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FINITE BIPRIMARY GROUPS WITH ALL 3-MAXIMAL SUBGROUPS $\mathfrak{U}$-SUBNORMAL

V. A. KOVALEVA[1] and X. YI[2,*,†]

[1]Department of Mathematics, Francisk Skorina Gomel State University, Gomel 246019, Belarus
e-mail: vika.kovalyova@rambler.ru

[2]Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China
e-mail: yxlyixiaolan@163.com

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Abstract. A complete classification of finite biprimary groups in which every 3-maximal subgroup is $\mathfrak{U}$-subnormal is given.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use $\mathfrak{U}$ to denote the class of all supersoluble groups; $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups N of G with $G/N \in \mathfrak{U}$. The symbol $\pi(G)$ denotes the set of prime divisors of the order of G. If $|\pi(G)| = 2$, then G is called *biprimary.*

A subgroup H of G is called 2-*maximal* or *second maximal* in G if H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define 3-*maximal subgroups,* and so on.

In the paper [1], B. Huppert proved that if every 3-maximal subgroup of G is normal in G, then the commutator subgroup G' of G is nilpotent and the chief rank of G is at most 2. Later, this result was generalized and developed by many other authors. In particular, M. Asaad [2] obtained the same result for strongly 3-maximal subgroups (that are 3-maximal subgroups, which are not 4-maximal). In [3], R. Schmidt described groups in which every 3-maximal subgroup is a modular element of the subgroup lattice.

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† Corresponding author.

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Among the recent papers in this line of research we can mention the paper of W. Guo, Yu. V. Lutsenko and A. N. Skiba [4] which gives a description of nonnilpotent groups under the condition that every two 3-maximal subgroups are permutable. A description of the groups all of whose 3-maximal subgroups are subnormal was obtained in [5]. In [6], V. N. Kniahina and V. S. Monakhov studied those groups G in which every 3-maximal subgroup permutes with each Schmidt subgroup.

Recall that a subgroup H of G is said to be $\mathfrak{U}$-*subnormal* in G if there exists a chain of subgroups $H = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$ such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \ldots, n$. If G is soluble, then $H \neq G$ is $\mathfrak{U}$-subnormal in G if and only if there is a chain $H = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$, where $|H_i : H_{i-1}|$ is a prime for all $i = 1, \ldots, n$. It is evident that every subnormal subgroup of a soluble group is $\mathfrak{U}$-subnormal. The inverse, in general, is not true since the class of all supersoluble groups is wider than the class of all nilpotent groups. This elementary observation and the results in [3,5,7] make natural the following question:

QUESTION 1.1. *What is the structure of G under the condition that every 3-maximal subgroup of G is $\mathfrak{U}$-subnormal?*

Note that since each subgroup of every supersoluble group is $\mathfrak{U}$-subnormal, we need, in fact, only consider the case when G is not supersoluble. But in this case, in view of [7, Theorem A], $\left|\pi(G)\right| \leqq 4$. Partly, the answer to Question 1.1 in the case when $\left|\pi(G)\right| = 3$ and $\left|\pi(G)\right| = 4$ was given in [8]. In this paper, we give the complete answer to this question in the case when $\left|\pi(G)\right| = 2$.

Before continuing, recall that G is called $\mathfrak{U}$-*critical* or *minimal nonsupersoluble* if G is not supersoluble but all proper subgroups of G are supersoluble. $\mathfrak{U}$-critical groups were described by B. Huppert [1] and K. Doerk [9]. We say that G is a *special Doerk–Huppert group* or an *SDH-group* if G is an $\mathfrak{U}$-critical group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.

It is known (see Theorem D in [7]) that every 2-maximal subgroup of a nonsupersoluble group G is $\mathfrak{U}$-subnormal if and only if G is an SDH-group. Based on this result we prove the following

THEOREM 1.2. *Let G be a nonsupersoluble biprimary group. Let p and q be distinct prime divisors of $|G|$, P and Q be a Sylow p-subgroup and q-subgroup of G, respectively. Every 3-maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if the following hold*:

(I) *If G has no normal Sylow subgroups and $O^p(G) \neq G$, then $G^{\mathfrak{U}} \leqq P$, Q is a cyclic group such that $[Q^q, G^{\mathfrak{U}}] = 1$ and p divides $q - 1$. Moreover, in this case, $G^{\mathfrak{U}}Q$ is a maximal subgroup of G and Q induces on $G^{\mathfrak{U}}$ an irreducible group of automorphisms.*

(II) *If P is normal in G, then the following statements are true*:

(i) *Every* 2*-maximal subgroup of* Q *induces an abelian group of automorphisms of exponent dividing* $p-1$ *on* P. *Every maximal subgroup of* Q *induces on* P *a group of automorphisms which is either irreducible or an abelian of exponent dividing* $p-1$.

