroup  $H$  of  $G$  is said to be  *$S$ -semipermutable in  $G$*  if  $H$  permutes with all Sylow subgroups  $P$  of  $G$  satisfying  $(|H|, |P|) = 1$ . Thus  $H$  is  $S$ -semipermutable in  $G$  if and only if it is  $\sigma$ -semipermutable in  $G$  where  $\sigma = \{\{2\}, \{3\}, \dots\}$ .

The  $S$ -semipermutability condition can be found in many known results (see for example Section 3 in [3, VI], Chapter 3 in [4] and also the recent papers [5, 6, 7]).

Before continuing, let's make the following remark.

**Remarks 1.3.** Let  $G = AB$  by a product of subgroups  $A$  and  $B$  and  $K \leq B$ . Suppose that  $A$  permutes with  $K^b$  for all  $b \in B$ . Then:

(i) For any  $x = ab$ , where  $a \in A$  and  $b \in B$ , we have  $AK^x = Aa(K^b)a^{-1} = a(K^b)a^{-1}A = K^x A$  and hence  $A$  permutes with all conjugates of  $K$ .

(ii)  $A^x K = K A^x$  for all  $x \in G$ . Indeed,  $(A^x K)^{x^{-1}} = A K^{x^{-1}} = K^{x^{-1}} A$  by Part (i), so  $(A K^{x^{-1}})^x = A^x K = K A^x$ .

**Example 1.4.** A subgroup  $H$  of  $G$  is said to be  *$SS$ -quasinormal* if  $G$  has a subgroup  $T$  such that  $HT = G$  and  $H$  permutes with all Sylow subgroups of  $T$ . If  $P$  is a Sylow subgroups of  $T$  satisfying  $(|H|, |P|) = 1$ , then  $P$  is a Sylow subgroups of  $G$  and so  $H$  is  $\sigma$ -semipermutable in  $G$ , where  $\sigma = \{\{2\}, \{3\}, \dots\}$ , by Example 1.2 and Remark 1.3(i). Various applications of  $SS$ -quasinormal subgroups can be found in [8, 9, 10] and in many other papers.

**Example 1.5.** In [11], Huppert proved that if a Sylow  $p$ -subgroup  $P$  of  $G$  of order  $|P| > p$  has a complement  $T$  in  $G$  and  $T$  permutes with all maximal subgroups of  $P$ , then  $G$  is  $p$ -soluble. In view of Remark 1.3 the condition " $T$  permutes with all maximal subgroups of  $P$ " is equivalent to the condition "all maximal subgroups of  $P$  are  $\sigma$ -semipermutable in  $G$  with respect to  $\{P, T\}$ ", where  $\sigma = \{\{p\}, \{p\}'\}$ . The result of Huppert was developed in the papers [12, 13], where instead of maximal subgroups were considered the subgroups of  $P$  of fixed order  $p^k$ .

Further, the results in [11, 12, 13] were generalized in [14, 15], where instead of a Sylow  $p$ -subgroup of  $G$  was considered a Hall subgroup of  $G$  (see Section 4 below).

Finally, note that all the above-mentioned results deal with two special cases: a "binary" case, when  $\sigma = \{\pi, \pi'\}$ , and an "n-ary" case, when  $\sigma = \{\{2\}, \{3\}, \dots\}$ .

In this paper, we consider the  $\sigma$ -semipermutability condition for arbitrary partition  $\sigma$  of  $\mathbb{P}$ .

In fact, our main results are the following two observations.

**Theorem A.** *Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that  $H_1$  is  $p$ -supersoluble of order divisible by  $p$ . Suppose also that there is a natural number  $k$  such that  $p^k < |P|$  and every subgroup of  $P$  of order  $p^k$  and every cyclic subgroup of  $P$  of order 4 (if  $p^k = 2$  and  $P$  is non-abelian) are  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Then  $G$  is  $p$ -supersoluble.*

**Theorem B.** *Let  $X \leq E$  be normal subgroups of  $G$ . Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H}$  such that every member of  $\mathcal{H}$  is supersoluble. Suppose also that for every non-cyclic Sylow subgroup  $P$  of  $X$  there is a natural number  $k = k(P)$  such that  $p^k < |P|$  and every subgroup of  $P$  of order  $p^k$  and every cyclic subgroup of  $P$  of order 4 (if  $p^k = 2$  and  $P$  is non-abelian) are  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . If  $X = E$  or  $X = F^*(E)$ , then every chief factor of  $G$  below  $E$  is cyclic.*

In this theorem  $F^*(E)$  denotes the generalized Fitting subgroup of  $E$ , that is, the product of all normal quasinilpotent subgroups of  $E$ .

We prove Theorems A and B in Section 3. In Section 4 we discuss some applications of these two results.

All unexplained notation and terminology are standard. The reader is referred to [16], [17], [18] or [4] if necessary.

## 2 Base lemmas

Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$ . For any subgroup  $H$  of  $G$  we write  $H \cap \mathcal{H}$  to denote the set  $\{H \cap H_1, \dots, H \cap H_t\}$ . If  $H \cap \mathcal{H}$  is a complete Hall  $\sigma$ -set of  $H$ , then we say that  $\mathcal{H}$  *reduces into*  $H$ .

**Lemma 2.1.** *Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that a subgroup  $H$  of  $G$  is  $\sigma$ -semipermutable with respect to  $\mathcal{H}$ . Let  $R$  be a normal subgroup of  $G$  and  $H \leq L \leq G$ . Then:*

(1)  $\mathcal{H}_0 = \{H_1 R/R, \dots, H_t R/R\}$  is a complete Hall  $\sigma$ -set of  $G/R$ . Moreover, if for every prime  $p$  dividing  $|H|$  and for a Sylow  $p$ -subgroup  $H_p$  of  $H$  we have  $H_p \not\leq R$ , then  $HR/R$  is  $\sigma$ -semipermutable in  $G/R$  with respect to  $\mathcal{H}_0$ .

