# КОНЕЧНЫЕ ГРУППЫ, ВСЕ $n$-МАКСИМАЛЬНЫЕ $(n=2,3)$ ПОДГРУППЫ КОТОРЫХ $K$ - $\mathfrak{U}$-СУБНОРМАЛЬНЫ 

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# FINITE GROUPS WITH ALL $n$-MAXIMAL ( $n=2,3$ ) SUBGROUPS $K$ - $\mathfrak{U}$-SUBNORMAL 

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Приведена полная классификация конечных групп, все $n$-максимальные $(n=2,3)$ подгруппы которых являются $K$ - $\mathfrak{U}$-суб-
нормальными.

Ключевые слова: п-максимальная подгруппа, К- $\mathfrak{U}$-субнормальная подгруппа, $\mathfrak{U}$-субнормальная подгруппа, сверхразрешимая группа, минимальная несверхразрешимая группа, SDH-группа.

A full classification of finite groups with all $n$-maximal $(n=2,3)$ subgroups $K$ - $\mathfrak{U}$-subnormal is given.
Keywords: n-maximal subgroup, $K$ - $\mathfrak{U}$-subnormal subgroup, $\mathfrak{U}$-subnormal subgroup, supersoluble group, minimal nonsupersoluble group, SDH-group.

## Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use $\mathfrak{U}$ to denote the class of all supersoluble groups; $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups $N$ of $G$ with $G / N \in \mathfrak{U}$. The symbol $\pi(G)$ denotes the set of prime divisors of the order of $G$.

A subgroup $H$ of $G$ is called a 2-maximal (second maximal) subgroup of $G$ whenever $H$ is a maximal subgroup of some maximal subgroup $M$ of $G$. Similarly we can define 3-maximal subgroups, and so on. If $H$ is $n$-maximal in $G$ but not $n$-maximal in any proper subgroup of $G$, then $H$ is said to be a strictly $n$-maximal subgroup of $G$.

One of the interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its $n$-maximal subgroups. The earliest publications in this direction are the articles of L. Rédei [1] and B. Huppert [2]. L. Rédei described the nonsoluble groups with abelian two maximal subgroups. B. Huppert established the supersolubility of $G$ whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of $G$ are normal in $G$, then the commutator subgroup $G^{\prime}$ of $G$ is nilpotent and the chief rank of $G$ is at most 2 . These results were developed by many authors. In partiqular, L.Ja. Poljakov [3] proved that $G$ is supersoluble if every 2-maximal subgroup of $G$ is permutable with every maximal
subgroup of $G$. He also established the solubility of $G$ in the case when every maximal subgroup of $G$ permutes with every 3-maximal subgroups of $G$. Some later, R.K. Agrawal [4] proved that $G$ is supersoluble if any 2-maximal subgroup of $G$ is permutable with every Sylow subgroup of $G$. In [5], Z. Janko described the groups whose 4-maximal subgroups are normal. A description of nonsoluble groups with all 2-maximal subgroups nilpotent was obtained by M. Suzuki [6] and Z. Janko [7]. In [8], T.M. Gagen and Z. Janko gave a description of simple groups whose 3 -maximal subgroups are nilpotent. V.A. Belonogov [9] studied those groups in which every 2 -maximal subgroup is nilpotent. Continuing this, V.N. Semenchuk [10] obtained a description of soluble groups whose all 2-maximal subgroups are supersoluble. A. Mann [11] studied the structure of the groups whose $n$-maximal subgroups are subnormal. He proved that if all $n$-maximal subgroups of a soluble group $G$ are subnormal and $|\pi(G)| \geq n+1$, then $G$ is nilpotent; but if $|\pi(G)| \geq n-1$, then $G$ is $\varphi$-dispersive for some ordering $\varphi$ of the set of all primes. Finally, in the case $|\pi(G)|=n$, Mann described $G$ completely. A.E. Spencer [12] studied groups in which every $n$-maximal chain contains subnormal subgroup. In partiqular, Spencer proved that $G$ is a Schmidt group with abelian Sylow subgroups if every 2-maximal chain of $G$ contains subnormal subgroup. In [13], M. Asaad studied groups whose strictly
n-maximal subgroups are normal. P. Flavell [14] obtained an upper bound for the number of maximal subgroups containing a strictly 2 -maximal subgroup and classify the extremal examples.