(ii) *If* P *is a minimal normal subgroup of* G *and* q *does not divide* $p-1$, *then* Q *is cyclic and* $Z(G)$ *is a subgroup of* Q *such that* $\big|Q:Z(G)\big| \in \{q,q^2\}$. *Moreover, in this case, if* G *is not an* $\mathfrak{U}$*-critical group, then* q^2 *divides* $p^{q-1}-1$.

(iii) *If* $\Phi(P)\neq 1$, *then* $G^{\mathfrak{U}}=P$ *and* $P/\Phi(P)$ *is a non-cyclic chief factor of* G. *Moreover, if* G *is an* $\mathfrak{U}$*-critical group, then* $\big|\Phi(P)\big|=p$. *If* G *is not an* $\mathfrak{U}$*-critical group, then* $\Phi(P)Q$ *is an SDH-group and hence* $\Phi(P)$ *is a minimal normal subgroup of* G.

(iv) *If* P *is not a minimal normal subgroup of* G *and* $\Phi(P)=1$, *then* $P=P_1\times P_2$, *where* P_1 *and* P_2 *are minimal normal subgroups of* G *and at least one of these subgroups is non-cyclic.*

All unexplained notation and terminology are standard. The reader is referred to [10] or [11], if necessary.

2. Preliminaries

We use the following results.

LEMMA 2.1. *Let* H *and* K *be subgroups of* G *and* H *is* $\mathfrak{U}$*-subnormal in* G.

(1) $H\cap K$ *is* $\mathfrak{U}$*-subnormal in* K [11, 6.1.7(2)].

(2) *If* N *is a normal subgroup in* G, *then* HN/N *is* $\mathfrak{U}$*-subnormal in* G/N [11, 6.1.6(3)].

(3) *If* K *is an* $\mathfrak{U}$*-subnormal subgroup of* H, *then* K *is* $\mathfrak{U}$*-subnormal in* G [11, 6.1.6(1)].

(4) *If* $G^{\mathfrak{U}}\leqq K$, *then* K *is* $\mathfrak{U}$*-subnormal in* G [11, 6.1.7(1)].

(5) *If* $K\leqq H$ *and* H *is supersoluble, then* K *is* $\mathfrak{U}$*-subnormal in* G.

LEMMA 2.2. *If every* n*-maximal subgroup of* G *is* $\mathfrak{U}$*-subnormal in* G, *then every* $(n-1)$*-maximal subgroup of* G *is supersoluble and every* $(n+1)$*-maximal subgroup of* G *is* $\mathfrak{U}$*-subnormal in* G.

PROOF. We first show that every $(n-1)$-maximal subgroup of G is supersoluble. Let H be an $(n-1)$-maximal subgroup of G and K any maximal subgroup of H. Then K is an n-maximal subgroup of G and so, by hypothesis, K is $\mathfrak{U}$-subnormal in G. Hence K is $\mathfrak{U}$-subnormal in H by Lemma 2.1(1), so $|H:K|$ is a prime. It follows that H is supersoluble.

Now, let E be an $(n+1)$-maximal subgroup of G, and let E_1 and E_2 be an n-maximal and an $(n-1)$-maximal subgroup of G, respectively, such that

$E \leqq E_1 \leqq E_2$. Then, by the above, E_2 is supersoluble, so E_1 is supersoluble. Hence E is $\mathfrak{U}$-subnormal in E_1. By hypothesis, E_1 is $\mathfrak{U}$-subnormal in G. Therefore E is $\mathfrak{U}$-subnormal in G by Lemma 2.1(3). □

In fact, the following lemma is a corollary of Lemma 2.1.

