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RESEARCH ARTICLE

A note on c-normal subgroups of finite groups

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Abstract. Let G be a finite group. We fix in every non-cyclic Sylow subgroup P of G some its subgroup D satisfying $1 < |D| < |P|$ and study the structure of G under assumption that all subgroups H of P with $|H| = |D|$ are c-normal in G.

Introduction

Throughout this paper, all groups are finite. Following [1], we say that a subgroup H of a group G is c-normal in G if there exists a normal in G subgroup T such that $G = HT$ and $T \cap H \leq H_G$, where H_G is the largest normal subgroup of G contained in H. Several authors have investigated the structure of a group G under the assumption that certain maximal or minimal subgroups of Sylow subgroups of G are c-normal in G. Remind, in particular, that Wang [1] proved that G is supersoluble if either all maximal subgroups of the Sylow subgroups of G are c-normal in G or all minimal subgroups and all cyclic subgroups with order 4 are c-normal in G. Later on Li and Guo [2] obtained the analogous results by limiting the c-normality of maximal or minimal subgroups to the Fitting subgroups of a soluble groups. By using the theory of formation, Wei [3] extended further the results to a saturated formation containing the class of supersoluble groups. In the connection with these results the following natural question arises: Is the group G supersoluble if for any Sylow subgroup P of G at least one of the following conditions holds: (1) The maximal subgroups of P are c-normal in G; (2) The minimal

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subgroups of P and all its cyclic subgroups with order 4 are c-normal in G? In this paper we prove the following theorem which gives the positive answer to this question.

Theorem 0.1. *Let G be a group and E be a normal subgroup of G with supersoluble quotient G/E. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups E of P satisfying $|H| = |D|$ and all its cyclic subgroups with order 4 (in the case $|D| = 2$) are c-normal in G. Then G is supersoluble.*

As one of applications of this theorem we prove also the following generalisation of the results from [2].

Theorem 0.2. *Let G be a group and E be a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that every non-cyclic Sylow subgroup P of $F(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P satisfying $|H| = |D|$ and all its cyclic subgroups with order 4 (in the case $|D| = 2$) are c-normal in G. Then G is supersoluble.*

1. Preliminaries

In this paper we use $\mathcal{U}$ to denote the class of all supersoluble groups, $Z_{\infty}^{\mathcal{U}}(G)$ to denote the $\mathcal{U}$-hypercenter of a group G that is the product of all such normal subgroups H of G whose G-chief factors have prime order.

Lemma 1.1. *[5, II, Theorem 9.15]. $G/C_G(Z_{\infty}^{\mathcal{U}}(G)) \in \mathcal{U}$.*

Lemma 1.2. *Let G a group and $P = P_1 \times \ldots \times P_t$ be a p-subgroup of G where $t > 1$ and $P_1, \ldots, P_t$ are minimal normal subgroups of G. Assume that P has a subgroup D such that $1 < |D| < |P|$ and every subroup H of P with $|H| = |D|$ is normal in G. Then the order of every subgroup P_i is prime.*

Proof. Assume that this is false and let G be a counterexample with minimal order. Then for some i we have $|P_i| > p$ and $|D| > p$, by minimality of P_i. Thus if for some j we have $|P_j| = p$, the hypothesis is still true for G/P_j and its subgroup P/P_j and so every G-chief factor of between P_j and P has prime order. But then all subgroups $P_1, \ldots, P_t$ are simple, contradiction. Therefore $|P_k| > p$ for all $k = 1, \ldots, t$. Without loss of generality we may suppose that $D = P_1 \ldots P_k$ for some $k < t$. Let M be a maximal subgroup of P_1 and $H = MP_2 \ldots P_k Z$ where $|Z| = p$ and $Z \leq P_{k+1}$. Then $|H| = |D|$ and so by hypothesis, H is normal in G. But then $D \cap H = MP_2 \ldots P_k(D \cap Z) = MP_2 \ldots P_k$ is normal in G and so $M = P_1 \cap MP_2 \dots P_k$ is normal in G. This contradicts the minimality of P_1. $\square$