Among the recent results on $n$-maximal subgroups we can mention the paper of X.Y. Guo and K.P. Shum [15]. In this paper the authors proved that $G$ is soluble if all its 2 -maximal subgroups enjoy the cover-avoidance property. W. Guo, K.P. Shum, A.N. Skiba and Li Baojun [16,17,18] gave new characterizations of supersoluble groups in terms of 2-maximal subgroups. Li Shirong [19] obtained a classification of nonnilpotent groups whose all 2-maximal subgroups are $T I$-subgroups. In the paper [20], W. Guo, H.V. Legchekova and A.N. Skiba described the groups whose every 3 -maximal subgroup permutes with all maximal subgroups. In [21], W. Guo, Yu.V. Lutsenko and A.N. Skiba gave a description of nonnilpotent groups in which every two 3-maximal subgroups are permutable. Yu.V. Lutsenko and A.N. Skiba [22] obtained a description of the groups whose all 3-maximal subgroups are $S$-quasinormal. Subsequently, this result was strengthened by Yu.V. Lutsenko and A.N. Skiba in [23] to provide a description of the groups whose all 3-maximal subgroups are subnormal. Developing some of the above-mention results, D.P. Andreeva and A.N. Skiba [24] obtained a description of the groups in which every 3-maximal chain contains a proper $S$-qausinormal subgroup. Moreover, in [25], W. Guo, D.P. Andreeva and A.N. Skiba obtained the description of the groups in which every 3-maximal chain contains a proper subnormal subgroup. In [26], A. Ballester-Bolinches, L.M. Ezquerro and A.N. Skiba obtained a full classification of the groups in which the second maximal subgroups of the Sylow subgroups cover or avoid the chief factors of some of its chief series. In [27], V.N. Kniahina and V.S. Monakhov studied those groups $G$ in which every $n$-maximal subgroup permutes with each Schmidt subgroup. In partiqular, it was be proved that if $n=1,2,3$, then $G$ is metanilpotent; but if $n \geq 4$ and $G$ is soluble, then the nilpotent length of $G$ is at most $n-1$.

Another interesting results on $n$-maximal subgroups were obtained by V.A. Kovaleva and A.N. Skiba in [28], [29] and V.S. Monakhov and V.N. Kniahina in [30]. In [28], the authors described the groups whose all $n$-maximal subgroups are $\mathfrak{U}$-subnormal. In [29] a description of the groups with all $n$-maximal subgroups $\mathfrak{F}$-subnormal for some saturated formation $\mathfrak{F}$ was obtained. In [30], the groups with all 2-maximal subgroups $\mathbb{P}$-subnormal were studied.

Recall that a subgroup $H$ of $G$ is said to be:
(i) $\mathfrak{U}$-subnormal in $G$ if there exists a chain of subgroups

$$
H=H_{0} \leq H_{1} \leq \cdots \leq H_{n}=G
$$

such that $H_{i} /\left(H_{i-1}\right)_{H_{i}} \in \mathfrak{U}$, for $i=1, \ldots, n$;
(ii) $\mathfrak{U}$-subnormal (in the sense of Kegel [31]) or $K$ - $\mathfrak{U}$-subnormal [32, p. 236] in $G$ if there exists a chain of subgroups