LEMMA 2.3. *Let $\mathfrak{F}$ be the class of groups all of whose 3-maximal subgroups are $\mathfrak{U}$-subnormal. The following hold*:
(i) *$\mathfrak{F}$ is closed with respect to quotient groups and subgroups*;
(ii) *All supersoluble groups belong to $\mathfrak{F}$.*

Fix some ordering ϕ of the set of all primes. The record $p\phi q$ means that p precedes q in ϕ and $p \neq q$. A group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is called *ϕ-dispersive* if $p_1 \phi p_2 \phi \cdots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$. Furthermore, if ϕ is such that $p\phi q$ always implies $p > q$, then a ϕ-dispersive group is called *Ore dispersive.*

LEMMA 2.4. *Let G be an $\mathfrak{U}$-critical group. Then the following hold*:
(1) *G is soluble and* $\left|\pi(G)\right| \leqq 3$ [1];
(2) *If G is not a Schmidt group, then G is Ore dispersive* [1];
(3) *$G^{\mathfrak{U}}$ is the unique normal Sylow subgroup of G* [1,9];
(4) *$G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})$ is a non-cyclic chief factor of G* [9].

LEMMA 2.5 (Theorem D in [7]). *Every 2-maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if G is either supersoluble or an SDH-group.*

3. Proof of Theorem 1.2

A maximal subgroup M of G is said to be *$\mathfrak{U}$-normal* in G if $G/M_G \in \mathfrak{U}$, otherwise it is said to be *$\mathfrak{U}$-abnormal* in G. Note that if G is soluble, then M is $\mathfrak{U}$-normal in G if and only if $|G : M|$ is a prime.

Recall that G is called a *Schmidt group* if G is not nilpotent but all proper subgroups of G are nilpotent.

PROOF OF THEOREM 1.2. By Burnside's $p^a q^b$-theorem, G is soluble.

Necessity. Let W be a maximal subgroup of G. In view of hypothesis and Lemma 2.1(1), every 2-maximal subgroup of W is $\mathfrak{U}$-subnormal in W. Therefore, by Lemma 2.5, W is either supersoluble or an SDH-group. In particular, all 2-maximal subgroups of G are supersoluble.

(I) Assume that G has no normal Sylow subgroups and $O^p(G) \neq G$. Then there exists a maximal subgroup M of G such that M is normal in G and $|G : M| = p$. Let $P_1 = M \cap P$. Then $M = P_1 Q$ and $|P : P_1| = p$.

(1) *P_1 is normal in G and $P_1 \nleqq \Phi(G)$.* Since M is either supersoluble or an SDH-group, one of the Sylow subgroups of M is normal in M. If Q is normal in M, then Q is a characteristic subgroup of M and so Q is normal in G, a contradiction. Consequently, P_1 is normal in M,

hence P_1 is normal in G. If $P_1 \leqq \Phi(G)$, then $M = \Phi(G)Q$. Since M is a normal subgroup of G, $G = MN_G(Q)$ by the Frattini Argument, so $G = \Phi(G)QN_G(Q) = \Phi(G)N_G(Q)$ and hence $G = N_G(Q)$, a contradiction. Hence we have (1).

Choose a maximal subgroup L of G such that $N_G(Q) \leqq L$. Then $L = P_2Q$, where P_2 is a Sylow p-subgroup of L and $P_1 \nleqq L$.

(2) *P_2 is not normal in L and Q is normal in L.* If P_2 is normal in L, then $P_1P_2 = P$ by [12, VI, 4.6], so $G = PQ \leqq N_G(P)$, a contradiction. Therefore P_2 is not normal in L. But since L is either supersoluble or an SDH-group, it has a normal Sylow subgroup. Therefore Q is normal in L.

(3) $p < q$. Assume that $p > q$. Then

(a) *$Q = L^{\mathfrak{U}}$ is a minimal normal subgroup of L. Hence P_2 is a maximal subgroup of L.* Indeed, since $p > q$ and a Sylow p-subgroup P_2 of L is not normal in L by Claim (2), L is not supersoluble and so L is an SDH-group. Hence $Q = L^{\mathfrak{U}}$ is a minimal normal subgroup of L by Claim (2).