Lemma 1.3. *Let p be odd prime and P be a normal p-subgroup of a group G. Assume that $\Omega_1(P) \leq Z_{\mathcal{U}}(G) \cap Z(P)$. Then $P \subseteq Z_{\mathcal{U}}(G)$.*

Proof. Let $P_0 = \Omega_1(P)$, x in G have order p^2 and $g \in G$. Then $(x^g)^p = (x^p)^g = (x^p)^i = (x^i)^p$ for some integer i and so by [8, Theorem 1 (iv)], $(x^g x^{-i})^p = 1$, i.e. $x^g = x^{ip}u$ for some $u \in P_0$. Thus in P/P_0 every subgroup of order p is normal in G/P_0. Therefore $\Omega_1(G/P_0) \subseteq Z_{\mathcal{U}}(G/P_0) \cap Z(P/P_0)$ and so by induction $P/P_0 \subseteq Z_{\mathcal{U}}(G/P_0)$. Now it follows that $P \subseteq Z_{\mathcal{U}}(G)$. $\square$

Lemma 1.4. *[1, Lemma 2.1]. Let G be a group and $H \leq K \leq G$. Then*

(i) If H normal in G , then H is c-normal in G.

(ii) If H is c-normal in G, the H c-normal in K.

(iii) Suppose that H is normal in G. Then K/H is c-normal in G if and only if K is c-normal in G.

(iv) Suppose that H is normal in G. Then for every c-normal in G subgroup T with $(|H|, |T|) = 1$ the subgroup HT/H is c-normal in G/H (see [4, Lemma 2.2]).

A saturated formation is a homomorph $\mathcal{F}$ of groups such that each group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient is still in $\mathcal{F}$.

Lemma 1.5. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let G be a group with soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p, besides, if every cyclic subgroup of P with prime order and order 4 (if $p = 2$) is c-normal in G, then $|P/\Phi(P)| = p$.*

Proof. By [5, VI, Theorem 24.2], P is a p-group for some prime p and the following hold: (1) $P/\Phi(P)$ is a G-chief factor of P; (2) P is a group of exponent p or exponent 4 (if $p = 2$ and P is non-abelian). Assume that every cyclic subgroup of P with prime order and order 4 is c-normal in G. Let $X/\Phi(P)$ be a subgroup of $P/\Phi(G)$ with prime order, $x \in X \diagdown \Phi(P)$ and $L =< x >$. Then $|L| = p$ or $|L| = 4$ and so L is c-normal in G. Besides, $L\Phi(P)/\Phi(P) = X/\Phi(P)$. First assume that L is not normal in G. Then G has a normal subgroup T such that $LT = G$ and $|G : T| = p$. In this case $L \not\subseteq T$. But on the other hand since G/T is a p-group, $L \leq P \leq T$, by the definition of P, a contradiction. Thus L is normal in G. But then $L\Phi(P)/\Phi(P) = X/\Phi(P)$ is normal in $G/\Phi(P)$. Therefore $|P/\Phi(P)| = p$. $\square$

Lemma 1.6. *[5, II, Lemma 7.9]. Let P be a nilpotent normal subgroup of a group G. If $P \cap \Phi(G) = 1$, P is the direct product of some minimal normal subgroup of G.*

Lemma 1.7. *[4, III, Theorem 3.5]. Let A, B be normal subgroups of a group G and $A \leq \Phi(G)$. Suppose that $A \leq B$ and B/A is nilpotent. Then B is nilpotent.*

Lemma 1.8. *[8, I, p.34]. Let p be a prime. Then the class of all p-closed groups is a saturated formation.*

Lemma 1.9. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

Proof. It is enough to consider the case when E is a minimal normal subgroup of G. Clearly $E \not subseteq \Phi(G)$. Let M be a maximal subgroup of G such that $G = [E]M$ and let $C = C_G(E)$. Then $M_G = C \cap M$ and so $G/M_G = [EM_G/M_G](M/M_G)$ is supersoluble, since $M/M_G \simeq G/C$ is an abelian group. Thus $G \simeq G/E \cap M_G \in \mathcal{F}$. □

2. The proof of Theorem 0.1

Proof. Suppose that this is false and consider a counterexample for which $|G||E|$ is minimal.