(2) If  $\mathcal{H}$  reduces into  $L$ , then  $H$  is  $\sigma$ -semipermutable in  $L$  with respect to  $L \cap \mathcal{H}$ . In particular, if  $L$  is normal in  $G$ , then  $H$  is  $\sigma$ -semipermutable in  $L$  with respect to  $L \cap \mathcal{H}$ .

(3) If  $L \leq H_i$ , for some  $i$ , then  $\mathcal{H}$  reduces into  $LR$ .

(4) If  $H \leq H_i$ , for some  $i$ , then  $H$  is  $\sigma$ -semipermutable in  $HR$ .

(5) If  $H$  is a  $p$ -group, where  $p \in \pi(H_i) \subseteq \sigma_i$  and  $R$  is a  $\sigma_i$ -group, then  $|G : N_G(H \cap R)|$  is a

$\sigma_i$ -number.

**Proof.** Without loss of the generality we can assume that  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ .

(1) It is clear that  $\mathcal{H}_0 = \{H_1R/R, \dots, H_tR/R\}$  is a complete Hall  $\sigma$ -set of  $G/R$ . Let  $i \in \{1, \dots, t\}$  such that  $(|HR/R|, |H_iR/R|) = 1$ . Let  $p \in \pi(H)$  and  $H_p$  a Sylow  $p$ -subgroup of  $H$ . Assume that  $p$  divides  $|H_i|$ . Then  $H_i$  contains a Sylow  $p$ -subgroup of  $G$  since it is a Hall subgroup of  $G$  and so  $H_p \leq R$ , contrary to the hypothesis. Hence  $(|H|, |H_i|) = 1$ . By hypothesis,  $HH_i^x = H_i^xH$  for all  $x \in G$ . Then

$$\begin{aligned} (HR/R)(H_iR/R)^{xR} &= HH_i^xR/R \\ &= H_i^xHR/R = (H_iR/R)^{xR}(HR/R), \end{aligned}$$

so  $HR/R$  is  $\sigma$ -semipermutable in  $G/R$  with respect to  $\mathcal{H}_0$ .

(2) Let  $L_i = H_i \cap L$  for all  $i = 1, \dots, t$  and  $\mathcal{L} = \{L_1, \dots, L_t\}$ . By hypothesis,  $\mathcal{L}$  is a complete  $\sigma$ -Hall set of  $L$ . Let  $i \in \{1, \dots, t\}$  such that  $(|H|, |L_i|) = 1$  and let  $a \in L$ . Then  $(|H|, |H_i|) = 1$ . Hence, by hypothesis,  $HH_i^a = H_i^aH$  for all  $a \in L$ , so  $L \cap HH_i^a = H(L \cap H_i^a) = H(L \cap H_i)^a = HL_i^a = L_i^aH$ . This shows that  $H$  is  $\sigma$ -semipermutable in  $L$  with respect to  $L \cap \mathcal{H}$ .

(3) Since  $H_i \cap R$  is a Hall  $\sigma_i$ -subgroup of  $R$  and  $H_i \cap LR = L(H_i \cap R)$ , we have  $|LR : H_i \cap LR| = |R : H_i \cap R|$ . Hence  $H_i \cap LR$  is a Hall  $\sigma_i$ -subgroup of  $LR$ . It is clear also that  $H_j \cap LR = H_j \cap R$  is a Hall  $\sigma_j$ -subgroup of  $LR$  for all  $j \neq i$ . Hence  $\mathcal{H}$  reduces into  $LR$ .

(4) This follows from Parts (2) and (3).

(5) For any  $j \neq i$ ,  $H_jH = HH_j$  is a subgroup of  $G$  and  $HH_j \cap R = (H \cap R)(H_j \cap R) = H \cap R$ , so  $H_j \leq N_G(H \cap R)$ . Hence  $|G : N_G(H \cap R)|$  is a  $\sigma_i$ -number.

**Lemma 2.2** (See Kegel [19]). *Let  $A$  and  $B$  be subgroups of  $G$  such that  $G \neq AB$  and  $AB^x = B^xA$ , for all  $x \in G$ . Then  $G$  has a proper normal subgroup  $N$  such that either  $A \leq N$  or  $B \leq N$ .*

**Lemma 2.3.** *Let  $P$  be a Sylow  $p$ -subgroup of  $G$  and  $\mathcal{H} = \{H_1, \dots, H_t\}$  a complete Hall  $\sigma$ -set of  $G$  such that  $p \in \pi(H_1)$ . Suppose that for any  $x \in G$ ,  $P^xH_i$  is a  $p$ -soluble subgroup of  $G$  for all  $i = 2, \dots, t$ . Then  $G$  is  $p$ -soluble.*

**Proof.** Assume that this is false and let  $G$  be a counterexample of minimal order. First note that the hypothesis holds for every normal subgroup  $R$  of  $G$ . Therefore every proper normal subgroup of  $G$  is  $p$ -soluble by the choice of  $G$ . Moreover, the choice of  $G$  and the hypothesis imply that  $PH_i \neq G$  for all  $i = 2, \dots, t$ . By Lemma 2.2, we have either  $P^G \neq G$  or  $(H_2)^G \neq G$ . Hence  $G$  has a proper non-identity normal subgroup  $R$ . But then  $R$  is  $p$ -soluble. On the other hand, the hypothesis holds for  $G/R$ , so  $G/R$  is also  $p$ -soluble by the choice of  $G$ . This implies that  $G$  is  $p$ -soluble.

A group  $G$  is said to be *strictly  $p$ -closed* [20, p.5] whenever  $G_p$ , a Sylow  $p$ -subgroup of  $G$ , is normal in  $G$  with  $G/G_p$  abelian of exponent dividing  $p - 1$ . A normal subgroup  $H$  of  $G$  is called *hypercyclically embedded* in  $G$  if every chief factor of  $G$  below  $H$  is cyclic.

