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LAWS OF THE LATTICES OF PARTIALLY COMPOSITION FORMATIONS

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Abstract: We prove that every law of the lattice of all formations of finite groups is fulfilled in the lattice of all n-multiply ω-composition formations of finite groups for every nonempty set of primes ω and every natural n.

Keywords: finite group, formation of groups, ω-composition satellite of a formation, n-multiply ω-composition formation, lattice of formations, law of a lattice, modular lattice, inductive lattice of formations, $\mathfrak{X}$-separated lattice of formations

Introduction

All groups in this paper are finite. In what follows, ω denotes some nonempty set of primes and $\omega' = \mathbb{P} \setminus \omega$. We write $C^p(G)$ to denote the intersection of the centralizes of those chief factors of G whose composition factors have a prime order p (if G has no such factors, then we assume that $C^p(G) = G$). If $\mathfrak{X}$ is a collection of groups, then $\mathrm{Com}(\mathfrak{X})$ denotes the class of all simple abelian groups A such that $A \simeq H/K$ for some composition factor H/K of $G \in \mathfrak{X}$. Also, $R_\omega(G)$ denotes the $\mathfrak{S}_\omega$-radical of G, that is, the product of all soluble normal subgroups of G which are ω-groups.

A *formation* is a class of groups closed under taking homomorphic images and finite subdirect products. In the theory of formations, a distinguished place is occupied by the so-called ω-saturated formations. A formation $\mathfrak{F}$ is *ω-saturated* if $\mathfrak{F}$ contains every group G satisfying $G/L \in \mathfrak{F}$, where $L \subseteq \Phi(G) \cap O_\omega(G)$. The interest in ω-saturated formations increased significantly in the recent years and gave rise to some natural generalizations (ω-composition formations [1], $\mathfrak{X}$-local formations [2], and others).

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}. \tag{1}$$

By [1], for every function f, we define the class of groups

$$CF_\omega(f) = (G \mid G/R_\omega(G) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for some function f of the form (1), then f is said to be *ω-composition* and f is said to be an *ω-composition satellite* of $\mathfrak{F}$ [1].

Every formation is 0-multiply ω-composition by definition. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply ω-composition* [1] if $\mathfrak{F} = CF_\omega(f)$ and all nonempty values of f are $(n-1)$-multiply ω-composition formations. The set of all n-multiply ω-composition formations c_n^ω ordered by inclusion $\subseteq$ forms a complete lattice [1]. It is worth noting that ω-composition (in particular, n-multiply ω-composition) formations, in contrast to the classes of saturated formations, ω-saturated formations, and composition formations, are not generalized the theory of $\mathfrak{X}$-local formations which was developed by Ballester-Bolinches, Calvo,

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and Shemetkov in [2]. On the other hand, n-multiply ω-composition formations, as well as n-multiply ω-saturated formations, are most useful to various applications of the theory of formations.

The books [3, 4], as well as the recent books [5, 6], demonstrate that the constructions and theorems of lattice theory are useful tools to study groups and formations of groups. In 1986 Skiba [7] established that the lattice of all (saturated) formations is modular. This fact was widely used later for analyzing the structure of saturated formations (see [3, Chapter 4; 4, Chapters 4 and 5; 5, Chapter 4]). In [4] Skiba proved that for all natural m and n the lattice of all τ-closed m-multiply saturated formations and the lattice of all τ-closed n-multiply saturated formations have the same systems of laws. Wenbin Guo and Skiba [8] showed that for any infinite set of primes ω and any distinct naturals m and n, the laws of the lattice of all m-multiply ω-saturated formations and those of the lattices of all n-multiple ω-saturated formations coincide. Shemetkov, Skiba, and Vorob′ev extended in [9] this result to the lattices of functorialy closed n-multiply ω-saturated formations.

Wenbin Guo and K. P. Shum [10] described nonnilpotent totally saturated formations $\mathfrak{F}$ with a Boolean lattice $\mathfrak{F}/_{\infty}\mathfrak{F}\cap\mathfrak{N}$ of all totally saturated formations lying between $\mathfrak{F}$ and $\mathfrak{F}\cap\mathfrak{N}$. Then Wenbin Guo in [11] described τ-closed n-multiply saturated formations $\mathfrak{F}$ with a Boolean lattice $\mathfrak{F}/^{\tau}_{n}\mathfrak{F}\cap\mathfrak{N}$ of all τ-closed n-multiply saturated formations lying between $\mathfrak{F}$ and $\mathfrak{F}\cap\mathfrak{N}$.

Finally, we note that V. A. Vedernikov and his students (see, for example, [12–15]) studied analogous questions within an original theory of foliated formations.

One of the open problems in this area is the following question posed in [1, Problem 3, p. 796]: *Is it true that for all nonnegative integers m and n and every nonempty set of primes ω, the lattices c^{ω}_{m} and c^{ω}_{n} have the same system of laws?*

The main goal of the present paper is to solve the problem in the case of an infinite set of primes ω. An important step towards this goal is provided in [16] where the lattice of all τ-closed n-multiply ω-composition formations was proved to be inductive (see [4, Definition 4.1]).

We use the standard terminology of [1, 3–6, 17].

1. Preliminaries

Recall that a *semiformation* [3] is a class of groups closed under taking homomorphic images. Let $\mathfrak{X}$ be a collection of groups. We write $\operatorname{form}\mathfrak{X}$ to denote the smallest formation containing $\mathfrak{X}$.

Recall also some known facts that will be needed for proving the main results.

Lemma 1 [4, Corollary 1.2.26]. *Let $\mathfrak{X}$ be a semiformation and $A \in \mathfrak{F} = \operatorname{form}\mathfrak{X}$. If A is a monolithic group and $A \notin \mathfrak{X}$, then $\mathfrak{F}$ contains a group H with normal subgroups N, $N_1, \dots, N_t$ and M, $M_1, \dots, M_t$ $(t \geqslant 2)$ such that the following statements hold:*

(1) *$H/N \simeq A$, $M/N = \operatorname{Soc}(H/N)$;*

(2) *$N_1 \cap \dots \cap N_t = 1$;*

(3) *H/N_i is a monolithic $\mathfrak{X}$-group with the monolith M_i/N_i, which is H-isomorphic to M/N;*

(4) *$M_1 \cap \dots \cap M_t \subseteq M$.*

Lemma 2 [1, Lemma 2]. *Let $\mathfrak{F} = \bigcap_{i\in I}\mathfrak{F}_i$, where $\mathfrak{F}_i = CF_\omega(f_i)$. Then $\mathfrak{F} = CF_\omega(f)$, where $f = \bigcap_{i\in I} f_i$.*

A nonempty set of formations Θ is a *complete lattice of formations* [4] if the intersection of every collection of formations in Θ belongs to Θ, and Θ contains a formation $\mathfrak{F}$ such that $\mathfrak{H} \subseteq \mathfrak{F}$ for every other formation $\mathfrak{H} \in \Theta$.