(b) *P is a maximal subgroup of G.* Let V be a maximal subgroup of G such that $P \leqq V$. It is clear that $|G : L \cap V| = |G : L|\,|G : V|$, so for a Sylow p-subgroup P_0 of $L \cap V$ we have $|P_0| = |P_2|$. Hence $P_0 = (P_2)^x$ for some $x \in L$, so P_0 is a maximal subgroup of L by Claim (a). Therefore $P_0 = V \cap L$. Hence $P = V$.

(c) $|G : L| = p$. Indeed, if $|G : L| \geqq p^2$, then L is an $\mathfrak{U}$-abnormal subgroup of G. Since $|P : P_2| = |G : L| \geqq p^2$, P_2 is a t-maximal subgroup of P, where $t \geqq 2$. Therefore P_2 is at least a $(t+1)$-maximal subgroup of G in view of Claim (b). Hence P_2 is $\mathfrak{U}$-subnormal in G by hypothesis and Lemma 2.2. Thus there is a maximal subgroup H of G such that $P_2 \leqq H$ and $G/H_G \in \mathfrak{U}$. Then $G^{\mathfrak{U}} \leqq H$. Since $L = QP_2 = L^{\mathfrak{U}}P_2$ and $L^{\mathfrak{U}} \leqq G^{\mathfrak{U}}$, $L \leqq H$. But then $L = H$ is $\mathfrak{U}$-normal in G, a contradiction. Thus $|G : L| = p$.

(d) $P_1 \cap P_2 \neq 1$. Suppose that $P_1 \cap P_2 = 1$. Since $|G : L| = p$ by Claim (c), $G^{\mathfrak{U}} \leqq L$. Consequently, $G^{\mathfrak{U}} \leqq M \cap L = P_1Q \cap P_2Q = Q(P_1Q \cap P_2)$ $= Q(P_1 \cap P_2) = Q$. Since Q is a minimal normal subgroup of L by Claim (a), it follows that $G^{\mathfrak{U}} = Q$ and so Q is normal in G, a contradiction. Thus $P_1 \cap P_2 \neq 1$.

Final contradiction for (3). Since $p > q$ and L is not Ore dispersive, L is a Schmidt group in view of Lemma 2.4(2). Hence P_2 is a cyclic group by [12, IV, 5.4]. Clearly, $P_1 \cap P_2 = \Phi(P_2) = \Phi(L)$ and so $L \leqq N_G(P_1 \cap P_2)$. Since $|P : P_2| = |G : L| = p$, P_2 is a maximal subgroup of P. Hence P_2 is normal in P. But $P_1 \cap P_2 = \Phi(P_2)$ is a characteristic subgroup of P_2. Therefore $P_1 \cap P_2$ is normal in P and hence $G = PL \leqq N_G(P_1 \cap P_2)$. Consider the quotient group $G/P_1 \cap P_2$. Since $P_1 \cap P_2$ is cyclic, $G/P_1 \cap P_2$ is not supersoluble. Hence $(G/P_1 \cap P_2)^{\mathfrak{U}} \neq 1$. Arguing as in Claim (d), we can get that $(G/P_1 \cap P_2)^{\mathfrak{U}} = Q(P_1 \cap P_2)/P_1 \cap P_2 \simeq Q$. Therefore $Q(P_1 \cap P_2)$ is a normal subgroup of G. Hence $G = Q(P_1 \cap P_2)N_G(Q) = N_G(Q)$ by the Frattini Argument, since clearly $P_1 \cap P_2 \leqq \Phi(G)$. This final contradiction shows that $p < q$.

Claims (1) and (3) imply that M is not supersoluble. Hence M is an SDH-group. Therefore $P_1 = M^{\mathfrak{U}}$ is a minimal normal subgroup of M, so $P_1 \cap L = 1$. Hence $|P_2| = |G : M| = p$. Moreover, by Lemma 2.4(2), M is a Schmidt group. Hence Q is a cyclic group by [12, IV, 5.4] and $[Q^q, G^{\mathfrak{U}}] = 1$. Therefore $L = QP_2$ is supersoluble and so $G^{\mathfrak{U}} = P_1$.