**Lemma 2.4** *A normal  $p$ -subgroup  $P$  of  $G$  is hypercyclically embedded in  $G$  if and only if  $G/C_G(P)$  is strictly  $p$ -closed.*

**Proof.** If  $P$  is hypercyclically embedded in  $G$ , then for any chief factor  $H/K$  of  $G$  below  $P$ ,

$G/C_G(H/K)$  is abelian of exponent dividing  $p - 1$ . Hence  $G/C$ , where  $C$  the intersection the centralizers of all such factors, is also an abelian group of exponent dividing  $p - 1$ . On the other hand,  $C/C_G(P)$  is a  $p$ -group by [21, Ch.5, Corollary 3.3]. Hence  $G/C_G(P)$  is strictly  $p$ -closed.

Now assume that  $G/C_G(P)$  is strictly  $p$ -closed and let  $H/K$  be any chief factor below  $P$ . Since  $C_G(P) \leq C_G(H/K)$ ,  $G/C_G(H/K)$  is strictly  $p$ -closed. But since  $O_p(G/C_G(H/K)) = 1$  [16, Ch.A, Lemma 13.6],  $G/C_G(H/K)$  is abelian of exponent dividing  $p - 1$ . It follows from [20, Ch.1, Theorem 1.4] that  $|H/K| = p$ . Thus  $P$  is hypercyclically embedded in  $G$ .

Let  $P$  be a  $p$ -group. If  $P$  is not a non-abelian 2-group, then we use  $\Omega(P)$  to denote the subgroup  $\Omega_1(P)$ . Otherwise,  $\Omega(P) = \Omega_2(P)$ .

**Lemma 2.5** (See [22, Lemma 2.12]). *Let  $P$  be a normal  $p$ -subgroup of  $G$  and  $D = \Omega(C)$ , where  $C$  is a Thompson critical subgroup of  $P$ . If either  $P/\Phi(P)$  is hypercyclically embedded in  $G/\Phi(P)$  or  $D$  is hypercyclically embedded in  $G$ , then  $P$  is also hypercyclically embedded in  $G$ .*

**Lemma 2.6.** *Let  $C$  be a Thompson critical subgroup of a  $p$ -group  $P$ . Then the group  $D = \Omega(C)$  is of exponent  $p$  if  $p$  is odd prime or exponent 4 if  $P$  is non-abelian 2-group. Moreover, every non-trivial  $p'$ -automorphism of  $P$  induces a non-trivial automorphism of  $D$ .*

**Proof.** The first assertion follows from [21, Ch.5, Theorem 3.11] and [22, Lemma 2.11]. The second one directly follows from [21, Ch. 5, Theorem 3.11].

**Lemma 2.7.** *Let  $E$  be a normal subgroup of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $E$  such that  $(p - 1, |G|) = 1$ . If either  $P$  is cyclic or  $G$  is  $p$ -supersoluble, then  $E$  is  $p$ -nilpotent and  $E/O_{p'}(E) \leq Z_\infty(G/O_{p'}(E))$ .*

**Proof.** First note that in view of [3, Ch.IV, Theorem 5.4] and the condition  $(p - 1, |G|) = 1$ ,  $E$  is  $p$ -nilpotent. Let  $H/K$  be any chief factor of  $G$  such that  $O_{p'}(E) \leq K < H \leq E$ . Then  $|H/K| = p$ , so  $G/C_G(H/K)$  divides  $p - 1$ . But by hypothesis,  $(p - 1, |G|) = 1$ , so  $C_G(H/K) = G$ . Thus  $E/O_{p'}(E) \leq Z_\infty(G/O_{p'}(E))$ .

The following lemma is well-known (see, for example, [18, Lemma 2.1.6]).

**Lemma 2.8.** *If  $G$  is  $p$ -supersoluble and  $O_{p'}(G) = 1$ , then  $p$  is the largest prime dividing  $|G|$ ,  $G$  is supersoluble and  $F(G) = O_p(G)$  is a Sylow  $p$ -subgroup of  $G$ .*

**Lemma 2.9** (See [23]). *Let  $H$ ,  $K$  and  $N$  be pairwise permutable subgroups of  $G$  and  $H$  is a Hall subgroup of  $G$ , then  $N \cap HK = (N \cap H)(N \cap K)$ .*

The following fact is also well-known (see for example [4, Ch.1, Lemma 5.35(6)]).

**Lemma 2.10** *If  $H$  is a subnormal  $\pi$ -subgroup of  $G$ , then  $H \leq O_\pi(G)$ .*

**Lemma 2.11** (See [24, Theorem C]). *Let  $E$  be a normal subgroup of  $G$ . If  $F^*(E)$  is hypercyclically embedded in  $G$ , then  $E$  is hypercyclically embedded in  $G$ .*

### 3 Proofs of Theorems A and B

Theorem A is a corollary of the following two general results.

**Theorem 3.1.** *Let  $E$  be a  $p$ -soluble normal subgroup of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $E$ . Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that  $H_1$  is  $p$ -supersoluble of order divisible by  $p$ . Suppose also that there is a natural number  $k$  such that  $p^k < |P|$  and every subgroup of  $P$  of order  $p^k$  and every cyclic subgroup of  $P$  of order 4 (if  $p^k = 2$  and  $P$  is non-abelian) are  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Then  $E/O_{p'}(E)$  is hypercyclically embedded in  $G/O_{p'}(E)$ .*

**Theorem 3.2.** *Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Suppose that  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that  $H_1$  is  $p$ -supersoluble of order divisible by  $p$ . Suppose also that there is a natural number  $k$  such that  $p^k < |P|$  and every subgroup of  $P$  of order  $p^k$  and every cyclic subgroup of  $P$  of order 4 (if  $p^k = 2$  and  $P$  is non-abelian) are  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Then  $G$  is  $p$ -soluble.*

**Proof of Theorem 3.1.** Assume that this theorem is false and let  $G$  be a counterexample with  $|G| + |E|$  minimal. Let  $|P| = p^n$ . Then:

(1)  $O_{p'}(N) = 1$  for every subnormal subgroup  $N$  of  $E$ . Hence  $O_p(G) \neq 1$ .

