УДК 512.542

КОНЕЧНЫЕ ГРУППЫ С ЗАДАННЫМИ СИСТЕМАМИ КВАЗИПЕРЕСТАНОВОЧНЫХ ПОДГРУПП

В.А. Ковалева, Чжао Сяоюй

Гомельский государственный университет им. Ф. Скорины, Гомель

FINITE GROUPS WITH GIVEN SYSTEMS OF QUASIPERMUTABLE SUBGROUPS

V.A. Kovalyova, Zhao Xiaoyu

F. Scorina Gomel State University, Gomel

Пусть G – конечная группа. Подгруппа A группы G называется квазиперестановочной в G, если A либо покрывает, либо изолирует каждую максимальную пару (K,H) из G. Изучаются конечные группы с заданными системами квазиперестановочных подгрупп.

Ключевые слова: *конечная группа, максимальная пара, (слабо) квазиперестановочная подгруппа, обобщенная подгруппа Фиттинга, p-нильпотентная группа, $\mathcal{U}$-гиперцентр.*

Let G be a finite group. A subgroup A of G is said to be quasipermutable in G if A either covers or avoids every maximal pair (K,H) of G. We study the finite groups with given systems of quasipermutable subgroups.

Keywords: *finite group, maximal pair, (weakly) quasipermutable subgroup, generalized Fitting subgroup, p-nilpotent group, $\mathcal{U}$-hypercentre.*

Introduction

Throughout this paper, all groups considered are finite. We write $\mathcal{U}$ to denote the class of all supersoluble groups.

Let A be a subgroup of a group G, $K \le H \le G$. Then we say that A *covers the pair* (K,H) if $AH = AK$; A *avoids* (K,H) if $A \cap H = A \cap K$. A subgroup H of G is said to be quasinormal or permutable in G if $HE = EH$ for all subgroups E of G. The permutable subgroups have many interesting properties. In particular, if E is a permutable subgroup of G, then for every *maximal* pair of G, that is, a pair (K,H), where K is a maximal subgroup of H, E either covers or avoids (K,H). This observation leads us to the following generalizations of permutability.

Definition. Let A be a subgroup of a group G. We say that:

(1) A is *quasipermutable in* G if A either covers or avoids every maximal pair (K,H) of G.

(2) A is *weakly quasipermutable in* G if G has a subgroup T and a quasipermutable subgroup C such that $G = AT$ and $T \cap A \le C \le A$.

In this paper we continue the research of the paper [1] and study the finite groups with given systems of quasipermutable subgroups and weakly quasipermutable subgroups. Our main goal here is to prove the following result.

***Theorem* A.** *Let G be a group. The following statements are equivalent:*

(1) *G is supersoluble.*

(2) *Every subgroup of $F^*(G)$ is quasipermutable in G.*

(3) *Every cyclic subgroup of $F^*(G)$ with prime order and order 4 is weakly quasipermutable in G.*

In this theorem $F^*(G)$ denotes the generalized Fitting subgroup of G, that is, the product of all normal quasinilpotent subgroups of G; see [2, Chapter X].

The proof of Theorem A consists of a large number steps and the following results are the main stages of it.

***Theorem* B.** *Let G be a group and p a prime dividing $|G|$ such that $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if for Sylow p-subgroup P of G either all maximal subgroups of P or all cyclic subgroup of P with prime order and order 4 (if P is a non-abelian 2-group) are weakly quasipermutable in G.*

A chief factor H/K of a group G is called $\mathcal{U}$-central provided $H/K \rtimes (G/C_G(H/K)) \in \mathcal{U}$. The product of all normal subgroups of a group G whose G-chief factors are $\mathcal{U}$-central in G is called the $\mathcal{U}$-hypercentre of G and denoted by $Z_{\mathcal{U}}(G)$ [3].

***Theorem* C.** *Let E be a normal subgroup of a group G. Suppose that all cyclic subgroups of E of prime order and order 4 are weakly quasipermutable in G. Then $E \le Z_{\mathcal{U}}(G)$.*

Note that Theorem C is independently interesting because it generalizes the main results of some papers (see, for example, [17]–[28]).

All unexplained notations and terminologies are standard. The reader is reffered to [3]–[5] if necessary.

1 Preliminaries

We need the following properties of weakly quasipermutable subgroups.