(1) *The hypothesis is still true for every Hall subgroup of E and for G/E_π where E_π is any normal Hall subgroup of E* (this directly follows from Lemma 1.4).

(2) *E has a non-cyclic Sylow subgroup.*

Suppose that all Sylow subgroups of E are cyclic and let P be a Sylow subgroup of E where p is the largest prime divisor of $|E|$. Then by [7, IV, Theorem 2.9], E is supersoluble and so P is normal in G, as it is a characteristic subgroup of E. Thus by (1) the hypothesis is still true for G respectively its normal subgroup P. But by Lemma 1.9, $|P| < |E|$, since G is not supersoluble, and so $|P||G| < |E||G|$, a contradiction.

Now we fix some non-cyclic Sylow q-subgroup Q of E and let D be a subgroup of Q such that $1 < |D| < |Q|$, every subroup H of Q satisfying $|H| = |D|$ is c-normal in G and if $|D| = 2$, then also every subgrou with order 4 is c-normal in G.

(3) *Assume that $|Q : D| = q$ and let N be a minimal normal subgroup of G contained Q. Then E/N is p-closed where p is the largest prime divisor of $|E/N|$. Besides, if $|Q : N| \neq 4$, then E/N is supersoluble.*

If $|Q : N| \neq 4$, then by Lemma 1.4 the hypothesis is still true for E/N and so E/N is supersoluble, by the choice of G. Assume that $|Q : N| = 4$.

Then either Q/N is normal in G/N or Q/N is cyclic (in these cases the hypothesis is still true for E/N and so E/N is supersoluble) or G has a subgroup T with $|G : T| = 2$ and $N \leq T$. In the last case the hypotesis is true for T/N and its normal subgroup $E \cap T/N$, since a Sylow 2-subgroup of $E \cap T/N$ has order 2. Thus a Sylow p-subgroup of $E \cap T/N$ is normal in $E \cap T/N$ and hence E/N is p-closed.

(4) *Assume that $|Q : D| > q, |D| > q$ and all subgroups H of Q satisfying $|H| = |D|$ are normal in G. Suppose also that some subgroup L of Q satisfying $|L| = q$ is not normal in G. Then Q has at least two different maximal subgroups which are normal in G, besides, each non-cyclic maximal subgroup of Q is normal in G.*

Let N be a minimal normal subgroup of G contained in Q. First assume that $|N| = q$. Then $N \neq L$ and so $X = LN$ is a non-cyclic group. Let $X \leq H \leq K$ where $|H| = |D|$ and where $|K : H| = q$. Then K is not cyclic and hence it has a maximal subgroup M different from H. Then K is normal in G as the product of normal subgroups H and M. Analogously one can show that every subgroup of Q containing H is normal in G. If $|N| > q$, N is not cyclic and as above one can show that every subgroup of P containing N is normal in G.

(5) *If $|Q : D| > p$, then every subgroup H of Q satisfying $|H| = |D|$ is normal in G.*

Indeed, assume that some subgroup H of Q satisfying $|H| = |D|$ is not normal in G. Then G has a normal subgroup T such that $DT = G$ and $T \cap H \leq H_G$. Let M be a normal maximal subgroup of G such that $T \leq M$. Then $|G : M| = q$ and so $G/E \cap M$ is supersoluble. Hence the hypothesis is still true for G respectively its normal subgroup $E \cap M$, since $|Q : D| > p$. But $|G||E \cap M| < |G||E|$, contrary to the choice of G and its normal subgroup E. Hence we have (4).