Suppose that for some subnormal subgroup  $N$  of  $G$  contained in  $E$  we have  $O_{p'}(N) \neq 1$ . Then  $O_{p'}(N)$  is subnormal in  $G$  and so  $O_{p'}(N) \leq O_{p'}(G)$  by Lemma 2.10. On the other hand, by Lemma 2.1(1), the hypothesis holds for  $(G/(E \cap O_{p'}(G)), E/(E \cap O_{p'}(G))) = (G/O_{p'}(E), E/O_{p'}(E))$ . Hence  $E/O_{p'}(E)$  is hypercyclically embedded in  $G/O_{p'}(E)$  by the choice of  $G$ , a contradiction. Thus we have (1).

(2) Let  $U = O_p(E)$ . Then  $U$  is not hypercyclically embedded in  $G$ .

Assume that  $U$  is hypercyclically embedded in  $G$ . Since  $E$  is  $p$ -soluble by hypothesis and  $O_{p'}(E) = 1$  by Claim (1),  $U \neq 1$  and  $C_E(U) \leq U$  by the Hall-Higman lemma [3, Ch.VI, Lemma 6.5]. But since  $U$  is hypercyclically embedded in  $G$ ,  $G/C_G(U)$  is strictly  $p$ -closed by Lemma 2.4 and so  $G/C_G(U)$  is supersoluble by [20, Ch.1, Theorem 1.9]. Now in view of the  $G$ -isomorphism  $EC_G(U)/C_G(U) \simeq E/E \cap C_G(U)$ , we conclude that  $E$  is hypercyclically embedded in  $G$ , a contradiction.

(3)  $k > 1$ .

Assume that  $k = 1$ . We show that in this case  $U$  is hypercyclically embedded in  $G$ . Assume that this is false. Let  $U/R$  be a chief factor of  $G$ . Then by the choice of  $G$  we have  $R$  is hypercyclically embedded in  $G$ , so for any normal subgroup  $V$  of  $G$  such that  $V < U$  we have  $V \leq R$  and  $U/R$  is not cyclic. Let  $B$  be a Thompson critical subgroup of  $U$  and  $\Omega = \Omega(B)$ . We claim that  $\Omega = U$ . Indeed, if  $\Omega < U$ , then  $\Omega \leq R$  and so  $\Omega$  is hypercyclically embedded in  $G$ . Hence  $U$  is hypercyclically embedded in  $G$  by Lemma 2.5, a contradiction. Thus  $\Omega = U$ . Since  $U \leq H_1$  and  $H_1$  is  $p$ -supersoluble by hypothesis, there is a subgroup  $L/R \leq U/R$  of order  $p$  such that  $L/R$  is normal in  $H_1/R$ . Let  $x \in L \setminus R$  and  $H = \langle x \rangle$ . Since  $\Omega = U$  and  $L \leq U$ ,  $|H|$  is either prime or 4. Then, by hypothesis,  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Hence  $HR/R$  is  $\sigma$ -semipermutable in  $G/R$  with respect to  $\{H_1R/R, \dots, H_tR/R\}$  by Lemma 2.1(1). Then, by Lemma 2.1(5),  $|G/R : N_{G/R}(HR/R)| = |G/R : N_{G/R}(L/R)|$  is a  $\pi(H_1)$ -number. It follows that  $L/R$  is normal in  $G/R$ , and so  $U/R = L/R$  is cyclic, a contradiction. This shows that  $U$  is hypercyclically embedded in  $G$ , contrary to Claim (2). Hence we have (3).

(4)  $|N| \leq p^k$  for any minimal normal subgroup  $N$  of  $G$  contained in  $P$ .

Indeed, suppose that  $|N| > p^k$ . Then there exists a non-identity proper subgroup  $H$  of  $N$  such that  $H$  is normal in  $H_1$  and  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . But then  $H$  is normal in  $G$  by Lemma 2.1(5), which contradicts the minimality of  $N$ .

(5) If  $P$  is a non-abelian 2-group, then  $k > 2$ .

Assume that  $k = 2$ . We shall show that in this case every subgroup  $H$  of  $P$  of order 2 is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . This means that  $k = 1$  is possible, which will contradict Claim (3).

First show that for any subgroup  $V = A \times B \leq P$  where  $|A| = 2 = |B|$ , if both  $V$  and  $A$  are  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ , then  $B$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Indeed, let  $i > 1$  and  $x \in G$ . Then  $AH_i^x$  and  $VH_i^x$  are subgroups of  $G$  and  $|VH_i^x : AH_i^x| = 2$ . Hence  $VH_i^x$  is 2-nilpotent, so  $H_i^x B = H_i^x B$  since  $H_i^x$  is normal in  $H_i^x V$ . Similarly, if  $V = \langle a \rangle$  is a cyclic subgroup of order 4, then  $\langle a^2 \rangle$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ .

Since  $P$  is a non-abelian 2-group,  $P$  has a cyclic subgroup  $H = \langle a \rangle$  of order 4. Then  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$  by hypothesis, so  $A = \langle a^2 \rangle$  is also  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Then every subgroup  $B$  of  $Z(P)$  of order 2 is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ , and so every subgroup of  $P$  of order 2 is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ .

(6) If  $N$  is a minimal normal subgroup of  $G$  contained in  $P$ , then  $(E/N)/O_{p'}(E/N)$  is hypercyclically embedded in  $(G/N)/O_{p'}(E/N)$ .

It is enough to show that the hypothesis holds for  $G/N$ . Since  $E/N$  is  $p$ -soluble, we can assume that  $|P/N| > p$ .

If either  $p > 2$  and  $|N| < p^k$  or  $p = 2$  and  $|N| < 2^{k-1}$ , then it is clear by Lemma 2.1(1). Now let either  $p > 2$  and  $|N| = p^k$  or  $p = 2$  and  $|N| \in \{2^k, 2^{k-1}\}$ .

In view of Claim (3),  $k > 1$ . Suppose that  $|N| = p^k$ . Then  $N$  is non-cyclic and so every subgroup of  $G$  containing  $N$  is not cyclic. Let  $N \leq K \leq P$ , where  $|K : N| = p$ . Since  $K$  is non-cyclic, it has a maximal subgroup  $L \neq N$ . Consider  $LN/N$ . Since  $L$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ ,  $LN/N$  is also  $\sigma$ -semipermutable in  $G/N$  with respect to  $\{H_1 R/R, \dots, H_t R/R\}$  by Lemma 2.1(1). Therefore, if  $P/N$  is abelian, the hypothesis is true for  $(G/N, P/N)$ . Next suppose that  $P/N$  is a non-abelian 2-group.