$$
H=H_{0} \leq H_{1} \leq \cdots \leq H_{t}=G
$$

such that either $H_{i-1}$ is normal in $H_{i}$ or $H_{i} /\left(H_{i-1}\right)_{H_{i}} \in \mathfrak{U}$ for all $i=1, \ldots, t$. It is evident that every subnormal subgroup is $K-\mathfrak{U}$-subnormal. The inverse, in general, it is not true. For example, in the group $S_{3}$ a subgroup of order 2 is $K-\mathfrak{U}$-subnormal and at the same time it is not subnormal. This elementary observation and the results in [23], [25] make natural the following question:
I. What is the structure of $G$ under the condition that every 2-maximal subgroup of $G$ is K- $\mathfrak{U}$-subnormal?
II. What is the structure of $G$ under the condition that every 3-maximal subgroup of $G$ is K- $\mathfrak{U}$-subnormal?

In this paper we given the solutions of these two questions.

## 1 Preliminary results

The solutions of Question I and Question II are based on the following results.

Lemma 1.1. Let $H$ and $K$ be subgroups of $G$ such that $H$ is $K-\mathfrak{U}$-subnormal in $G$.
(1) $H \cap K$ is $K-\mathfrak{U}$-subnormal in $K$ [32, Lemma 6.1.7 (2)].
(2) If $N$ is a normal subgroup in $G$, then $H N / N$ is $K$ - $\mathfrak{U}$-subnormal in $G / N$ [32, Lemma 6.1.6 (3)].
(3) If $K$ is $K-\mathfrak{U}$-subnormal in $H$, then $K$ is $K-\mathfrak{U}$-subnormal in $G$ [32, Lemma 6.1.6(1)].
(4) If $G^{\mathfrak{U}} \leq K$, then $K$ is $K-\mathfrak{U}$-subnormal in $G$ [32, Lemma 6.1.7 (1)].

The next lemma is evident.
Lemma 1.2. If $G$ is supersoluble, then every subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$.

Lemma 1.3. If every n-maximal subgroup of $G$ is $K$ - $\mathfrak{U}$-subnormal in $G$, then every $(n-1)$-maximal subgroup of $G$ is supersoluble and every $(n+1)$-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$.

Proof. We first show that every ( $n-1$ ) -maximal subgroup of $G$ is supersoluble. Let $H$ be an ( $n-1$ )-maximal subgroup of $G$ and $K$ any maximal subgroup of $H$. Then $K$ is an $n$-maximal subgroup of $G$ and so, by hypothesis, $K$ is $K-\mathfrak{U}$-subnormal in $G$. Hence $K$ is $K-\mathfrak{U}$-subnormal in $H$ by Lemma 1.1 (1). Therefore either $K$ is normal in $H$ or $H / K_{H} \in \mathfrak{U}$. If $K$ is normal in $H$, then $|H: K|$ is a prime in view of maximality of $K$ in $H$. Let $H / K_{H} \in \mathfrak{U}$. Then we also get that

$$
|H: K|=\left|H / K_{H}: K / K_{H}\right|
$$

is a prime. Thus $H$ is supersoluble.
Now, let $E$ be an $(n+1)$-maximal subgroup of $G$, and let $E_{1}$ and $E_{2}$ be an $n$-maximal and an ( $n-1$ ) -maximal subgroup of $G$, respectively, such that $E \leq E_{1} \leq E_{2}$.

Then, by the above, $E_{2}$ is supersoluble, so $E_{1}$ is supersoluble. Hence it is easy to see that $E$ is $K-\mathfrak{U}$-subnormal in $E_{1}$. By hypothesis, $E_{1}$ is $K-\mathfrak{U}$-subnormal in $G$. Therefore $E$ is $K-\mathfrak{U}$-subnormal in $G$ by Lemma 1.1 (3). The lemma is proved.

Lemma 1.4. If every 3-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$, then $G$ is soluble.