Finally, since L is supersoluble, $L/O_{q',q}(L)$ is an abelian group of exponent dividing $q-1$ by [13, Ch. 1, 1.4] and [13, Appendixes, 3.2]. But $O_{q',q}(L) = O_{q',q}(QP_2) = QC_{P_2}(Q) = Q$. Hence $L/O_{q',q}(L) = QP_2/Q \simeq P_2$ and so p divides $q-1$.

(II) Now suppose that P is normal in G. Then Q is not normal in G, since G is not supersoluble by hypothesis.

(i) Let $V < E < Q$, where E is a maximal subgroup of Q and V is a maximal subgroup of E. Then PE is a maximal subgroup of G and PV is a maximal subgroup of PE. Hence PV is supersoluble.

Assume that P is not a minimal normal subgroup of PE. Then PE is not an SDH-group, so PE is supersoluble. Hence $PE/O_{p',p}(PE)$ is an abelian group of exponent dividing $p-1$ by [13, Ch. 1, 1.4] and [13, Appendixes, 3.2]. Moreover, $O_{p',p}(PE) = PC_E(P)$ and hence $PE/O_{p',p}(PE) \simeq E/C_E(P)$. Thus E induces an abelian group of automorphisms of exponent dividing $p-1$ on P.

(ii) Assume that P is a minimal normal subgroup of G. Suppose that q does not divide $p-1$. First we show that in this case Q is cyclic. If G has a supersoluble nonnilpotent subgroup K, then $K = PQ_1$, where $Q_1 \neq 1$ is a subgroup of Q. Since PQ_1 is not nilpotent, $O_{p'}(PQ_1) < Q_1$, so $O_{p',p}(PQ_1) < PQ_1$. But PQ_1 is supersoluble and so $PQ_1/O_{p',p}(PQ_1) \simeq Q_1/C_{Q_1}(P)$ is a non-identity abelian group of exponent dividing $p-1$ by [13, Ch. 1, 1.4] and [13, Appendixes, 3.2]. Therefore we may assume that every supersoluble subgroup of G is nilpotent.

By the above, every 2-maximal subgroup of G is supersoluble and so is nilpotent. Therefore every maximal subgroup of G is either nilpotent or a Schmidt group. If all maximal subgroups of G are nilpotent, then Q is cyclic by [12, IV, 5.4] and $Z(G)$ is a subgroup of Q such that $\left|Q : Z(G)\right| = q$. Assume that there is a maximal subgroup M of G such that M is a Schmidt group. Then $M = PV$, where V is a maximal subgroup of Q. As above, we get that V is cyclic and so $\Phi(V)$ is a maximal subgroup of V. Since $\Phi(V)$ is characteristic in V and V is normal in Q, $\Phi(V) \leqq \Phi(Q)$ is a normal subgroup of Q. Moreover, it is clear that $C_Q(P) = \Phi(V)$. Since $\left|Q/C_Q(P)\right| = q^2$, $Q/C_Q(P)$ is an abelian group. Therefore $Q/C_Q(P)$ is cyclic by [12, II, 3.10]. Hence $Q/\Phi(Q) \simeq \big(Q/\Phi(V)\big)/\big(\Phi(Q)/\Phi(V)\big)$ is cyclic and so Q is cyclic. Furthermore, $Z(G) = C_Q(P)$ and so $\left|Q : Z(G)\right| = q^2$.

Finally, we show that in the case when G is not $\mathfrak{U}$-critical, q^2 divides $p^{q-1}-1$. By the above, Q is cyclic and $\left|Q : Z(G)\right| = q^2$. Hence $P\Phi(Q)$ is a Schmidt group. Therefore, in view of [12, II, 3.10], q divides $p^n - 1 = |P| - 1$, where n is the least number with such property. Moreover, since $\left|Q/C_Q(P)\right| = q^2$, q^2 divides $p^n - 1$ by [12, II, 3.10]. Note also that in view of the Euler Theorem, q divides $p^{q-1} - 1$. It follows that n divides $q - 1$. But then q^2 divides $p^{q-1} - 1$.

(iii) Assume that $\Phi(P) \neq 1$. Since $\Phi(P)$ is characteristic in P, this subgroup is normal in G and so in this case every maximal subgroup of G containing P is supersoluble.