Then  $P$  is non-abelian and so  $k > 2$  by Claim (5). Since  $|P/N| > 2$ ,  $n - k \geq 2$ . We may, therefore, let  $N \leq K \leq V \leq P$  such that  $|V : N| = 4$ ,  $V/N$  is cyclic and  $|V : K| = 2$ . Since  $V/N$  is not elementary,  $N \not\leq \Phi(V)$ . Hence for some maximal subgroup  $K_1$  of  $V$  we have  $V = K_1 N$ . Suppose that  $K_1$  is cyclic. Then  $|K_1 \cap N| = 2$  and  $2 = |V : K_1| = |K_1 N : K_1| = |N : K_1 \cap N|$ . This implies that  $|N| = 4$ . But then  $k = 2$ , a contradiction. Hence  $K_1$  is not cyclic. Let  $S$  and  $R$  be two different maximal subgroups of  $K_1$ . Then  $K_1 = SR$ . If  $SN \leq K$  and  $RN \leq K$ , then  $K_1 = SR \leq K$ , which contradicts the choice of  $K_1$ . Now since  $N/N < K/N < V/N$  where  $K/N$  is a maximal subgroup of  $V/N$ , we have that  $V/N = K_1 N/N = SRN/N = (SN/N)(RN/N)$ . But since  $V/N$  is cyclic, either  $V/N = SN/N$  or  $V/N = RN/N$ . Without loss of generality, we may assume that  $NS = V$ . Since

$S$  is a maximal subgroup of  $K_1$  and  $K_1$  is a maximal subgroup of  $V$ ,  $|S| = |N| = p^k$ . Then  $S$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$ . Hence by Lemma 2.1(1),  $V/N$  is  $\sigma$ -semipermutable in  $G/N$  with respect to  $\{H_1R/R, \dots, H_tR/R\}$ . This shows that the hypothesis is true for  $(G/N, P/N)$ .

Now suppose that  $2^{k-1} = |N|$ . If  $|N| > 2$ , then  $N$  is not cyclic and as above one can show that every subgroup  $\bar{H}$  of  $P/N$  with order 2 and every cyclic subgroup of  $P/N$  of order 4 (if  $P/N$  is a non-abelian 2-group) is  $\sigma$ -semipermutable in  $G/N$  with respect to  $\{H_1R/R, \dots, H_tR/R\}$ . Finally, if  $|N| = 2$  and  $P/N$  is non-abelian, then  $P$  is non-abelian and  $k = 2$ , which contradicts Claim (5). Thus (6) holds.

$$(7) \Phi(U) = 1.$$

Assume that for some minimal normal subgroup  $N$  of  $G$  we have  $N \leq \Phi(U)$ . Then, by Claim (6), every chief factor of  $G/N$  between  $O_{p'}(E/N)$  and  $E/N$  is cyclic. Note that if  $V/N = O_{p'}(E/N) \neq 1$  and  $W$  is a  $p$ -complement in  $V$ , then by the Frattini argument,  $G = VN_G(W) = NW N_G(W) = N_G(W)$  since  $N \leq \Phi(O_p(E)) \leq \Phi(G)$ . Hence  $W = 1$  by Claim (1). Therefore every chief factor of  $G$  between  $E$  and  $N$  is cyclic. Now applying Lemma 2.5, we deduce that  $E$  is hypercyclically embedded in  $G$ , a contradiction. Hence we have (7).

*Final contradiction.* In view of Claims (2) and (7),  $U$  is an elementary group and for some minimal normal subgroup  $N$  of  $G$  contained in  $U$  we have  $|N| > p$ . Let  $S$  be a complement of  $N$  in  $U$ . Since  $N \leq H_1$  and  $|N| \leq p^k$  by (4), there are a maximal subgroup  $V$  of  $N$  and a subgroup  $W$  of  $S$  such that  $V$  is normal in  $H_1$  and  $|VW| = p^k$ . Then  $VW$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\mathcal{H}$  by hypothesis, so  $V = VW \cap N$  is normal in  $G$  by Lemma 2.1(5). Thus  $V = 1$ , and so  $|N| = p$ . This final contradiction completes the proof of the result.

**Proof of Theorem 3.2.** Assume that this theorem is false and let  $G$  be a counterexample of minimal order. Without loss of generality we can assume that  $P \leq H_1$  and  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ . Let  $|P| = p^n$  and  $V$  be a normal subgroup of  $G$  such that  $G/V$  is a simple group.

(1)  $O_{p'}(N) = 1$  for any subnormal subgroup  $N$  of  $G$  (See Claim (1) in the proof of Theorem 3.1).

(2)  $P \not\leq N$  for any proper normal subgroup  $N$  of  $G$  (In view of Lemma 2.1(4), this follows from the choice of  $G$ ).

(3) If the hypothesis holds for  $V$ , then  $G/V$  is non-abelian,  $O_p(V)$  is a Sylow  $p$ -subgroup of  $V$  and  $O_p(V)$  is hypercyclically embedded in  $G$ .

The choice of  $G$  implies that  $V$  is  $p$ -soluble. Hence  $V$  is  $p$ -supersoluble by Theorem A. Since  $O_{p'}(V) = 1$  by Claim (1),  $V$  is supersoluble and  $O_p(V)$  is a Sylow  $p$ -subgroup of  $V$  by Lemma 2.8. It is clear that  $O_p(V)$  is normal in  $G$ , so  $O_p(V)$  is hypercyclically embedded in  $G$  by Theorem 3.1.

(4)  $k > 1$ .

Assume that  $k = 1$ . Then:

(a) For a Sylow  $p$ -subgroup  $V_p$  of  $V$  we have  $V_p \not\leq Z_\infty(G)$ .