Proof. Suppose that lemma is false and let $G$ be a counterexample with $|G|$ minimal. Since every 3-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$, every 2-maximal subgroup of $G$ is supersoluble by Lemma 1.3. Hence every maximal subgroup of $G$ is either supersoluble or a minimal nonsupersoluble group. Therefore all proper subgroups of $G$ are soluble in view of [2]. Assume that all 3-maximal subgroups of $G$ are identity. Then all 2 -maximal subgroups of $G$ have prime orderes and so every maximal subgroup of $G$ is supersoluble. Hence $G$ is either supersoluble or a minimal nonsupersoluble group. Thus in view of [2], $G$ is soluble, a contradiction. Hence there is a 3 -maximal subgroup $T$ of $G$ such that $T \neq 1$. Since $T$ is $K-\mathfrak{U}$-subnormal in $G$, there exists a proper subgroup $H$ of $G$ such that $T \leq H$ and either $G / H_{G} \in \mathfrak{U}$ or $H$ is normal in $G$. If $G / H_{G} \in \mathfrak{U}$, then $G$ is soluble in view of solubility of $H_{G}$, a contradiction. Therefore $H$ is normal in $G$. Let $E / H$ be any 3-maximal subgroup of $G / H$. Then $E$ is a 3 -maximal subgroup of $G$, hence $E$ is $K$ - $\mathfrak{U}$-subnormal in $G$. Hence $E / H$ is $K-\mathfrak{U}$-subnormal in $G / H$ by Lemma 1.1 (2). Thus the hypothesis holds for $G / H$. Hence $G / H$ is soluble by the choice of $G$. Therefore $G$ is soluble. This contradiction completes the proof of the lemma.

2 Description of groups with all 2-maximal or all 3-maximal subgroups $K$ - $\mathfrak{U}$-subnormal

Recall that $G$ is called a minimal nonsupersoluble group provided $G$ does not belong to $\mathfrak{U}$ but every proper subgroup of $G$ belongs to $\mathfrak{U}$. Such groups were described by B. Huppert [2] and K. Doerk [33]. We say that $G$ is a special Doerk-Huppert group or an SDH-group if $G$ is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of $G$.

The solution of Question I originates to [28], [29], where, in particular, the following theorem was proved.

Theorem $\mathbf{A}^{*}$. Every 2-maximal subgroup of $G$ is $\mathfrak{U}$-subnormal in $G$ if and only if $G$ is either supersoluble or an SDH-group.

If every 2-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal, then every maximal subgroup of $G$ is supersoluble by Lemma 1.3. Therefore in this case $G$ is either supersoluble or a minimal nonsupersoluble group, hence $G$ is soluble by [2]. Thus we get the following

Theorem A. Every 2-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$ if and only if $G$ is either supersoluble or an SDH-group.

The solution of Question II is more complete. Note that since each subgroup of every supersoluble group is $K-\mathfrak{U}$-subnormal, we need, in fact, only consider the case when $G$ is not supersoluble. But in this case, in view of [28] or [29], $|\pi(G)| \leq 4$.