(6) *E is soluble.*

By (1) we have only to consider the case $N = G$. Besides, we may assume that G is not p-closed for each prime divisor p of its order. First assume that G is p-nilpotent for some prime divisor p of $|G|$ and let T be a normal p-complement of G. Since $|T| < |G|$ and by (1), the hypothesis is still true for T, T is supersoluble and hence G is soluble. Next suppose that G is not p-nilpotent where p is the smallest prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. Then P is not cyclic [9, 10.1.9] and so by hypothesis, P has a subgroup D such that $1 < |D| < |P|$, every subroup H satisfying $|H| = |D|$ is c-normal in G and if $|D| = 2$, then also every subgroup with order 4 is c-normal in G. Besides, since G is not p-nilpotent, it has a p-closed Schmidt subgroup (see [4, IV, Theorem 5.4]). Thus by Lemma 1.5, $|D| > p$. Let $|P : D| = p$. Suppose that for some maximal subgroup H of P we have $H_G = 1$ and let T be a normal complement of H in G. Then the order of a Sylow p-subgroup of T is equel to p and so the hypothesis is still true for T. Therefore T is supersoluble, by the choice of G and hence G soluble. Hence for some maximal subgroup H of P we have $H_G \neq 1$. Let N be a minimal normal subgroup of G contained in P. If $|P : N| \neq 4$, then G/N is supersoluble by (3) and so G is soluble. Assume that $|P : N| = 4$. Then either P/N is normal in G/N or is cyclic (in these cases the hypothesis is still true for G/N and so G/N is supersoluble and hence G is soluble) or G has a subgroup T with $|G : T| = 2$ and $N \leq T$. In the last case the hypotesis is true T/N, since a Sylow 2-subgroup of T/N has order 2. Thus T/N is supersoluble and hence G is soluble. Finaly, assume that $|P : D| > p$. Then by (5) every subgroup H of P satisfying $|H| = |D|$ is normal in G. Suppose that for some minimal normal subgroup N of G contained in P we have $|N| = p$. If $|H/N| > p$, the hypothesis is true for G/N and so G is supersoluble, since $|N| = p$. But this contradicts the choice of G and hence N is a maximal subgroup of H. From above we have known that some minimal subgroup L of P is not normal in G and so by (4), P has at least two different maximal subgroups which are normal in G. If for some minimal normal subgroup N of G contained in P we have $|N| > p$, then by (4) every maximal subgroup M of P containing N normal in G. Thus P is normal in G subgroup as the product of normal subgroups. It follows that G is soluble.

(7) *E is p-closed where p is the largest prime divisor of $|E|$.*

By (1) and the choice of G we have only to consider the case $E = G$. Moreover, since by Lemma 1.4 the hypothesis is still true for all Hall subgroups of G, we may suppose that G is biprimary, i.e. $|G| = p^a q^b$ for some prime q and some $a, b \in \mathbb{N}$. Assume that G is not p-closed. Then a Sylow q-subgroup Q of G is non-cyclic. Hence by hypothesis, Q has a subgroup D such that $1 < |D| < |Q|$, every subroup H satisfying $|H| = |D|$ is c-normal in G and if $|D| = 2$, then also every cyclic subgroup with order 4 is c-normal in G Since G is not p-nilpotent, it has a minimal non-p-nilpotent subgroup S, say, and by [7, IV, Theorem 5.4], S is a Schmidt group. Hence by Lemma 1.5, $|D| > p$. First assume that $|Q : D| = q$. Then by (2), G/N is p-closed for every minimal normal subgroup N of G contained in Q. Thus by Lemma 1.8, $N \nsubseteq \Phi(G)$ and N is the only minimal normal subgroup of G contained in Q. By [4, III, Lemma 3.3] for some maximal subgroup V of P we have $N \nsubseteq V$. Let $L = V_G$ and T be a normal subgroup of G such that $VT = G$ and $T \cap V \leq L$. First assume that $L = 1$. Then $|T| = qp^a$ and so the hypothesis is still true for T. Therefore T is p-closed, since p is the largest prime divisor of $|T|$. Hence G is closed, a contradiction. Thus $L \neq 1$ and so $N \leq L \leq V$, a contradiction.