Let Θ be a complete lattice of formations. If $\mathfrak{M}$, $\mathfrak{H} \in \Theta$, then $\mathfrak{M}\cap\mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M},\mathfrak{H}\}$ in Θ and $\mathfrak{M}\vee_\Theta\mathfrak{H}$ is the least upper bound for $\{\mathfrak{M},\mathfrak{H}\}$ in Θ. A satellite f is called Θ-*valued* [1] if all values of f belong to Θ. By [4], we write Θ^{ω_c} to denote the collection of all formations having a Θ-valued ω-composition satellite. In [1, pp. 786, 789] it was shown that Θ^{ω_c} and c^{ω}_{n} are complete lattices of formations.

Let Θ be a complete lattice of formations and let $\mathfrak{X} \subseteq \mathfrak{F} \in \Theta$ be a collection of groups. We write $\Theta\operatorname{form}\mathfrak{X}$ to denote the intersection of all formations of Θ containing all groups of $\mathfrak{X}$. In particular, we write $\Theta\,\mathrm{form}\,G$ when $\mathfrak{X} = \{G\}$. Thus $c_n^\omega\,\mathrm{form}\,\mathfrak{X}$ is the intersection of all n-multiply ω-composition formations containing all groups of $\mathfrak{X}$.

Let $\{f_i \mid i \in I\}$ be the set of all c_{n-1}^ω-valued ω-composition satellites of a formation $\mathfrak{F}$. By Lemma 2 we see that $f = \bigcap_{i\in I} f_i$ is a c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F}$. The satellite f is called the *minimal* c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F}$. The following lemma provides a method for constructing the minimal c_{n-1}^ω-valued satellite of $\mathfrak{F} = c_n^\omega\,\mathrm{form}\,\mathfrak{X}$.

Lemma 3 [1, Lemma 11]. *Let $\mathfrak{X}$ be a nonempty collection of groups, $\mathfrak{F} = c_n^\omega\,\mathrm{form}\,\mathfrak{X}$, where $n \geqslant 1$, $\pi = \omega\cap\pi(\mathrm{Com}(\mathfrak{X}))$, and let f be the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F}$. Then the following statements hold:*

(1) $f(\omega') = c_{n-1}^\omega\,\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{X})$;

(2) $f(p) = c_{n-1}^\omega\,\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$;

(3) $f(p) = \varnothing$ *for all* $p \in \omega \setminus \pi$;

(4) *if* $\mathfrak{F} = CF_\omega(h)$ *and the satellite* h *is* c_{n-1}^ω*-valued, then for all* $p \in \pi$

$$f(p) = c_{n-1}^\omega\,\mathrm{form}(A \mid A \in h(p) \cap \mathfrak{F},\ O_p(A) = 1),$$
$$f(\omega') = c_{n-1}^\omega\,\mathrm{form}(A \mid A \in h(\omega') \cap \mathfrak{F},\ R_\omega(A) = 1).$$

Lemma 4 [4, Lemma 4.1.3]. *Let $N_1 \times \cdots \times N_t = \mathrm{Soc}(G)$, where N_i is a minimal normal subgroup of a group G $(i = 1, \dots, t)$, $t > 1$, and $O_p(G) = 1$. Let M_i be the largest normal subgroup of G containing $N_1 \times \cdots \times N_{i-1} \times N_{i+1} \times \cdots \times N_t$ but not containing N_i $(i = 1, \dots, t)$. Then the following statements hold:*

(1) *for every $i \in \{1, \dots, t\}$ the factor group G/M_i is monolithic, its monolith N_iM_i/M_i is G-isomorphic to N_i, and $O_p(G/M_i) = 1$;*

(2) $M_1 \cap \cdots \cap M_t = 1$.

Lemma 5 [17, Theorem 2.2; 4, Lemma 1.2.22]. *If $\mathfrak{X}$ is a collection of groups, then* $\mathrm{form}\,\mathfrak{X} = \mathrm{QR}_0(\mathfrak{X})$.

The intersection of all semiformations that contain a given collection of groups $\mathfrak{X}$ is called the *semiformation generated by* $\mathfrak{X}$ [4].

Lemma 6 [4, Lemma 1.2.21]. *Let $\mathfrak{F}$ be a semiformation generated by $\mathfrak{X}$. Then $\mathfrak{F} = \mathrm{Q}\mathfrak{X}$.*

Recall that a class of groups $\mathfrak{F}$ is a *Fitting class* if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. With each Fitting class $\mathfrak{F}$, we can associate the smallest (by the inclusion) Fitting class $\mathfrak{F}^*$ containing $\mathfrak{F}$ and such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H. A Fitting class $\mathfrak{F}$ is called a *Lockett class* [18] if $\mathfrak{F} = \mathfrak{F}^*$.

Lemma 7 [18, X, Theorem 1.9]. *Let $\mathfrak{F}$ be a Fitting class. The following conditions are equivalent:*

(1) $\mathfrak{F} = \mathfrak{F}^*$;

(2) $(G \times H)_{\mathfrak{F}} = G_{\mathfrak{F}} \times H_{\mathfrak{F}}$ *for all* G *and* H.

Lemma 8 [19, Lemma 2]. *Let Z_p be a group of prime order p and let G be a group with $O_p(G) = 1$. Then the base group of the regular wreath product $T = Z_p \wr G$ is equal to $C^p(T) = O_p(T)$.*

Lemma 9 [1, Lemma 4]. *If $\mathfrak{F} = CF_\omega(f)$ and $G/O_p(G) \in f(p) \cap \mathfrak{F}$ for some $p \in \omega$, then $G \in \mathfrak{F}$.*

Let Θ be a complete lattice of formations. Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of Θ-formations. We denote

$$\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\,\mathrm{form}\Big(\bigcup_{i\in I}\mathfrak{F}_i\Big).$$

Let $\{f_i \mid i \in I\}$ be a collection of Θ-valued satellites. Then $\vee_\Theta(f_i \mid i \in I)$ denotes the satellite f such that

$$f(a) = \Theta\,\mathrm{form}\Big(\bigcup_{i\in I} f_i(a)\Big)$$

for every $a \in \omega \cup \{\omega'\}$.

A complete lattice of formations Θ^{ω_c} is *inductive* [4] if for every collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ of Θ^{ω_c} and for every collection $\{f_i \mid i \in I\}$ of inner Θ-valued ω-composition satellites f_i, where f_i is an ω-composition satellite of $\mathfrak{F}_i$, we have

$$\vee_{\Theta^{\omega_c}}(\mathfrak{F}_i \mid i \in I) = CF_\omega(\vee_\Theta(f_i \mid i \in I)).$$

Lemma 10 [16, Theorem 2.1]. *The lattice of all n-multiply ω-composition formations c_n^ω is inductive.*

Lemma 11 [11, Lemma 3.4.3]. *For every variety of groups $\mathfrak{M}$ the map* fin *of the form $\mathfrak{M} \to \operatorname{fin}\mathfrak{M}$ is an embedding of the lattice and semigroup of locally finite varieties into the algebra of all formations.*

2. $\mathfrak{G}$-Separability of the Lattice c_n^ω

Lemma 12. *Let A be a monolithic group with a nonabelian monolith R, let $\mathfrak{M}$ be a semiformation, and $A \in c_n^\omega \operatorname{form}\mathfrak{M}$, $n \geqslant 0$. Then $A \in \mathfrak{M}$.*

Proof. We proceed by induction on n. Suppose that $n = 0$. Then $A \in c_0^\omega \operatorname{form}\mathfrak{M} = \operatorname{form}\mathfrak{M}$. Let $A \notin \mathfrak{M}$. By Lemma 1 the formation $\operatorname{form}\mathfrak{M}$ contains a group H with normal subgroups $N, N_1, \dots, N_t$ and $M, M_1, \dots, M_t$ $(t \geqslant 2)$ such that the following statements hold:

(1) $H/N \simeq A$, $M/N = \operatorname{Soc}(H/N)$;

(2) H/N_i is a monolithic $\mathfrak{M}$-group with the monolith M_i/N_i and $M_i/N_i \overset{H}{\simeq} M/N$, $i = 1, \dots, t$.