The following theorems are proved.
Theorem B. Let $G$ be a nonsupersoluble group with $|\pi(G)|=2$. Let $p, q$ be distinct prime divisors of $|G|$ and $G_{p}, G_{q}$ be Sylow p-subgroup and $q$ subgroup of $G$ respectively. Every 3-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$ if and only if $G$ is a soluble group of one of the following types:
I. $G$ is a minimal nonsupersoluble group and either $\left|\Phi\left(G^{\mathfrak{U}}\right)\right|$ is a prime or $\Phi\left(G^{\mathfrak{U}}\right)=1$.
II. $G=G_{p} \rtimes G_{q}$, where $G_{p}$ is the unique minimal normal subgroup of $G$ and every 2-maximal subgroup of $G_{q}$ is an Abelian group of exponent dividing $p-1$. Moreover, every maximal subgroup of $G$ containing $G_{p}$ is either supersoluble or an SDH-group and at least one of the maximal subgroup of $G$ is not supersoluble.
III. $G=\left(G_{p} \times Q_{1}\right) \rtimes Q_{2}$, where $G_{q}=Q_{1} \rtimes Q_{2}$, $G_{p}$ and $Q_{1}$ are minimal normal subgroups of $G$, $\left|Q_{1}\right|=q, \quad G_{p} \rtimes Q_{2}$ is an SDH-group and every maximal subgroup of $G$ containing $G_{p} \rtimes Q_{1}$ is supersoluble. Moreover, if $p<q$, then every 2-maximal subgroup of $G$ is nilpotent.
IV. $G=G_{p} \rtimes G_{q}$, where $G_{p}$ is a minimal normal subgroup of $G, O_{q}(G) \neq 1, \quad \Phi(G) \neq 1$, every maximal subgroup of $G$ containing $G_{p}$ is either supersoluble or an SDH-group and $G / \Phi(G)$ is a group one of types II or III.
V. $G=\left(P_{1} \times P_{2}\right) \rtimes G_{q}$, where $G_{p}=P_{1} \times P_{2}, \quad P_{1}$, $P_{2}$ are minimal normal subgroups of $G$, every maximal subgroup of $G$ containing $G_{p}$ is supersoluble, $P_{1} \rtimes G_{q}$ is an SDH-group and $P_{2} \rtimes G_{q}$ is either an SDH-group or a supersoluble group with $\left|P_{2}\right|=p$.
VI. $G=G_{p} \rtimes G_{q}, \Phi\left(G_{p}\right)$ is a minimal normal subgroup of $G$, every maximal subgroup of $G$ containing $G_{p}$ is supersoluble and $\Phi\left(G_{p}\right) \rtimes G_{q}$ is an SDH-group.
VII. Every of the subgroups $G_{p}$ and $G_{q}$ is not normal in $G$ and the following hold:
(i) if $p<q$, then $G=P_{1} \rtimes\left(G_{q} \rtimes P_{2}\right)$, where $G_{p}=P_{1} \rtimes P_{2}, \quad P_{1}$ is a minimal normal subgroup of $G,\left|P_{2}\right|=p, G_{q}=\langle a\rangle$ is a cyclic group and $\left\langle a^{q}\right\rangle$ is normal in $G$. Moreover, $G$ has precisely three classes of maximal subgroups whose representatives are $P_{1} \rtimes G_{q}, \quad G_{q} \rtimes P_{2},\left\langle a^{q}\right\rangle \rtimes G_{p}$, where $P_{1} \rtimes G_{q}$ is an SDH-group;
(ii) if $p>q$, then $G=P_{1}\left(G_{q} \rtimes P_{2}\right)$, where $G_{p}=P_{1} P_{2}, P_{1}$ is a normal subgroup of $G, P_{2}=\langle b\rangle$ is a cyclic group and $1 \neq P_{1} \cap P_{2}=\left\langle b^{p}\right\rangle$. Moreover, $G$ has precisely three classes of maximal subgroups whose representatives are $P_{1} \rtimes G_{q}, \quad G_{q} \rtimes P_{2}, \quad G_{p}$, where $\left|G: G_{q} \rtimes P_{2}\right|=p, \quad P_{1} \rtimes G_{q}$ is a supersoluble group and $G_{q} \rtimes P_{2}$ is an SDH-group.