Indeed, assume that  $V_p \leq Z_\infty(G)$ . By [3, Ch. IV, Theorem 5.4],  $G$  has a  $p$ -closed Schmidt subgroup  $A$  and  $A = A_p \rtimes A_q$ , where the Sylow subgroup  $A_p$  of  $A$  is of exponent  $p$  or exponent



4 (if  $p = 2$  and  $A_2$  is non-abelian), and if  $\Phi = \Phi(A_p)$ , then  $A_p/\Phi$  is a non-central chief factor of  $A$ . Without loss of generality, we may assume that  $A_p \leq P$ . Then  $V_p \cap A \leq Z_\infty(A) \cap A_p \leq \Phi$  and so there exists a subgroup  $H$  of  $A_p$  such that  $H \not\leq V$  and  $H$  is a cyclic group of order  $p$  or of order 4 (if  $p = 2$  and  $A_2$  is non-abelian). By hypothesis,  $H$  is  $\sigma$ -semipermutable in  $G$ , so  $HV/V$  is  $\sigma$ -semipermutable subgroup of  $G/V$  by Lemma 2.1(1). Note that  $G \neq HH_2$  (In fact, if  $|H| = p$ , it is clear since  $|P| > p$ . If  $HH_2 = G$  and  $H$  is a cyclic group of order 4, then  $G$  is  $p$ -soluble, contrary to the choice of  $G$ ). Hence  $G/V$  is not simple by Lemma 2.2, a contradiction. Hence we have (a).

(b) If  $|V_p| = p$ , then  $V$  is not  $p$ -soluble, and so  $H_1V = G$ .

Indeed, if  $V$  is  $p$ -soluble, then  $V_p$  is normal in  $G$  by Claim (1). Hence  $V_p$  and  $C_G(V_p)$  are normal in  $G$ . Claim (a) implies that  $P \leq C_G(V_p) < G$ , which contradicts Claim (2). Therefore  $V$  is not  $p$ -soluble. But since the hypothesis holds for  $H_1V$  by Lemma 2.1(2)(3), the choice of  $G$  implies that  $H_1V = G$ .

(c)  $|V_p| \neq p$ . Hence the hypothesis holds for  $V$  by Lemma 2.1(2) and  $|P| > p^2$ .

Assume that  $|V_p| = p$ . If  $V_p = V \cap P \leq \Phi(P)$ , then  $V$  is  $p$ -nilpotent by the Tate theorem [3, Ch. IV, Theorem 4.7], contrary to (1). Hence  $V_p$  has a complement  $W$  in  $P$ . Let  $L$  be a subgroup of order  $p$  in  $W$ . Assume that  $L < W$ . Then the hypothesis holds for  $VW$  by Lemma 2.1(2)(3), so  $VW$  is  $p$ -soluble, contrary to Claim (b). Therefore  $|W| = p$ , so  $|P| = p^2$  and  $P = V_pW$  is not cyclic.

Let  $E = (H_2 \cdots H_t)^G$ . Then in view of Claim (b), we can assume, without loss of generality, that  $E \leq V$ . We show that there is a subgroup  $W_0$  of  $P$  order  $p$  such that  $W_0 \not\leq V$  and  $W_0 \not\leq C_G(E)$ . Indeed, suppose that  $W \leq C_G(E)$ . Note that  $C_G(E) \neq G$  by Claim (1). Hence  $V_p \not\leq C_G(E)$  by Claim (2). It follows Claim (1) that  $C_G(E) \cap V = 1$ . Consequently  $G = C_G(E) \times V$ . Let  $W = \langle a \rangle$ ,  $V_p = \langle b \rangle$  and  $W_0 = \langle ab \rangle$ . Then  $W_0 \cap C_G(E) = 1 = W \cap V$ .

Now let  $i > 1$ . Then  $W_0H_i^x = H_i^xW_0$  for all  $x \in G$  by hypothesis. Let  $L = H_i^{W_0} \cap W_0^{H_i}$ . Then  $L$  is a subnormal subgroup of  $G$  by [25, Theorem 7.2.5]. Suppose that  $L \neq 1$  and let  $L_0$  be a minimal subnormal subgroup of  $G$  contained in  $L$ . Then  $S = L_0 \cap W_0$  is a Sylow  $p$ -subgroup of  $L_0$  since  $L \leq W_0H_i$ . Moreover, in view of Claim (1) and Lemma 2.10,  $S \neq 1$ , and so  $W_0 = S$ . If  $L_0$  is abelian, then  $S = W_0 \leq O_p(G)$ , where  $O_p(G) < P$  by Claim (2). Hence  $W_0 = O_p(G) \not\leq V$ . Consequently  $W_0 \leq C_G(V) \leq C_G(E)$ . This contradiction shows that  $L_0$  is non-abelian. But then  $L_0 = L_0^G$  is a minimal normal subgroup of  $G$  by Claim (2) since  $|P| = p^2$ , which again implies that  $W_0 \leq C_G(E)$ . This contradiction shows that  $L = 1$ . Therefore for every  $x \in G$  and every  $i > 1$  we have  $(H_i^x)^{W_0} \cap W_0^{H_i^x} = 1$ , and so

$$[W_0, H_i^x] \leq [(H_i^x)^{W_0}, W_0^{H_i^x}] = 1.$$

Therefore  $W_0 \leq C_G(E)$ , a contradiction. Hence we have (c).

*Final contradiction for (4).* Let  $C = C_G(V_p)$ . By Claims (3) and (c),  $V_p$  is normal in  $G$  and it is hypercyclically embedded in  $G$ . Hence  $G/C$  is strictly  $p$ -closed by Lemma 2.4. If  $V_p \not\leq Z(G)$ , then there is a normal maximal subgroup  $M$  of  $G$  such that  $C \leq M$ . But since  $|P| > p^2$ , the hypothesis holds for  $M$ , so  $M$  is  $p$ -soluble and so  $G$  does. This contradiction shows that  $V_p \leq Z(G)$ , which contradicts Claim (a). Hence we have (4).

(5)  $|N| \leq p^k$  for any minimal normal subgroup  $N$  of  $G$  contained in  $P$  (See Claim (4) in the proof of Theorem 3.1).

(6)  $k = n - 1$ .