Now we show that $P/\Phi(P)$ is a non-cyclic chief factor of G. If all maximal subgroups of G are supersoluble, this directly follows from Lemma 2.4(4). Otherwise, take a maximal nonsupersoluble subgroup V of G. Then $P \nleqq V$ and V is an SDH-group. Let V_p be a Sylow p-subgroup of V. Then $1 \neq \Phi(P) \leqq V_p = P \cap V$ is normal in V, so $V_p = V^{\mathfrak{U}} = \Phi(P)$ is a minimal normal subgroup of V. Thus $P/\Phi(P)$ is a non-cyclic chief factor of G. Hence $P = G^{\mathfrak{U}}$.

Suppose that G is an $\mathfrak{U}$-critical group and $\left|\Phi(P)\right| \geqq p^2$. Let M be a maximal subgroup of G such that $P \nleqq M$. Then $G = PM$ and $M = (P \cap M)Q = \Phi(P)Q$ since $P/\Phi(P)$ is a chief factor of G. Since M is supersoluble, there is a 2-maximal subgroup E of M such that $|M : E| = p^2$. Hence $M = \Phi(P)E$ and so $G = PE$. Since E is $\mathfrak{U}$-subnormal in G, there exists a maximal subgroup H of G such that $E \leqq H$ and $G/H_G \in \mathfrak{U}$. Therefore $P \leqq H$, hence $G = PE \leqq H$, a contradiction. Thus $\left|\Phi(P)\right| = p$.

Finally, suppose that G is not an $\mathfrak{U}$-critical group. Then, since every maximal subgroup of G containing P is supersoluble, there is a nonsupersoluble maximal subgroup M such that $PM = G$. Then M is an SDH-group and $M = \Phi(P)Q^x$ for some $x \in G$. Thus $\Phi(P)$ is a minimal normal subgroup of G. Hence we have (iii).

(iv) Suppose that P is not a minimal normal subgroup of G and $\Phi(P) = 1$. By Maschke's Theorem, $P = P_1 \times P_2$, where P_1 is a minimal normal subgroup of G and P_2 is a normal subgroup of G. Then $L = P_2Q$ is a maximal subgroup of G. We show that P_2 is also a minimal normal subgroup of G. If L is an SDH-group, then $P_2 = L^{\mathfrak{U}}$ is a minimal normal subgroup of L, so P_2 is a minimal normal subgroup of G. Assume that L is supersoluble. Then $G/P_1 \simeq L$ is a supersoluble group. If P_1Q is supersoluble, then $G/P_2 \simeq P_1Q$ is supersoluble and hence G is supersoluble, a contradiction. Thus P_1Q is not a supersoluble group. But every 2-maximal subgroup of G is supersoluble. Hence P_1Q is a maximal subgroup of G, so P_2 is a minimal normal subgroup of G.

Since G is not supersoluble, at least one of the subgroups $L = P_2Q$ or $T = P_1Q$ is not supersoluble. Let T be an SDH-group. Then $T^{\mathfrak{U}} = P_1$, so P_1 is not cyclic.

Sufficiency. Let E be any 3-maximal non-identity subgroup of G and M a maximal subgroup of G such that E is a 2-maximal subgroup of M. In order to prove that E is $\mathfrak{U}$-subnormal in G, in view of Lemmas 2.1(3) and 2.5, it is enough to find in G an $\mathfrak{U}$-normal maximal subgroup L such that $E \leqq L$ and L is either supersoluble or an SDH-group.

First assume that G has no normal Sylow subgroups and that $O^p(G) \neq G$. Then in view of Assertion (I), any maximal subgroup of G is a conjugate of one of the subgroups $M_1 = Q \rtimes P_1$, $M_2 = G^{\mathfrak{U}}Q$, or $M_3 = Q^q \rtimes P$, where P_1 is a subgroup of P such that $P = G^{\mathfrak{U}} \rtimes P_1$. It is clear that the subgroup M_3 is supersoluble and $\mathfrak{U}$-normal in G. Hence every its subgroup is $\mathfrak{U}$-subnormal in G. Assume that $M = M_2$. Then M is $\mathfrak{U}$-normal in G and it is an SDH-group, so we get that E is $\mathfrak{U}$-subnormal in G. Finally, let $M = M_1$. Then M is supersoluble. Since M_2 is $\mathfrak{U}$-normal in G, $|P_1| = |P : G^{\mathfrak{U}}| = p$. Therefore E is contained in a conjugate of M_3 and so is $\mathfrak{U}$-subnormal in G.