Since $R \simeq M/N$ is nonabelian, $C_H(M/N) = N$. Furthermore, $M_i/N_i \overset{H}{\simeq} M/N$. Therefore $N_i \subseteq N$. Hence $A \simeq H/N \in \mathfrak{M}$; a contradiction. Thus the assertion of the lemma holds for $n = 0$.

Let $n > 0$ and the lemma holds for $n-1$. Suppose that f is the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F} = c_n^\omega \operatorname{form}\mathfrak{M}$. Since R is a nonabelian group, $\pi(\operatorname{Com}(R)) = \varnothing$. Thus $R_\omega(A) = 1$. Hence by Lemma 3

$$A \simeq A/1 = A/R_\omega(A) \in f(\omega') = c_{n-1}^\omega \operatorname{form}(G/R_\omega(G) \mid G \in \mathfrak{M}).$$

It follows that

$$A \in c_{n-1}^\omega \operatorname{form}(G/R_\omega(G) \mid G \in \mathfrak{M}) \subseteq c_{n-1}^\omega \operatorname{form}\mathfrak{M}.$$

By induction, $A \in \mathfrak{M}$, and the lemma is proved. □

Lemma 13. *Let $\mathfrak{M}$ be a semiformation and $A \in c_n^\omega \operatorname{form}\mathfrak{M}$, $n \geqslant 0$. Then the following statements hold:*

(1) *if $O_p(A) = 1$ and $p \in \omega$, then $A \in c_n^\omega \operatorname{form}\mathfrak{M}_1$, where $\mathfrak{M}_1 = (G/O_p(G) \mid G \in \mathfrak{M})$;*

(2) *if $R_\omega(A) = 1$, then $A \in c_n^\omega \operatorname{form}\mathfrak{M}_2$, where $\mathfrak{M}_2 = (G/R_\omega(G) \mid G \in \mathfrak{M})$.*

Proof. If $A \in \mathfrak{M}$, then the lemma is obvious. So we assume that $A \notin \mathfrak{M}$.

Suppose that A is a monolithic group with the monolith R. We proceed by induction on n.

Let $n = 0$. Since $A \notin \mathfrak{M}$ and $A \in c_0^\omega \operatorname{form}\mathfrak{M} = \operatorname{form}\mathfrak{M}$, it follows from Lemma 1 that $\operatorname{form}\mathfrak{M}$ contains a group H with normal subgroups $N, N_1, \dots, N_t$ and $M, M_1, \dots, M_t$ $(t \geqslant 2)$ such that the following statements hold: (1) $H/N \simeq A$, $M/N = \operatorname{Soc}(H/N)$; (2) $N_1 \cap \dots \cap N_t = 1$; (3) H/N_i is a monolithic $\mathfrak{M}$-group with the monolith M_i/N_i, which is H-isomorphic to M/N.

As $O_p(A) = 1$ and $R_\omega(A) = 1$, by Lemma 1 we obtain

$$H \in \mathrm{R}_0(H/N_1, \dots, H/N_t) \subseteq \mathrm{R}_0\mathfrak{M}_j,$$

where $j = 1, 2$. By condition 1 of Lemma 1 and Lemma 5, this implies that

$$A \simeq H/N \in \mathrm{QR}_0(H/N_1, \dots, H/N_t) = \operatorname{form}(H/N_1, \dots, H/N_t) \subseteq \operatorname{form}\mathfrak{M}_j,$$

where $j = 1, 2$.

Let $n > 0$. Suppose that $O_p(A) = 1$ and $p \in \omega$. If R is a nonabelian group, then $A \in \mathfrak{M}$ by Lemma 12; a contradiction. Hence R is a q-group, where $q \in \mathbb{P} \setminus \{p\}$.

Let $\mathfrak{F} = c_n^\omega \operatorname{form} \mathfrak{M}$ and $\mathfrak{H}_j = c_n^\omega \operatorname{form} \mathfrak{M}_j$, $j = 1, 2$. Let f and h_j $(j = 1, 2)$ be the minimal c_{n-1}^ω-valued ω-composition satellites of formations $\mathfrak{F}$ and $\mathfrak{H}_j$, respectively. By Lemma 3

$$f(\omega') = c_{n-1}^\omega \operatorname{form}(G/R_\omega(G) \mid G \in \mathfrak{M}), \quad f(s) = c_{n-1}^\omega \operatorname{form}(G/C^s(G) \mid G \in \mathfrak{M})$$

for all $s \in \omega \cap \pi(\operatorname{Com}(\mathfrak{M}))$;

$$h_j(\omega') = c_{n-1}^\omega \operatorname{form}(G/R_\omega(G) \mid G \in \mathfrak{M}_j), \quad h_j(s) = c_{n-1}^\omega \operatorname{form}(G/C^s(G) \mid G \in \mathfrak{M}_j)$$

for all $s \in \omega \cap \pi(\operatorname{Com}(\mathfrak{M}_j))$, $j = 1, 2$.

For every group G,

$$G/R_\omega(G) \simeq (G/O_p(G))/(R_\omega(G)/O_p(G)) = (G/O_p(G))/R_\omega(G/O_p(G)).$$

This implies that $f(\omega') = h_j(\omega')$, $j = 1, 2$.

If $q \notin \omega$, then $R_\omega(A) = 1$. Hence

$$A \simeq A/1 = A/R_\omega(A) \in f(\omega') = h_1(\omega') \subseteq \mathfrak{H}_1.$$

Let $q \in \omega$. We show that $A/R \in \mathfrak{H}_1$. Since $A \in \mathfrak{F}$, it follows that $A/R_\omega(A) \in f(\omega') = h_1(\omega') \subseteq \mathfrak{H}_1$.

Let $O_p(A/R) = 1$. Since $|A/R| < |A|$, we have $A/R \in \mathfrak{H}_1$ by induction.

Suppose that $O_p(A/R) \neq 1$. Let $R \subseteq \Phi(A)$ and $D/R = O_p(A/R)$. Then D is nilpotent. Hence $D = D_p \times D_q$, where D_p is a Sylow p-subgroup of D and D_q is a Sylow q-subgroup of D. Therefore $D_p = O_p(A) = 1$; a contradiction. Thus $R \not\subseteq \Phi(A)$ and so $R = C_A(R) = C^q(A)$.