Assume that  $k < n - 1$ . Then  $VP \neq G$ . Indeed, if  $VP = G$ , then  $|G : V| = p$  and the hypothesis holds for  $V$ . Hence  $V$  is  $p$ -soluble by the choice of  $G$  and so  $G$  is  $p$ -soluble, a contradiction. By Lemma 2.1(4) the hypothesis holds for  $VP$ , so  $VP$  is  $p$ -soluble by the choice of  $G$  since  $VP \neq G$ . Therefore  $V$  is  $p$ -soluble, so  $O_p(V) \neq 1$  by Claim (1). Let  $N$  be a minimal normal subgroup of  $G$  contained in  $O_p(V)$ . It is clear that  $N \neq P$ . Since  $k < n - 1$ ,  $|P : N| > p$  by Claim (5). Now repeating some arguments in Claim (6) of the proof of Theorem A one can show that the hypothesis holds for  $G/N$ , so  $G/N$  is  $p$ -soluble by the choice of  $G$ . But then  $G$  is  $p$ -soluble, a contradiction. Hence we have (6).

(7) If  $O_p(G) \neq 1$ , then  $P$  is not cyclic.

Suppose that  $P$  is cyclic. Let  $L$  be a minimal normal subgroup of  $G$  contained in  $O_p(G) \leq P$ . Assume that  $C_G(L) = G$ . Then  $L \leq Z(G)$ . Let  $N = N_G(P)$ . If  $P \leq Z(N)$ , then  $G$  is  $p$ -nilpotent by Burnside's theorem [3, Ch. IV, Theorem 2.6], a contradiction. Hence  $N \neq C_G(P)$ . Let  $x \in N \setminus C_G(P)$  with  $(|x|, |P|) = 1$  and  $K = P \rtimes \langle x \rangle$ . By [3, Ch. III, Theorem 13.4],  $P = [K, P] \times (P \cap Z(K))$ . Since  $L \leq P \cap Z(K)$  and  $P$  is cyclic, it follows that  $P = P \cap Z(K)$  and so  $x \in C_K(P)$ . This contradiction shows that  $C_G(L) \neq G$ .

Since  $P$  is cyclic,  $|L| = p$ . Hence  $G/C_G(L)$  is a cyclic group of order dividing  $p - 1$ . But then  $P \leq C_G(L)$ , so  $C_G(L)$  is  $p$ -soluble by the choice of  $G$ . Hence  $G$  is  $p$ -soluble. This contradiction shows that we have (7).

(8)  $G \neq PH_i$  for any  $i > 1$ .

Without loss of generality, assume that  $G = PH_2$ . Let  $V_1, \dots, V_r$  be the set of all maximal subgroups of  $P$  and  $D_i = V_i^G$ . Then  $D_i = V_i^{PH_2} = V_i^{H_2} \leq V_i H_2 = H_2 V_i$  by Claim (6).

Suppose that for some  $i$ , say  $i = 1$ , we have  $D_1 P < G$ . Then  $D_1 P$  is  $p$ -soluble by the choice of  $G$ . Hence  $O_p(G) \neq 1$ . By Claim (7),  $P$  is not cyclic. Moreover, for any  $i > 1$ , we have that  $G = P^G = D_1 D_i$ . Hence for all such  $i > 1$ , we have that  $D_i P = G$  and so  $D_i = V_i H_2$ . It is also clear that  $V_2 \cap \dots \cap V_r = \Phi(P)$ . Let  $E = V_2 H_2 \cap \dots \cap V_r H_2$ . Then

$$P \cap E = (P \cap V_2 H_2) \cap \dots \cap (P \cap V_r H_2) = V_2 (P \cap H_2) \cap \dots \cap V_r (P \cap H_2) = V_2 \cap \dots \cap V_r = \Phi(P).$$

Hence  $E$  is  $p$ -nilpotent by the Tate theorem [3, Ch. IV, Theorem 4.7]. It follows that  $1 < H_2 \leq O_{p'}(G)$ , contrary to Claim (1). Hence we have (8).

(9)  $P^G = G$ , so  $P \not\leq H_i^G < G$  for all  $i > 1$ .

First note that  $P^G = G$  by Claim (2) and  $PH_i \neq G$  by Claim (8). If  $P$  is not cyclic, then  $PH_i^x = H_i^x P$  for all  $x \in G$ . Hence  $H_i^G < G$  by Lemma 2.2. Now assume that  $P$  is cyclic and  $V$  be a maximal subgroup of  $P$ . Lemma 2.2 implies that either  $V^G < G$  or  $H_i^G < G$ . But if  $V^G < G$ , then  $P \not\leq V^G$  and so  $V^G \cap P \leq \Phi(P)$ . Thus  $V^G$  is  $p$ -nilpotent by the Tate theorem [3, IV, 4.7], which implies that  $V^G = V$ , contrary to Claim (7). Hence  $H_i^G < G$ .

*Final contradiction.* Claim (8) implies that  $PH_i \neq G$  for all  $i = 2, \dots, t$ . Hence in view of Claim (9),  $H_2^G < G$ . Assume that  $P = KL$ , where  $K$  and  $L$  are different maximal subgroups of  $P$ . Then the hypothesis and claim (6) imply that  $PH_i = K L H_i = H_i K L = H_i P$  for all  $i$ . On the other hand, the hypothesis holds for  $PH_i$ , so  $PH_i$  is  $p$ -soluble by the choice of  $G$ . Now Lemma 2.3 implies that  $G$  is  $p$ -soluble. This contradiction shows that  $P$  is cyclic. But  $P \not\leq H_2^G$  by Claim (9), so  $H_2^G \cap P \leq \Phi(P)$ . Therefore  $H_2^G$  is  $p$ -nilpotent by the Tate theorem [3, Ch.IV, 4.7]. It follows from Claim (1) that  $H_2^G$  is a  $p$ -subgroup. This final contradiction completes the proof.