Now, assume that P is normal in G. First suppose that $P \leqq M$. Then $M = P \rtimes V$, where $V = M \cap Q$ is a maximal subgroup of Q. Hence V induces a group of automorphisms on P which is either irreducible or an abelian of exponent dividing $p - 1$ by Assertion (II)(i). If $V/C_V(P)$ is an abelian group of exponent dividing $p - 1$, then M is supersoluble by [13, Ch. 1, 1.4] and so E is $\mathfrak{U}$-subnormal in G since M is $\mathfrak{U}$-normal in G. If $V/C_V(P)$ is an irreducible automorphism group of P, then V is a maximal subgroup of PV and so in view of Assertion (II)(i), PV is an SDH-group. Now as above one can show that E is $\mathfrak{U}$-subnormal in G.

Now suppose that $P \nleqq M$. Without loss of generality we can assume that $Q \leqq M$ and that a Sylow q-subgroup E_q of E is contained in Q. If $\Phi(P) \neq 1$, then Q is a maximal subgroup of $M = \Phi(P)Q$ by Assertion (II)(iii). Hence q divides $|M : E|$, so for some maximal subgroup V of Q we have $E \leqq PV$. Since P is not a minimal normal subgroup of PV, PV is supersoluble in view of Assertion II(i). Hence E is $\mathfrak{U}$-subnormal in G. Now assume that $\Phi(P) = 1$. If P is a minimal normal subgroup of G, then $M = Q$ and so E is $\mathfrak{U}$-subnormal in G in view of Assertion (II)(i). Finally, consider the case when $P = P_1 \times P_2$, where P_1 and P_2 are minimal normal subgroups of G and at least one of these subgroups is non-cyclic. Without loss of generality we can assume that $M = P_1Q$. It is clear that P_1 is a minimal normal subgroup of M, so Q is a maximal subgroup of M. Hence q divides $|M : E|$. But then, as above, we can get that $E \leqq L$, where $|G : L| = q$, so E is $\mathfrak{U}$-subnormal in G. □

Finally, note that the following example shows that in the case when P is not a minimal normal subgroup of G and $P = P_1 \times P_2$, where P_1 and P_2 are minimal normal subgroups of G, there is a case when P_1 and P_2 are non-cyclic.

EXAMPLE 3.1. Let p and q be primes, where q divides $p - 1$. Let Q be a non-abelian group such that $|Q| = q^3$ and $\exp(Q) = q$. Let P be a faithful irreducible $\mathbb{F}_p Q$-module and $A = P \rtimes Q$. It is easy to see that A is an SDH-group. Let $A_1 = P_1 \rtimes Q_1$ and $A_2 = P_2 \rtimes Q_2$ be isomorphic copies of the group A. Let μ_i be an epimorphism of A_i on Q, for $i = 1, 2$. Finally, let $G = A_1 \curlywedge A_2 = \{ (a_1, a_2) \mid a_i \in A_i,\ a_1^{\mu_1} = a_2^{\mu_2} \}$ (see [12, I, 9.11]). Then there are an epimorphism $\alpha_1 : G \to A_1$ such that $\operatorname{Ker} \alpha_1 = \{ (1, n_2) \mid n_2 \in P_2 \} \simeq P_2$ and epimorphism $\alpha_2 : G \to A_2$ such that $\operatorname{Ker} \alpha_2 = \{ (n_1, 1) \mid n_1 \in P_1 \} \simeq P_1$. Moreover, there is an epimorphism $\beta : G \to Q$ such that $\operatorname{Ker} \beta = P_1 \times P_2$.

It is easy to see that for the group G we have $P = P_1 \times P_2 = G^{\mathfrak{U}}$ and all minimal normal subgroups of G are of the nonprime orders in G. Moreover, every 3-maximal subgroup of G is $\mathfrak{U}$-subnormal in G.

REMARK 3.2. In fact, Theorem 1.2 is an improved and revised version of Theorem B in [8]. Moreover, the referee called our attention to the fact that some of the assertions in Theorem B are incorrect.

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