Let $q \in \omega \cap \pi(\operatorname{Com}(A/R))$. For every group G,

$$G/C^q(G) \simeq (G/O_p(G))/(C^q(G)/O_p(G)) = (G/O_p(G))/C^q(G/O_p(G)).$$

This implies $f(q) = h_1(q)$. By hypothesis, $A \in \mathfrak{F}$ and so

$$A/R = A/C^q(A) \in f(q) = h_1(q) \subseteq \mathfrak{H}_1.$$

Thus in any case we have $A/R \in \mathfrak{H}_1$. Hence

$$A/C^r(A) \simeq (A/R)/(C^r(A)/R) = (A/R)/C^r(A/R) \in h_1(r)$$

for all $r \in \omega \cap \pi(\operatorname{Com}(A/R)) \setminus \{q\}$. Therefore $A/C^r(A) \in h_1(r)$ for all $r \in \omega \cap \pi(\operatorname{Com}(A))$. Furthermore, since $A \in \mathfrak{F}$, it follows that $A/R_\omega(A) \in f(\omega') = h_1(\omega')$. Thus $A \in \mathfrak{H}_1$. This proves assertion 1.

We now prove assertion 2. By hypothesis, $A \in \mathfrak{F}$. Since $R_\omega(A) = 1$,

$$A \simeq A/1 = A/R_\omega(A) \in f(\omega') = h_2(\omega') = c_{n-1}^\omega \operatorname{form}(G/R_\omega(G) \mid G \in \mathfrak{M}_2) \subseteq c_{n-1}^\omega \operatorname{form} \mathfrak{M}_2 \subseteq \mathfrak{H}_2.$$

Thus $A \in \mathfrak{H}_2$.

Suppose that A is not a monolithic group, that is, $\operatorname{Soc}(A) = N_1 \times \dots \times N_t$ with N_i a minimal normal subgroup of A and $t > 1$. Let M_i be the largest normal subgroup of A containing $N_1 \times \dots \times N_{i-1} \times N_{i+1} \times \dots \times N_t$ but not containing N_i, where $i = 1, \dots, t$. By Lemma 4 $A \in \mathrm{R}_0(A/M_1, \dots, A/M_t)$. By hypothesis, $A \in c_n^\omega \operatorname{form} \mathfrak{M}$. Therefore $A/M_i \in c_n^\omega \operatorname{form} \mathfrak{M}$. As we proved above, $A/M_i \in c_n^\omega \operatorname{form} \mathfrak{M}_1$. Thus $A \in c_n^\omega \operatorname{form} \mathfrak{M}_1$.

Considering the proof of Lemma 4, we replace the condition $O_p(A) = 1$ by the condition $R_\omega(A) = 1$ and conclude that for every $i \in \{1, \dots, t\}$ the factor group A/M_i is monolithic with the monolith $N_i M_i/M_i$ and $R_\omega(A/M_i) = 1$. As we proved above, $A/M_i \in c_n^\omega \operatorname{form} \mathfrak{M}_2$. Thus $A \simeq A/1 = A/(M_1 \cap \dots \cap M_t) \in c_n^\omega \operatorname{form} \mathfrak{M}_2$, as claimed. □

Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of n-multiply ω-composition formations. We denote

$$\vee_n^{\omega_c}(\mathfrak{F}_i \mid i \in I) = c_n^\omega \operatorname{form}\Big(\bigcup_{i \in I} \mathfrak{F}_i\Big).$$

Let $\{f_i \mid i \in I\}$ be a collection of c_n^ω-valued functions of the form

$$f_i : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}.$$

We write $\vee_n^{\omega_c}(f_i \mid i \in I)$ to denote the function f such that

$$f(\omega') = c_n^\omega \operatorname{form}\Big(\bigcup_{i \in I} f_i(\omega')\Big) \quad \text{and} \quad f(p) = c_n^\omega \operatorname{form}\Big(\bigcup_{i \in I} f_i(p)\Big)$$

for all $p \in \omega$.

The following lemma can be proved by direct calculations.

Lemma 14. *Let $n \geqslant 1$, and let f_i be the minimal c_{n-1}^{ω}-valued ω-composition satellite of an n-multiply ω-composition formation $\mathfrak{F}_i$, $i \in I$. Then $\vee_{n-1}^{\omega_c}(f_i \mid i \in I)$ is the minimal c_{n-1}^{ω}-valued ω-composition satellite of the formation $\mathfrak{F} = \vee_n^{\omega_c}(\mathfrak{F}_i \mid i \in I)$.*

Lemma 15. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be n-multiply ω-composition formations and $A \in \mathfrak{F}_1 \vee_n^{\omega_c} \mathfrak{F}_2$, $n \geqslant 0$. Then there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, 2)$ such that $A \in \big(c_n^{\omega} \operatorname{form} A_1\big) \vee_n^{\omega_c} \big(c_n^{\omega} \operatorname{form} A_2\big)$.*

Proof. We proceed by induction on n. Let $n = 0$. By Lemma 5

$$A \in \mathfrak{F}_1 \vee_0^{\omega_c} \mathfrak{F}_2 = c_0^{\omega} \operatorname{form}(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \operatorname{form}(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \mathrm{QR}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2).$$

Therefore $A \simeq H/N$, where $H \in \mathrm{R}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Thus H has normal subgroups $N_1, \dots, N_t$ $(t \geqslant 2)$ such that

$$\bigcap_{i=1}^{t} N_i = 1 \quad \text{and} \quad H/N_i \in \mathfrak{F}_1 \cup \mathfrak{F}_2, \quad i = 1, \dots, t.$$

Observe that $H^{\mathfrak{F}_1} \cap H^{\mathfrak{F}_2} = 1$. Hence $H \in \mathrm{R}_0(H/H^{\mathfrak{F}_1}, H/H^{\mathfrak{F}_2})$. Applying Lemma 5, we obtain

$$\begin{aligned} A \simeq H/N &\in \mathrm{QR}_0(H/H^{\mathfrak{F}_1}, H/H^{\mathfrak{F}_2}) = \operatorname{form}(H/H^{\mathfrak{F}_1}, H/H^{\mathfrak{F}_2}) \\ &= \operatorname{form}(H/H^{\mathfrak{F}_1}) \vee_0^{\omega_c} \operatorname{form}(H/H^{\mathfrak{F}_2}) \subseteq \mathfrak{F}_1 \vee_0^{\omega_c} \mathfrak{F}_2. \end{aligned}$$

Let $n > 0$, $\{p_1, \dots, p_t\} = \omega \cap \pi(\operatorname{Com}(A))$ and $A \in \mathfrak{F}_1 \vee_n^{\omega_c} \mathfrak{F}_2$. Then by Lemma 14

$$A/C^{p_i}(A) \in f_1(p_i) \vee_{n-1}^{\omega_c} f_2(p_i) \quad \text{and} \quad A/R_{\omega}(A) \in f_1(\omega') \vee_{n-1}^{\omega_c} f_2(\omega'),$$

where f_j is the minimal c_{n-1}^{ω}-valued ω-composition satellite of $\mathfrak{F}_j$ for $j = 1, 2$ and $i = 1, \dots, t$. By induction, there exist groups $A_{i_1} \in f_1(p_i)$, $A_{i_2} \in f_2(p_i)$, $T_1 \in f_1(\omega')$, and $T_2 \in f_2(\omega')$ such that

$$A/C^{p_i}(A) \in \big(c_{n-1}^{\omega} \operatorname{form} A_{i_1}\big) \vee_{n-1}^{\omega_c} \big(c_{n-1}^{\omega} \operatorname{form} A_{i_2}\big),$$

$$A/R_{\omega}(A) \in \big(c_{n-1}^{\omega} \operatorname{form} T_1\big) \vee_{n-1}^{\omega_c} \big(c_{n-1}^{\omega} \operatorname{form} T_2\big).$$