***Lemma* 1.1** [1, Lemma 2.15]. *Let H be a normal subgroup of a group G and K a weakly quasipermutable subgroup of G. Then:*

(1) *If $K \le E \le G$, then K is weakly quasipermutable in E.*

(2) *If $H \le K$, then K/H is weakly quasipermutable in G/H.*

(3) *If $(|H|,|K|)=1$, then HK/H is weakly quasipermutable in G/H.*

***Lemma* 1.2** [1, Lemma 2.15]. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and G a group with soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p. In addition, if every cyclic subgroup of P with prime order and order 4 (if $p=2$ and P is non-abelian) is weakly quasipermutable in G, then $|P|=p$ is not the smallest prime dividing $|G|$.*

***Lemma* 1.3** [1, Lemma 2.5]. *Every quasipermutable subgroup of a group G is subnormal in G.*

***Lemma* 1.4** [6, Lemma 2.9]. *Let G be a group, p the smallest prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

***Lemma* 1.5** [7]. *Let A be a subnormal π-subgroup of a group G. Then $A \le O_{\pi}(G)$.*

Let P be a p-group. If P is not a non-abelian 2-group we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P)=\Omega_2(P)$.

***Lemma* 1.6** [8]. *Let P be a p-group of class at most 2. Suppose that $\exp(P/Z(P))$ divides p.*

(1) *If $p>2$, then $\exp(\Omega(P))=p$.*

(2) *If P is a non-abelian 2-group, then* $\exp(\Omega(P))=4$.

***Lemma* 1.7** (See [9, II] or [3, IV, Chapter 6]). *Let P be a normal p-subgroup of a group G. If $\Omega \le Z_{\mathcal{U}}(G)$, then $P \le Z_{\mathcal{U}}(G)$.*

***Lemma* 1.8** [2, Chapter X]. *Let G be a group. If $F^*(G)$ is soluble, then $F^*(G)=F(G)$.*

***Lemma* 1.9** [10, Theorem C]. *Let E be a subgroup of a group G. If every G-chief factor below $F^*(G)$ is cyclic, then every G-chief factor below E is cyclic.*

2 Proof of Theorems B, C and A

***Proof of Theorem* B.** We only need to prove the "if part". Suppose that this is false and let G be a counterexample with minimal order.

(1) $O_{p'}(G)=1$.

Suppose that $O_{p'}(G)\neq 1$. By Lemma 1.1 (3), the hypothesis holds for $G/O_{p'}(G)$. Therefore $G/O_{p'}(G)$ is p-nilpotent by the choice of G, a contradiction. Hence $O_{p'}(G)=1$.

(2) *Every maximal subgroup of P is weakly quasipermutable in G.*

Suppose that this is false. Then by hypothesis every cyclic subgroup of P with prime order and order 4 (if P is a non-abelian 2-group) is weakly quasipermutable in G. Since G is not p-nilpotent, it has a p-closed Schmidt subgroup [11, IV, Theorem 5.4] $H = H_p \rtimes H_q$, where $H_p \le P$. By Lemma 1.1 (1), every cyclic subgroup of H_p with prime order and order 4 (if H_p is a non-abelian 2-group) is weakly quasipermutable in H. Then by Lemma 1.2, $|H_p|=p$. Hence $H/C_H(H_p)\simeq L \le Aut(H_p)$ and $Aut(H_p)$ is a cyclic subgroup with order $p-1$. This contradiction shows that every maximal subgroup of P is weakly quasipermutable in G.

(3) $O_p(G)=1$.

Suppose that $O_p(G)\neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then $N \le P$ and by Lemma 1.1 (2) the hypothesis holds for G/N. Therefore G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the only minimal normal subgroup of G contained in P, $N \not\subseteq \Phi(G)$ and $N \not\subseteq Z(G)$. Moreover, G is p-soluble. Since $(|G|,p-1)=1$, $|N|>p$. But $O_{p'}(G)=1$ by (1). Hence N is the only minimal normal subgroup of G. Let M be a maximal subgroup of G such that $N \not\subseteq M$. Then $G = N \rtimes M$ and $N = O_p(G)$. Indeed,

$$O_p(G)=O_p(G)\cap NM = N(O_p(G)\cap M).$$

Since $O_p(G)\le F(G)\le C_G(N)$ by [9], $O_p(G)\cap M$ is normal in G, so $O_p(G)\cap M = 1$. Hence $N = O_p(G)$.