Therefore we may assume that $|Q : D| > q$. Then by (5) every subgroup H of Q satisfying $|H| = |D|$ is normal in G. If some minimal subgroup L of Q satisfying $|L| = q$ is not normal in G, then by (4), Q has at least two different maximal subgroups, M and E, say, which are normal in G. Since $|Q : D| > q$, the hypothesis is true for MP and EP where P is a Sylow p-subgroup of G. But then $Q = ME \leq N_G(P)$ and so G is p-closed, a contradiction. Thus all minimal subgroups of Q are normal in G. Let N be a minimal normal subgrop of G contained in Q and let $N \leq H$ where $|H| = |D|$. We show that G/N is p-closed. If $|N| < |D|$ it can be shown as above. Thus let $|N| = |D|$. Then since $|D| > p$, N is non-cyclic and so the minimal subgroups of N are normal non-identity subgroups of G. This contradiction completes the proof of (7).

(8) *$E = Q$ is non-cyclic q-group.*

Indeed, let p be the largest prime divisor of $|E|$ and P be a Sylow p-subgroup of E. Then by (7), P is normal in N and so P is normal in G, as a characteristic subgroup of E. Besides, by (1), the hypothesis is true for G/P and so G/P is supersoluble, by the choice of G. Thus $P = E = Q$, otherwise $|G||P| < |G||E|$. From Lemma 1.9 it follows also that E is not cyclic.

(9) $O_{q'}(G) = 1$.

Indeed, assume that $O_{q'}(G) = 1 \neq 1$. Then $|G/O_{q'}(G)| < |G|$ and so since by Lemma 4(iv) the hypothesis is true for $|G/O_{q'}(G)|$, $G/O_{q'}(G)$ is supersoluble, by the choice of G. But then $G \simeq G/(Q \cap O_{q'}(G))$ is supersoluble, a contradiction.

(10) $|D| > q$.

Assume that $|D| = q$. Then since $G/Q = G/E$ is supersoluble, $G^{\mathcal{U}} \leq Q$ and so the hypothesis is still true for G and its normal subgroup $G^{\mathcal{U}}$. Hence $G^{\mathcal{U}} = Q$. Let M be an arbitrary maximal subgroup of G not containing P. Then $G/Q = MQ/Q \simeq M/M \cap Q$ is supersoluble and so for M and its normal subgroup $M \cap Q$ the hypothesis is still true. Hence M is supersoluble, by the choice of G. Now using Lemma 1.5 we see that $Q/\Phi(Q)$ is a chief factor of G and $|Q/\Phi(Q)| = q$. But then, by Lemma 1.5, $G/\Phi(Q)$ is supersoluble and so $Q \leq \Phi(Q)$, a contradiction. Thus we have (10).

(11) *G/N is supersoluble for each minimal normal subgroup N of G contained in Q.*

We have only to show that the hypothesis is still true for G/N. By (8) it is clear that the hypothesis is true for G/N if either $Q : D| = q$ or $|N| = q > 2$. Therefore we may suppose that $|Q : D| > q$ and that either $|N| = 2$ or N is not cyclic. By (4) every subgroup H of P satisfying $|H| = |D|$ is normal in G. If N is non-cyclic we conclude from (4) that each maximal subgroup of Q contaning N is normal in G and we again see that the hypothesis is true for G/N. So let $|N| = 2$. In this case the hypothesis is true for G/N if Q is a Sylow q-subgroup of G, so let Q be a proper subgroup of Sylow q-subgroup of G. Then G has a normal maximal subgroup M such that $Q \leq M$. Since the hypothesis is still true for M, M is supersoluble and so a Sylow p-subgroup M_p is normal in G where p is the largest prime divisor of $|M|$. But this $M_p \leq O_{q'}(G)$ and so $O_{q'}(G) \neq 1$, contrary to (9).