Clearly,

$$c_{n-1}^{\omega} \operatorname{form}(A_{i_1}, A_{i_2}) = \big(c_{n-1}^{\omega} \operatorname{form} A_{i_1}\big) \vee_{n-1}^{\omega_c} \big(c_{n-1}^{\omega} \operatorname{form} A_{i_2}\big),$$

$$c_{n-1}^{\omega} \operatorname{form}(T_1, T_2) = \big(c_{n-1}^{\omega} \operatorname{form} T_1\big) \vee_{n-1}^{\omega_c} \big(c_{n-1}^{\omega} \operatorname{form} T_2\big).$$

Let $\mathfrak{R}_1$ be a semiformation generated by A_{i_1}; $\mathfrak{R}_2$, a semiformation generated by A_{i_2}; $\mathfrak{Y}_1$, a semiformation generated by T_1; and $\mathfrak{Y}_2$, a semiformation generated by T_2.

By Lemma 6

$$\mathfrak{R}_1 = (B_1, \dots, B_s) \quad \text{and} \quad \mathfrak{R}_2 = (C_1, \dots, C_r)$$

for some $B_1, \dots, B_s \in \mathrm{Q}(A_{i_1})$ and $C_1, \dots, C_r \in \mathrm{Q}(A_{i_2})$;

$$\mathfrak{Y}_1 = (U_1, \dots, U_m) \quad \text{and} \quad \mathfrak{Y}_2 = (V_1, \dots, V_q)$$

for some $U_1, \dots, U_m \in \mathrm{Q}(T_1)$ and $V_1, \dots, V_q \in \mathrm{Q}(T_2)$.

Therefore

$$\begin{aligned} A/C^{p_i}(A) &\in c_{n-1}^{\omega} \operatorname{form}(A_{i_1}, A_{i_2}) = c_{n-1}^{\omega} \operatorname{form}(\mathfrak{R}_1 \cup \mathfrak{R}_2) \\ &= c_{n-1}^{\omega} \operatorname{form}(B_1, \dots, B_s, C_1, \dots, C_r), \end{aligned}$$

$$\begin{aligned} A/R_{\omega}(A) &\in c_{n-1}^{\omega} \operatorname{form}(T_1, T_2) = c_{n-1}^{\omega} \operatorname{form}(\mathfrak{Y}_1 \cup \mathfrak{Y}_2) \\ &= c_{n-1}^{\omega} \operatorname{form}(U_1, \dots, U_m, V_1, \dots, V_q). \end{aligned}$$

Since $O_{p_i}(A/C^{p_i}(A)) = 1$ and $R_{\omega}(A/R_{\omega}(A)) = 1$, we may suppose by Lemma 13 that $O_{p_i}(B_k) = 1 = O_{p_i}(C_l)$ and $R_{\omega}(U_x) = 1 = R_{\omega}(V_z)$ for all $k = 1, \dots, s$ and $l = 1, \dots, r$; $x = 1, \dots, m$ and $z = 1, \dots, q$.

Let $D_{i_1} = B_1 \times \dots \times B_s$, $D_{i_2} = C_1 \times \dots \times C_r$, $U = U_1 \times \dots \times U_m$, and $V = V_1 \times \dots \times V_q$. Since $\mathfrak{N}_{p_i}$ and $\mathfrak{S}_\omega$ are Lockett classes, Lemma 7 implies that

$$O_{p_i}(D_{i_1}) = (D_{i_1})_{\mathfrak{N}_{p_i}} = (B_1 \times \dots \times B_s)_{\mathfrak{N}_{p_i}} = (B_1)_{\mathfrak{N}_{p_i}} \times \dots \times (B_s)_{\mathfrak{N}_{p_i}}$$
$$= O_{p_i}(B_1) \times \dots \times O_{p_i}(B_s) = 1 \times \dots \times 1 = 1,$$

$$R_\omega(U) = U_{\mathfrak{S}_\omega} = (U_1 \times \dots \times U_m)_{\mathfrak{S}_\omega} = (U_1)_{\mathfrak{S}_\omega} \times \dots \times (U_m)_{\mathfrak{S}_\omega}$$
$$= R_\omega(U_1) \times \dots \times R_\omega(U_m) = 1 \times \dots \times 1 = 1.$$

Analogously, $O_{p_i}(D_{i_2}) = 1$ and $R_\omega(V) = 1$. Furthermore,

$$A/C^{p_i}(A) \in c^\omega_{n-1}\,\mathrm{form}(\mathfrak{R}_1 \cup \mathfrak{R}_2) = c^\omega_{n-1}\,\mathrm{form}(D_{i_1}, D_{i_2}) = c^\omega_{n-1}\,\mathrm{form}(A_{i_1}, A_{i_2}).$$

Let Z_{p_i} be a group of order p_i, $W_{i_1} = Z_{p_i} \wr D_{i_1}$, and $W_{i_2} = Z_{p_i} \wr D_{i_2}$. We show that $W_{i_1} \in \mathfrak{F}_1$. Let K be the base group of the regular wreath product W_{i_1}. By Lemma 8

$$W_{i_1}/K = W_{i_1}/O_{p_i}(W_{i_1}) \simeq D_{i_1} \in f_1(p_i) \cap \mathfrak{F}_1,$$

where $p_i \in \omega \cap \pi(\mathrm{Com}(A))$. By Lemma 9 we have $W_{i_1} \in \mathfrak{F}_1$. Analogously, $W_{i_2} \in \mathfrak{F}_2$.

Since $T_1 \in f_1(\omega')$ and f_1 is an inner ω-composition satellite of $\mathfrak{F}_1$, it follows that $T_1 \in \mathfrak{F}_1$. Thus $U_x \in \mathfrak{F}_1$ for all $x = 1, \dots, m$. Analogously, $V_z \in \mathfrak{F}_2$ for all $z = 1, \dots, q$.