**Proof of Theorem B.** Assume that this theorem is false and let  $G$  be a counterexample with  $|G| + |E|$  minimal.

First suppose  $X = E$ . Let  $p$  be the smallest prime dividing  $|E|$  and  $P$  a Sylow  $p$ -subgroup of  $E$ . Then  $E$  is  $p$ -nilpotent. Indeed, if  $|P| = p$ , it follows directly from Lemma 2.7. If  $|P| > p$ , then  $E$  is  $p$ -supersoluble by Theorems 3.1 and 3.2, so  $E$  is  $p$ -nilpotent again by Lemma 2.7. Let  $V = O_{p'}(E)$ . Since  $V$  is characteristic in  $E$ , it is normal in  $G$  and the hypothesis holds for  $(G, V)$  and  $(G/V, E/V)$  by Lemma 2.1(1)(4).

The choice of  $G$  and Theorem 3.1 implies that  $P \neq E$ . Hence  $V \neq 1$ , so  $E/V$  is hypercyclically embedded in  $G/V$  by the choice of  $(G, E)$ . It is also clear that  $V$  is hypercyclically embedded in  $G$ . Hence  $E$  is hypercyclically embedded in  $G$  by the Jordan-Hölder theorem for the chief series, a contradiction. Therefore in the case, when  $X = E$ , the theorem is true. Finally, if  $X = F^*(E)$ , then the assertion follows from Lemma 2.11. The result is proved.

## 4 Applications

Theorems A, B, Theorems 3.1 and 3.2 cover many known results. Here we list some of them.

**Corollary 4.1** (Gaschütz and N. Ito [3, Ch. IV, Theorem 5.7]). *If every minimal subgroup of  $G$  is normal in  $G$ , then  $G$  is soluble and  $G'$  has a normal Sylow 2-subgroup with nilpotent factor group.*

**Proof.** This follows from the fact that  $G$  is  $p$ -supersoluble for all odd prime  $p$  dividing  $|G|$  by Theorem A.

**Corollary 4.2** (Buckley [26]). *If every minimal subgroup of a group  $G$  of odd order is normal in  $G$ , then  $G$  is supersoluble.*

In view of Example 1.5 we get from Theorem 3.2 the following results.

**Corollary 4.3** (Huppert [11]). *Suppose that for a Sylow  $p$ -subgroup  $P$  of  $G$  we have  $|P| > p$ . Assume that  $G$  has a  $p$ -complement  $E$  such that  $E$  permutes with all maximal subgroups of  $P$ . Then  $G$  is  $p$ -soluble.*

**Corollary 4.4** (Sergienko [12], Borovikov [13]). *Suppose that for a Sylow  $p$ -subgroup  $P$  of  $G$  we have  $|P| > p$ . Assume that  $G$  has a  $p$ -complement  $E$  and there is a natural number  $k$  such that  $p^k < |P|$  and every subgroup of  $P$  of order  $p^k$  permutes with  $E$ . Suppose also that in the case when  $p = 2$  the Sylow 2-subgroups of  $G$  are abelian. Then  $G$  is  $p$ -supersoluble.*

**Corollary 4.5** (Guo, Shum and Skiba [14]). *Suppose that  $G = AT$ , where  $A$  is a Hall  $\pi$ -subgroup*

of  $G$  and  $T$  a nilpotent supplement of  $A$  in  $G$ . Suppose that  $A$  permutes with all subgroups of  $T$ . Then  $G$  is  $p$ -supersoluble for each prime  $p \notin \pi$  such that  $|T_p| > p$  for the Sylow  $p$ -subgroup  $T_p$  of  $T$ .

**Proof.** Let  $E$  be the Hall  $\pi'$ -subgroup of  $T$ . Then every subgroup  $H$  of  $E$  permutes with  $A^x$  for all  $x \in G$  by Remark 1.3. Hence  $H$  is  $\sigma$ -semipermutable in  $G$  with respect to  $\{A, E\}$ , so  $G$  is  $p$ -supersoluble by Theorem A.

**Corollary 4.6** (Guo, Shum and Skiba [15]). Suppose that  $G = AT$ , where  $A$  is a Hall  $\pi$ -subgroup of  $G$  and  $T$  a minimal nilpotent supplement of  $A$  in  $G$ . Suppose that  $A$  permutes with all maximal subgroups of any Hall subgroup of  $T$ . Then  $G$  is  $p$ -supersoluble for each prime  $p \notin \pi$  such that  $|T_p| > p$  for the Sylow  $p$ -subgroup  $T_p$  of  $T$ .

In view of Example 1.4 we get from Theorem A the following

**Corollary 4.7** (Wei, Guo [10]). Let  $p$  be the smallest prime dividing  $|G|$  and  $P$  be a Sylow  $p$ -subgroup of  $G$ . If there a subgroup  $D$  of  $P$  with  $1 < |D| < |P|$  such that every subgroup  $H$  of  $P$  with order  $|D|$  or order  $2|D|$  (if  $|D| = 2$ ) is  $SS$ -quasinormal in  $G$ , then  $G$  is  $p$ -nilpotent.

From Example 1.4 and Theorem B we get the following three results.

**Corollary 4.8** (Li, Shen and Liu [8]). Let  $\mathcal{F}$  be a saturated formation containing all supersoluble groups and  $E$  a normal subgroup of  $G$  such that  $G/E \in \mathcal{F}$ . Suppose that for every maximal subgroup of every non-cyclic Sylow subgroup of  $E$  is  $SS$ -quasinormal in  $G$ . Then  $G \in \mathcal{F}$ .

**Corollary 4.9** (Li, Shen and Kong [9]). Let  $E$  a normal subgroup of  $G$  such that  $G/E$  is supersoluble. Suppose that for every maximal subgroup of every Sylow subgroup of  $F^*(E)$  is  $SS$ -quasinormal in  $G$ . Then  $G$  is supersoluble.

**Corollary 4.10** (Li, Shen and Kong [9]). Let  $\mathcal{F}$  be a saturated formation containing all supersoluble groups and  $E$  a normal subgroup of  $G$  such that  $G/E \in \mathcal{F}$ . Suppose that for every maximal subgroup of every Sylow subgroup of  $F^*(E)$  is  $SS$ -quasinormal in  $G$ . Then  $G \in \mathcal{F}$ .

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