(12) *Final contradiction.*

Let N be a minimal normal subgroup of G contained in Q. Then by (9), N is the only minimal normal subgroup of G contained in Q and $N \not\subseteq Phi(G)$. Let M be a maximal subgroup of G such that $N \not\subseteq M$ and let $C = C_G(N)$. Then $G = [R]M$ and $Q \subseteq C$, by [5, I, Corollary 4.1.1]. Hence $Q \cap M$ is normal in G. But $Q = Q \cap NM = N(Q \cap M)$ and so $Q = N$. Since Q is not cyclic, it has such a subgroup D that $1 < |D| < |P|$ and every subgroup H satisfying $|D| = |H|$ is c-normal in G. But it is impossible because P is a minimal normal subgroup of G. This contradiction completes the proof of this theorem. □

3. Some corollaries of Theorem 0.1

In this section we consider some applications of Theorem 0.1.

As an immediate consequence of Theorem 0.1, we have:

Corollary 1. *Let G be a group. Suppose that for any non-cyclic Sylow subgroup P of G at least one of the following conditions holds:*

(1) The minimal subgroups of P and all its cyclic subgroups with order 4 are normal in G;

(2) The maximal subgroups of P are normal in G.

Then G is supersoluble.

Corollary 2. *(Buckley [8]) Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

Corollary 3. *(Srinivasan [9]). If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersoluble.*

Corollary 4. *Let G be a group. Suppose that for any non-cyclic Sylow subgroup P of G at least one of the following conditions holds:*

(1) The minimal subgroups of P and all its cyclic subgroups with order 4 are c-normal in G;

(2) The maximal subgroups of P are c-normal in G.

Then G is supersoluble.

Corollary 5. *(Wang [1]).If all subgroups of G of prime order or order 4 are c-normal in G, then G is supersoluble.*

Corollary 6. *(Wang [1]).If the maximal subgroups of the Sylow subgroups of G are c-normal in G, then G is supersoluble.*

Now using Theorem 0.1 we prove Theorem 0.2.

Proof. Assume that this is false and let G be a counterexample with minimal $|G||E|$. Let $F = F(E)$ and p be the largest prime divisor of $|F|$. Let P be the Sylow p-subgroup of F, $P_0 = \Omega_1(P)$ and $C = C_G(P_0)$. Clearly C is normal in G.

(1) *The hypothesis is true for E and for every normal subgroup of G having coprime order with $|E|$* (this directly follows from Lemma 1.4).

(2) $p > 2$.

Indeed, suppose that $p = 2$. Assume that $E \neq G$. Then E is supersoluble, by (1) and the choice of G. Hence a Sylow p-subgroup E_p of E is normal in E where p is the largest prime divisor of E. Hence $E_p = P = E$. But in this case G is supersoluble, by Theorem 0.1, a contradiction. Therefore $E = G$ is a soluble group and hence $C_G(F) \leq F$ is a 2-group, by [5, II, Theorem 7.12]. Let Q be a subgroup of G with prime order q where $q \neq 2$ and let $X = FQ$. Then the hypothesis is still true for X and so it is supersoluble, by the choice of G. But then Q is normal in X and so $Q \leq C_G(F)$, a contradiction. Hence we have (2).

(3) $P_0 \nleq Z_\infty^{\mathcal{U}}(G) \cap Z(P)$.

Suppose that $P_0 \leq Z_\infty^{\mathcal{U}}(G) \cap Z(P)$. We show that the hypothesis is true for G/P_0 and its normal subgroup C_E/P_0 where $C_E = C \cap E$. . By Lemma 1.1, G/C is supersoluble and hence G/C_E is supersoluble, since G/E is supersoluble, by hypothesis. Clearly $F = F(C_E)$ and so since $P_0 \leq Z(C_E), F(C_E/P_0) = F/P_0$. Now making use Lemma 1.3 we see that the hypothesis is still true for G/P_0 and so this quotient is supersoluble, by the choice of G. Since $P_0 \leq Z_\infty^{\mathcal{U}}(G)$, it follows that G is supersoluble, by Lemma 1.9, a contradiction. Thus we have (3).