Let $A_1 = W_{1_1} \times W_{2_1} \times \dots \times W_{t_1} \times U$ and $A_2 = W_{1_2} \times W_{2_2} \times \dots \times W_{t_2} \times V$. Then $A_1 \in \mathfrak{F}_1$ and $A_2 \in \mathfrak{F}_2$. Let $\mathfrak{F} = \left(c^\omega_n\,\mathrm{form}\,A_1\right) \vee^{\omega_c}_n \left(c^\omega_n\,\mathrm{form}\,A_2\right)$. We claim that $A \in \mathfrak{F}$. It suffices to prove that $A/R_\omega(A) \in f(\omega')$ and $A/C^{p_i}(A) \in f(p_i)$ for all $i = 1, \dots, t$, where f is the minimal c^ω_{n-1}-valued ω-composition satellite of $\mathfrak{F}$. Clearly, $W_{i_1} \in \mathfrak{F}$. Hence by Lemma 8 we see that $D_{i_1} \simeq W_{i_1}/K = W_{i_1}/C^{p_i}(W_{i_1}) \in f(p_i)$. Analogously, we deduce that $D_{i_2} \in f(p_i)$. Then $A/C^{p_i}(A) \in c^\omega_{n-1}\,\mathrm{form}(D_{i_1}, D_{i_2}) \subseteq f(p_i)$. Furthermore, $U, V \in \mathfrak{F}$. Therefore by Lemma 3

$$T_1 \simeq T_1/R_\omega(T_1) \in f(\omega') = c^\omega_{n-1}\,\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{F}).$$

Analogously, $T_2 \in f(\omega')$. Then $A/R_\omega(A) \in c^\omega_{n-1}\,\mathrm{form}(T_1, T_2) \subseteq f(\omega')$.

Thus $A \in \mathfrak{F}$. This proves the claim. $\square$

Let $\mathfrak{X}$ be a nonempty class of groups. A complete lattice of formations Θ is called $\mathfrak{X}$-*separated* [4] if for any term $\xi(x_1, \dots, x_m)$ of signature $\{\cap, \vee_\Theta\}$, any formations $\mathfrak{F}_1, \dots, \mathfrak{F}_m$ of Θ, and any group $A \in \mathfrak{X} \cap \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$, there exist $\mathfrak{X}$-groups $A_1 \in \mathfrak{F}_1, \dots, A_m \in \mathfrak{F}_m$ such that $A \in \xi(\Theta\,\mathrm{form}\,A_1, \dots, \Theta\,\mathrm{form}\,A_m)$.

We now prove the following result, which plays an essential role in the proof of our main results.

Proposition. *The lattice of all n-multiply ω-composition formations is $\mathfrak{G}$-separated for every nonnegative integer n.*

Proof. Let $\xi(x_1, \dots, x_m)$ be a term of signature $\{\cap, \vee^{\omega_c}_n\}$, let $\mathfrak{F}_1, \dots, \mathfrak{F}_m$ be n-multiply ω-composition formations, and suppose that $A \in \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$. By induction on the number r of occurrences of the symbols of $\{\cap, \vee^{\omega_c}_n\}$ in ξ, we prove that there exist $A_i \in \mathfrak{F}_i$ $(i = 1, \dots, m)$ such that $A \in \xi\left(c^\omega_n\,\mathrm{form}\,A_1, \dots, c^\omega_n\,\mathrm{form}\,A_m\right)$. If $r = 0$, then it is obvious that $A \in c^\omega_n\,\mathrm{form}\,A$.

We claim that the assertion is true for $r = 1$. If $A \in \mathfrak{F}_1 \cap \mathfrak{F}_2$, then $A \in c^\omega_n\,\mathrm{form}\,A \cap c^\omega_n\,\mathrm{form}\,A$. If $A \in \mathfrak{F}_1 \vee^{\omega_c}_n \mathfrak{F}_2$, then by Lemma 15 there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, 2)$ such that $A \in \left(c^\omega_n\,\mathrm{form}\,A_1\right) \vee^{\omega_c}_n \left(c^\omega_n\,\mathrm{form}\,A_2\right)$. Thus the assertion is proved for $r = 1$.

Let ξ have $r > 1$ occurrences of the symbols of $\{\cap, \vee^{\omega_c}_n\}$, and suppose that the assertion is proved for terms with less number of occurrences. Let ξ have the form $\xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b})$, where $\triangle \in \{\cap, \vee^{\omega_c}_n\}$, and let $\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}$. Let $\mathfrak{H}_1$ denote the formation $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$, and let $\mathfrak{H}_2$ denote the formation $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. There exist groups $A_1 \in \mathfrak{H}_1$ and $A_2 \in \mathfrak{H}_2$ such that $A \in c^\omega_n\,\mathrm{form}\,A_1 \triangle c^\omega_n\,\mathrm{form}\,A_2$. On the other hand, by induction, there exist groups $B_1, \dots, B_a$, $C_1, \dots, C_b$ such that $B_k \in \mathfrak{F}_{i_k}$, $C_k \in \mathfrak{F}_{j_k}$,

$$A_1 \in \xi_1\left(c^\omega_n\,\mathrm{form}\,B_1, \dots, c^\omega_n\,\mathrm{form}\,B_a\right), \quad A_2 \in \xi_2\left(c^\omega_n\,\mathrm{form}\,C_1, \dots, c^\omega_n\,\mathrm{form}\,C_b\right).$$

Suppose that $x_{i_1}, \dots, x_{i_t}$ are not contained in ξ_2, but $x_{i_{t+1}}, \dots, x_{i_a}$ are contained in ξ_2. Let $D_{i_k} = B_k$ if $k < t+1$ and $D_{i_k} = B_k \times C_q$, where q satisfies $x_{i_k} = x_{j_q}$ for all $k \geqslant t+1$. Let $D_{j_k} = C_k$ if $x_{j_k} \notin \{x_{i_{t+1}}, \dots, x_{i_a}\}$. Let $\mathfrak{R}_p$ denote the formation $c_n^\omega \operatorname{form} D_{i_p}$, and let $\mathfrak{X}_c$ denote the formation $c_n^\omega \operatorname{form} D_{j_c}$, $p = 1, \dots, a$, $c = 1, \dots, b$. It follows that $A_1 \in \xi_1(\mathfrak{R}_1, \dots, \mathfrak{R}_a)$ and $A_2 \in \xi_2(\mathfrak{X}_1, \dots, \mathfrak{X}_b)$. There exist formations $\mathfrak{L}_1 = c_n^\omega \operatorname{form} L_1, \dots, \mathfrak{L}_m = c_n^\omega \operatorname{form} L_m$ such that

$$A \in \xi_1(\mathfrak{L}_{i_1}, \dots, \mathfrak{L}_{i_a}) \triangle \xi_2(\mathfrak{L}_{j_1}, \dots, \mathfrak{L}_{j_b}) = \xi(\mathfrak{L}_1, \dots, \mathfrak{L}_m),$$

where $L_i \in \mathfrak{F}_i$, $i = 1, \dots, m$. Thus c_n^ω is $\mathfrak{G}$-separated. This proves the claim. □

In view of [1, Remark 3] we have

Corollary 1. *The lattice of all n-multiply $\mathfrak{L}$-composition formations is $\mathfrak{G}$-separated.*

In the case $n = 1$, we have

Corollary 2. *The lattice of all ω-composition formations is $\mathfrak{G}$-separated.*

When $\omega = \mathbb{P}$, we have

Corollary 3. *The lattice of all n-multiply composition formations $\mathfrak{G}$-separated.*

For $n = 1$ and $\omega = \mathbb{P}$, we obtain

Corollary 4. *The lattice of all composition formations $\mathfrak{G}$-separated.*

3. The Main Results

For every term ξ of signature $\{\cap, \vee_n^{\omega_c}\}$, we denote by $\bar{\xi}$ the term of signature $\{\cap, \vee_{n-1}^{\omega_c}\}$ obtained from ξ by replacing of every symbol $\vee_n^{\omega_c}$ by the symbol $\vee_{n-1}^{\omega_c}$.