Theorem C. Let $G$ be a nonsupersoluble group with $|\pi(G)|=3$. Let $p, q, r$ be distinct prime divisors of $|G|$ and $G_{p}, G_{q}, G_{r}$ be Sylow $p$-subgroup, $q$-subgroup and $r$-subgroup of $G$ respectively. Every 3-maximal subgroup of $G$ is $K-\mathfrak{U}$-subnormal in $G$ if and only if $G$ is a soluble group of one of the following types:
I. $G$ is a minimal nonsupersoluble group and either $\left|\Phi\left(G^{\mathfrak{L}}\right)\right|$ is a prime or $\Phi\left(G^{\mathfrak{U}}\right)=1$.
II. $G=G_{p} \rtimes\left(G_{q} \rtimes G_{r}\right)$, where $G_{p}$ is a minimal normal subgroup of $G$, every maximal subgroup of $G$ is either supersoluble or an SDH-group and at least one of the maximal subgroups of $G$ is not supersoluble. Moreover, the following hold:
(i) if $G_{p}$ is the unique minimal normal subgroup of $G$, then every 2-maximal subgroup of $G_{q} \rtimes G_{r}$ is an Abelian group of exponent dividing $p-1$;
(ii) if $G_{q} \rtimes G_{r}$ is an SDH-group, then every maximal subgroup of $G$ containing $G_{p} G_{q}$ is supersoluble and $G_{p} \rtimes G_{r}$ is either an SDH-group or a supersoluble group with $\left|G_{p}\right|=p$.
III. $G=\left(P_{1} \times P_{2}\right) \rtimes\left(G_{q} \rtimes G_{r}\right)$, where $G_{p}=P_{1} \times P_{2}$, $P_{1}, P_{2}$ are minimal normal subgroups of $G$ and $G_{q}, G_{r}$ are cyclic groups. Moreover, every maximal subgroup of $G$ containing $G_{p}$ is supersoluble, $P_{1} \rtimes\left(G_{q} \rtimes G_{r}\right)$ is an SDH-group and $P_{2} \rtimes\left(G_{q} \rtimes G_{r}\right)$
is either an SDH-group or a supersoluble group with $\left|P_{2}\right|=p$.
IV. $G=G_{p} \rtimes\left(G_{q} \rtimes G_{r}\right), \quad \Phi\left(G_{p}\right)$ is a minimal normal subgroup of $G$, every maximal subgroup of $G$ containing $G_{p}$ is supersoluble and $\Phi\left(G_{p}\right) \rtimes\left(G_{q} \rtimes G_{r}\right)$ is an SDH-group.

Theorem D. Let $G$ be a nonsupersoluble group with $|\pi(G)|=4$. Let $p, q, r, t$ be distinct prime divisors of $|G|(p>q>r>t)$ and $G_{p}, G_{q}$, $G_{r}, G_{t}$ be Sylow p-subgroup, $q$-subgroup, $r$-subgroup and $t$-subgroup of $G$ respectively. Every 3-maximal subgroup of $G$ is $K$ - $\mathfrak{U}$-subnormal in $G$ if and only if $G=G_{p} \rtimes\left(G_{q} \rtimes\left(G_{r} \rtimes G_{t}\right)\right)$ is a soluble group such that $G$ has precisely three classes of maximal subgroups whose representatives are $G_{q} G_{r} G_{t}, \quad G_{p} G_{q} G_{r} \Phi\left(G_{t}\right), \quad G_{p} G_{q} \Phi\left(G_{r}\right) G_{t} \quad$ and $G_{p} \Phi\left(G_{q}\right) G_{r} G_{t}$, and every nonsupersoluble maximal subgroup of $G$ is an SDH-group, $G_{r}$ and $G_{t}$ are cyclic groups and following hold:
(1) if $G_{q} G_{r} G_{t}$ is an SDH-group, then $G^{\mathfrak{L}}=G_{p} \times G_{q}$, $G_{q}$ is a minimal normal subgroup of $G$, the subgroups $G_{p} G_{q} G_{r} \Phi\left(G_{t}\right)$ and $G_{p} G_{q} \Phi\left(G_{r}\right) G_{t}$ are supersoluble and $G_{p} G_{r} G_{t}$ is either an SDH-group or a supersoluble group with $\left|G_{p}\right|=p$;
(2) if $G_{q} G_{r} G_{t}$ is a soluble group, then $G_{q}$ is cyclic.

The classes of groups which are described in Theorems B and C are pairwise disjoint. It is easy to construct examples to show that all classes of the groups in this theorems and in Theorems A and D are not empty. Note also that Theorems B, C and D show that the class of the groups with all 3-maximal subgroups $K-\mathfrak{U}$-subnormal is essentially wider then the class of the groups with all 3-maximal subgroups subnormal [23].

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