By (3), P is not cyclic and so by hypothesis P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is c-normal in G.

(4) $|D| > p$.

Suppose tat $|D| = p$. By (3), P has a subgroup H with prime order which is not normal in G. Let T be a normal complement of H in G. Then the hypothesis is true for G and its subgroup $V = T \cap E$. Indeed, evidently G/T is supersoluble and $F(V) \leq F(E)$. On the other hand, since $|G : T| = p$, every Sylow p-subgroup of F where $q \neq p$ is contained in T. Thus the hypothesis is still true for G and its subgroup

V, by Lemma 1.4. But since T is a proper subgroup of G and evidently $ET = G$, $|T| < |E|$, which contradicts the choice of G and its normal subgroup E. This contradiction completes the proof of (4).

(5) *If L is a minimal normal subgroup of G and $L \leq P$, $|L| > p$.*

Assume that $|L| = p$. Let $C_0 = C_G(L)$. Then the hypothesis is true for G/L and its subgroup C_0/L. Indeed, since $L \leq Z(C_0)$, we have $F(C_0/L) = F/L$. On the other hand, if H/L is a subgroup of G/L such that $|H| = |D|$, we have $1 < |H/L| < |P/L|$, by (4). Besides, H/L is c-normal in G/L we have only to use Lemma1.4(iv).

(6) $\Phi(G) \cap P = 1$.

Suppose that $\Phi(G) \cap P \neq 1$ and let L be a minimal normal subgroup of G contaned in $\Phi(G) \cap P$. Then by (5), L is non-cyclic and so every subgroup of G, containing L is non-cyclic. Clearly $|L| \leq |D|$. We show that the hypothesis is true for G/L and its normal subgroup E/L. First of all note that by Lemma 1.8, $F(E/L) = F/L$. By Lemma 1.4(iv) we may assume that $|P : L| > p$. Let $|L| = |D|$ and let $L \leq K, M \leq K$ where $M \neq L$ and L, M are maximal subgroups of K. We have only to show that K is c-normal in G. It is evident if M is normal in G. Let $L = K_G$ and T be a normal subgroup of G such that $MT = G$ and $T \cap M \leq M_G \neq M$. Let S be a normal subgroup of G such that $|G : S| = p$ and $T \leq S$. Then evidently $KS = G$ and $S \cap K \leq K_G = L$, since $L \leq \Phi(G) \leq S$. Therefore K is c-normal in G. Thus the hypothesis is still true for G/L and hence G/L is supersoluble, by the choice of G. Since the class supersoluble groups is a saturated formation, it follows that G is supersoluble, a contradiction.

(7) *P is the direct product of some minimal normal subgroups of G* (this directly follows from (6) and Lemma 1.6).

(8) *Final contradiction.*

If every subgroup H of G with $|H| = |D|$ is normal in G, then by (7) and Lemma 1.3, $P \leq Z_\infty^{\mathcal{U}}(G)$, which contradicts (3). Thus for some of such subgroups H we have $H \neq H_G$ and so G has a normal maximal subgroup T such that $PT = G$. In this case a chief factor $P/P \cap T$ of G has prime order and so by (7) for some minimal normal subgroup L of G contained in P we have $|P| = p$, which contradicts (5). $\square$

The following corollaries are consequences of Theorem 0.2.

Corollary 7. *Let G a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that for any non-cyclic Sylow subgroup P of $F(E)$ at least one of the following conditions holds:*

(1) The minimal subgroups of P and all its cyclic subgroups with order 4 are c-normal in G;

(2) The maximal subgroups of P are c-normal in G.
Then G is supersoluble.

Corollary 8. *(Li and Guo [2]). Let G a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 9. *(Li and Guo [2]). Let G a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all subgroups of $F(E)$ of prime order or order 4 are c-normal in G, then G is supersoluble.*

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