Lemma 16. *Let $\xi(x_1, \dots, x_m)$ be a term of signature $\{\cap, \vee_n^{\omega_c}\}$ and let f_i be an inner c_{n-1}^ω-valued ω-composition satellite of a formation $\mathfrak{F}_i$, where $i = 1, \dots, m$ and $n \geqslant 1$. Then*

$$\xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = CF_\omega(\bar{\xi}(f_1, \dots, f_m)).$$

Proof. We proceed by induction on the number r of occurrences of the symbols in $\{\cap, \vee_n^{\omega_c}\}$ into ξ. The case $r = 1$ follows from Lemmas 2 and 10.

Let ξ have $r > 1$ occurrences of the symbols of $\{\cap, \vee_n^{\omega_c}\}$. Let

$$\xi(x_1, \dots, x_m) = \xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b}),$$

where $\triangle \in \{\cap, \vee_n^{\omega_c}\}$, and

$$\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}.$$

Assume that the lemma holds for the terms ξ_1 and ξ_2. Then

$$\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = CF_\omega(\bar{\xi}_1(f_{i_1}, \dots, f_{i_a})), \quad \xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = CF_\omega(\bar{\xi}_2(f_{j_1}, \dots, f_{j_b})).$$

It is clear that $\bar{\xi}_1(f_{i_1}, \dots, f_{i_a})$ and $\bar{\xi}_2(f_{j_1}, \dots, f_{j_b})$ are inner c_{n-1}^ω-valued ω-composition satellites of the formations $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$ and $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, respectively. Hence,

$$\begin{aligned} \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) &= \xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) \triangle \xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) \\ &= CF_\omega(\bar{\xi}_1(f_{i_1}, \dots, f_{i_a}) \bar{\triangle} \bar{\xi}_2(f_{j_1}, \dots, f_{j_b})) = CF_\omega(\bar{\xi}(f_1, \dots, f_m)), \end{aligned}$$

where $\bar{\triangle} = \cap$ if $\triangle = \cap$ and $\bar{\triangle} = \vee_{n-1}^{\omega_c}$ if $\triangle = \vee_n^{\omega_c}$, as claimed. □

Theorem 1. *Let $n \geqslant 1$. Then every law of the lattice of all formations c_0^ω is fulfilled in the lattice of all n-multiply ω-composition formations c_n^ω.*

Proof. Fix a law

$$\xi_1(x_{i_1},\dots,x_{i_a}) = \xi_2(x_{j_1},\dots,x_{j_b}) \tag{2}$$

of signature $\{\cap, \vee_n^{\omega_c}\}$. Let

$$\bar\xi_1(x_{i_1},\dots,x_{i_a}) = \bar\xi_2(x_{j_1},\dots,x_{j_b}) \tag{3}$$

be the same law of signature $\{\cap, \vee_{n-1}^{\omega_c}\}$.

Suppose that law (3) is true in the lattice c_{n-1}^ω. Let $\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}$ and $\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b}$ be arbitrary n-multiply ω-composition formations. We show that $\xi_1(\mathfrak{F}_{i_1},\dots,\mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b})$.

Let f_{i_c} be the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F}_{i_c}$ (where $c = 1,\dots,a$) and let f_{j_d} be the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{F}_{j_d}$ (where $d = 1,\dots,b$). By Lemma 16 we have

$$\xi_1(\mathfrak{F}_{i_1},\dots,\mathfrak{F}_{i_a}) = CF_\omega(\bar\xi_1(f_{i_1},\dots,f_{i_a})), \quad \xi_2(\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b}) = CF_\omega(\bar\xi_2(f_{j_1},\dots,f_{j_b})).$$

For every prime $p \in \omega$, the formations $f_{i_1}(p),\dots,f_{i_a}(p);\ f_{j_1}(p),\dots,f_{j_b}(p)$ and the formations $f_{i_1}(\omega')$, $\dots, f_{i_a}(\omega');\ f_{j_1}(\omega'),\dots,f_{j_b}(\omega')$ belong to c_{n-1}^ω. By induction,

$$\bar\xi_1(f_{i_1},\dots,f_{i_a})(p) = \bar\xi_1(f_{i_1}(p),\dots,f_{i_a}(p)) = \bar\xi_2(f_{j_1}(p),\dots,f_{j_b}(p)) = \bar\xi_2(f_{j_1},\dots,f_{j_b})(p)$$

and

$$\bar\xi_1(f_{i_1},\dots,f_{i_a})(\omega') = \bar\xi_1(f_{i_1}(\omega'),\dots,f_{i_a}(\omega')) = \bar\xi_2(f_{j_1}(\omega'),\dots,f_{j_b}(\omega')) = \bar\xi_2(f_{j_1},\dots,f_{j_b})(\omega').$$

Hence, $\xi_1(\mathfrak{F}_{i_1},\dots,\mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b})$. Thus law (2) is true in the lattice c_n^ω, and the result is proved. □

Corollary 5 [20, Theorem 5; 1, Theorem 4]. *The lattice of all n-multiply ω-composition formations c_n^ω is modular but not distributive.*

Proof. Since c_0^ω is modular (see [7]), Theorem 1 implies that c_n^ω is modular.

We show now that c_n^ω is not distributive. Let $\mathfrak{M}$ be the class of locally finite groups whose exponents divide a given prime $p \neq 2$. By [21] $\mathfrak{M}$ is a variety. Let $L(\mathfrak{M})$ be the lattice of the subvarieties of $\mathfrak{M}$. Then by Higman's result [22] (see also [23, § 54.24]) $L(\mathfrak{M})$ is not distributive. By Lemma 11 we see that $L(\mathfrak{M})$ can be embedded into c_0^ω. By Theorem 1, c_n^ω is not distributive. □

Lemma 17. *Let Θ be an $\mathfrak{X}$-separated lattice of formations and let η be a sublattice of Θ such that η contains all one-generated Θ-subformations of the form $\Theta\,\mathrm{form}\,A$, where $A \in \mathfrak{X}$, of every formation $\mathfrak{F} \in \eta$. Suppose that a law $\xi_1 = \xi_2$ of signature $\{\cap, \vee_\Theta\}$ is true for all one-generated Θ-formations belonging to η. Then the law $\xi_1 = \xi_2$ is true for all Θ-subformations belonging to η.*

Proof. Let $x_{i_1},\dots,x_{i_a}$ be the arguments occurring in the term ξ_1, let $x_{j_1},\dots,x_{j_b}$ be the arguments occurring in the term ξ_2, and let $\mathfrak{F}_{i_1},\dots,\mathfrak{F}_{i_a};\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b} \in \eta$. We show that

$$\mathfrak{F} = \xi_1(\mathfrak{F}_{i_1},\dots,\mathfrak{F}_{i_a}) \subseteq \xi_2(\mathfrak{F}_{j_1},\dots,\mathfrak{F}_{j_b}) = \mathfrak{M}.$$

Without loss of generality, we may suppose that $x_{j_1},\dots,x_{j_t} \in \{x_{i_1},\dots,x_{i_a}\}$ but $\{x_{j_{t+1}},\dots,x_{j_b}\} \cap \{x_{i_1},\dots,x_{i_a}\} = \varnothing$. Let $A \in \mathfrak{F}$. Then by assumption there exist $\mathfrak{X}$-groups $A_{i_1},\dots,A_{i_a}$ such that $A_{i_k} \in \mathfrak{F}_{i_k}$ (where $k = 1,\dots,a$) and $A \in \xi_1(\Theta\,\mathrm{form}\,A_{i_1},\dots,\Theta\,\mathrm{form}\,A_{i_a})$.

Let $\mathfrak{H}_{i_k} = \Theta\,\mathrm{form}\,A_{i_k}$, and let

$$\mathfrak{H}_{j_k} = \begin{cases} \mathfrak{H}_{i_c}, & \text{where } x_{j_k} = x_{i_c} \\ & \text{for some } c \in \{1,\dots,a\} \text{ and for all } k \in \{1,\dots,t\}, \\ \Theta\,\mathrm{form}\,B_{j_k} & \text{for some group } B_{j_k} \in \mathfrak{F}_{j_k} \text{ provided that } k > t. \end{cases}$$

By assumption, $\xi_1(\mathfrak{H}_{i_1},\dots,\mathfrak{H}_{i_a}) = \xi_2(\mathfrak{H}_{j_1},\dots,\mathfrak{H}_{j_b})$. But $\xi_2(\mathfrak{H}_{j_1},\dots,\mathfrak{H}_{j_b}) \subseteq \mathfrak{M}$. Hence $A \in \mathfrak{M}$. Thus $\mathfrak{F} \subseteq \mathfrak{M}$. The inverse inclusion can be proved analogously. Therefore $\mathfrak{F} = \mathfrak{M}$, which completes the proof of this lemma. □

Theorem 2. *Let $n \geqslant 1$. If ω is an infinite set, then the law system of the lattice c_0^ω coincides with the law system of the lattice c_n^ω.*

Proof. Fix a law

$$\xi_1(x_{i_1}, \dots, x_{i_a}) = \xi_2(x_{j_1}, \dots, x_{j_b}) \tag{4}$$

of signature $\{\cap, \vee_n^{\omega_c}\}$. Let

$$\bar{\xi}_1(x_{i_1}, \dots, x_{i_a}) = \bar{\xi}_2(x_{j_1}, \dots, x_{j_b}) \tag{5}$$

be the same law of signature $\{\cap, \vee_{n-1}^{\omega_c}\}$.

Suppose that law (4) is true in the lattice c_n^ω. We show that law (5) is true in c_{n-1}^ω. By Lemma 17 and the proposition, it suffices to prove that if $\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}$; $\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}$ are arbitrary one-generated $(n-1)$-multiply ω-composition formations, then $\bar{\xi}_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \bar{\xi}_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. Let

$$\mathfrak{F}_{i_1} = c_{n-1}^\omega \operatorname{form} A_{i_1}, \dots, \mathfrak{F}_{i_a} = c_{n-1}^\omega \operatorname{form} A_{i_a};$$
$$\mathfrak{F}_{j_1} = c_{n-1}^\omega \operatorname{form} A_{j_1}, \dots, \mathfrak{F}_{j_b} = c_{n-1}^\omega \operatorname{form} A_{j_b}.$$

We choose prime $p \in \omega$ such that $p \notin \pi(A_{i_1}, \dots, A_{i_a}, A_{j_1}, \dots, A_{j_b})$. Let

$$B_{i_1} = Z_p \wr A_{i_1}, \dots, B_{i_a} = Z_p \wr A_{i_a}, \quad B_{j_1} = Z_p \wr A_{j_1}, \dots, B_{j_b} = Z_p \wr A_{j_b},$$

where Z_p is a group of order p. Since formations

$$\mathfrak{M}_{i_1} = c_n^\omega \operatorname{form} B_{i_1}, \dots, \mathfrak{M}_{i_a} = c_n^\omega \operatorname{form} B_{i_a},$$
$$\mathfrak{M}_{j_1} = c_n^\omega \operatorname{form} B_{j_1}, \dots, \mathfrak{M}_{j_b} = c_n^\omega \operatorname{form} B_{j_b}$$

belong to c_n^ω, it follows that

$$\mathfrak{F} = \xi_1(\mathfrak{M}_{i_1}, \dots, \mathfrak{M}_{i_a}) = \xi_2(\mathfrak{M}_{j_1}, \dots, \mathfrak{M}_{j_b}) = \mathfrak{M}.$$

Let f_{i_c} be the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{M}_{i_c}$ (where $c = 1, \dots, a$) and let f_{j_d} be the minimal c_{n-1}^ω-valued ω-composition satellite of $\mathfrak{M}_{j_d}$ (where $d = 1, \dots, b$). By Lemma 16

$$\xi_1(\mathfrak{M}_{i_1}, \dots, \mathfrak{M}_{i_a}) = CF_\omega(\bar{\xi}_1(f_{i_1}, \dots, f_{i_a})),$$
$$\xi_2(\mathfrak{M}_{j_1}, \dots, \mathfrak{M}_{j_b}) = CF_\omega(\bar{\xi}_2(f_{j_1}, \dots, f_{j_b})).$$

Let f and m be the minimal c_{n-1}^ω-valued ω-composition satellites of $\mathfrak{F}$ and $\mathfrak{M}$, respectively. Then by Lemmas 3 and 14 we see that

$$\bar{\xi}_1(f_{i_1}, \dots, f_{i_a})(p) = \bar{\xi}_1(f_{i_1}(p), \dots, f_{i_a}(p)) = f(p),$$
$$\bar{\xi}_2(f_{j_1}, \dots, f_{j_b})(p) = \bar{\xi}_2(f_{j_1}(p), \dots, f_{j_b}(p)) = m(p).$$

Hence $\bar{\xi}_1(f_{i_1}(p), \dots, f_{i_a}(p)) = \bar{\xi}_2(f_{j_1}(p), \dots, f_{j_b}(p))$. Since $O_p(A_{i_c}) = 1$, it follows from Lemma 3 that $f_{i_c}(p) = \mathfrak{F}_{i_c}$, where $c = 1, \dots, a$. Analogously, $f_{j_d}(p) = \mathfrak{F}_{j_d}$, where $d = 1, \dots, b$.

Therefore $\bar{\xi}_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \bar{\xi}_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, that is, law (5) is true in the lattice c_{n-1}^ω. Thus every law in the lattice c_n^ω is also true in the lattice c_0^ω. Using Theorem 1, we have the result. □

Corollary 6. *Let ω be an infinite set. Let m and n be nonnegative integers. Then the law systems of lattices c_m^ω and c_n^ω coincide.*

In the case $\omega = \mathbb{P}$, we obtain

Corollary 7. *Let m and n be nonnegative integers. Then the law systems of lattices c_m and c_n coincide.*

Finally, we note that Jakubík proved in [24] that the collection of all formations of lattice ordered groups is a complete Brouwerian lattice.

The following natural question arises from Theorem 1: *Is it true that for any natural m and n, where $m > n$, the lattice of all m-multiply ω-composition formations is not a sublattice of the lattice of all n-multiply ω-composition formations?*

This is true for n-multiply ω-saturated formations (see [25]).

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