Since $G = N \rtimes M$, $AM = G$ for every maximal subgroup A of P containing N. Moreover, $M \simeq G/N$ is a p-nilpotent supplement of A in G. Therefore by Lemma 1.4, some maximal subgroup V of P neither contains N nor has a p-nilpotent supplement in G. Then by (2), V is weakly quasipermutable in G. Let C and T be subgroups of G such that $VT = G$, C is quasipermutable in G and $T \cap V \le C \le V$. By Lemma 1.3, C is subnormal in G. Hence by Lemma 1.5, $C \le N$. If $C = 1$, then $T \cap V = 1$, so $|T_p| = |P \cap T| = p$. Hence T is p-nilpotent by [12, Theorem 10.1.9], a contradiction. Therefore $C \ne 1$. Since C is quasipermutable in G, $CM = G$, so $C = N$, a contradiction. Thus $O_p(G) = 1$.

Final contradiction. Let V be a maximal subgroup of P. By (2), V is weakly quasipermutable in G. Let C and T be subgroups of G such that $VT = G$, C is quasipermutable in G and $T \cap V \le C \le V$. By Lemma 1.3, C is subnormal in G. Hence by Lemma 1.5, $C \le O_p(G) = 1$. Then $T \cap V = 1$, so $|T_p| = |P \cap T| = p$ and T is p-nilpotent. Therefore by Lemma 1.4, G is p-nilpotent. This contradiction completes the proof of Theorem B.

Proof of Theorem **C.** Suppose that theorem is false and consider a counterexample (G, E) for which $|G||E|$ is minimal. Let P be a Sylow p-subgrop of E, where p is the smallest prime dividing $|E|$.

(1) *If X is a Hall subgroup of E, the hypothesis is still true for (X, X). If, in addition, X is normal in G, then the hypothesis also holds for (G, X) and $(G/X, E/X)$.*

This follows directly from Lemma 1.1.

(2) *If X is a non-identity normal Hall subgroup of E, then $X = E$.*

Since X is a characteristic subgroup of E, it is normal in G. Hence by (1) the hypothesis is still true for $(G/X, E/X)$ and (G, X). If $X \ne E$, then $E/X \le Z_{\mathcal{U}}(G/X)$ and $X \le Z_{\mathcal{U}}(G)$ by the choice of (G, E). Hence $E \le Z_{\mathcal{U}}(G)$, a contradiction. Thus $X = E$.

(3) $E = P$.

By Theorem B, E is p-nilpotent. If H is a Hall p'-subgroup of E, then H is normal in E. Suppose that $E \ne P$. Then $H \ne E$, which contradicts (2). Thus $E = P$.

(4) *P is not cyclic.*

This follows from (3) and [13, 7, Theorem 6.1].

(5) *G has a normal subgroup $R \le P$ such that P/R is a non-cyclic chief factor of G, $R \le Z_{\mathcal{U}}(G)$ and $V \le R$ for any normal subgroup $V \ne P$ of G contained in P.*

Let P/R be a chief factor of G. Then by Lemma 1.1 the hypothesis holds for (G, R). Hence $R \le Z_{\mathcal{U}}(G)$, so P/R is non-cyclic by the choice of (G, E). Let $V \ne P$ be any normal subgroup of G contained in P. Then $V \le Z_{\mathcal{U}}(G)$ by the choice of (G, E) and Lemma 1.1. If $V \not\le R$, then by [14, Lemma 2.3], $P = VR \le Z_{\mathcal{U}}(G)$, a contradiction. Hence $V \le R$.

(6) $\Omega(P) = P$.

Suppose that $\Omega(P) < P$. Then by (5), $\Omega(P) \le Z_{\mathcal{U}}(G)$. Hence by Lemma 1.7, $P \le Z_{\mathcal{U}}(G)$, a contradiction. Thus $\Omega(P) = P$.

The final contradiction. Let H/R be any minimal subgroup of $P/R \cap Z(G_p/R)$, where G_p is a Sylow p-subgroup of G. Let $x \in H \setminus R$ and $L = \langle x \rangle$. Then $|L| = p$ or $|L| = 4$ by (6) and Lemma 1.6. Hence by hypothesis L is weakly quasipermutable in G. Let C and T be subgroups of G such that $LT = G$, C is quasipermutable in G and $T \cap L \le C \le L$. If $T \ne G$, then G has a maximal normal subgroup M such that $G = LM$. Hence $P = L(P \cap M)$, so $p = |G/M| = |P/P \cap M|$. Therefore $P/P \cap M$ is cyclic and $P \cap M \le R \le Z_{\mathcal{U}}(G)$. Hence $P \le Z_{\mathcal{U}}(G)$, a contradiction. Thus $T = G$ and L is quasipermutable in G. Let W be a maximal subgroup of G such that $G = P \rtimes W$. Since L is quasipermutable in G, L either covers or avoids (W, G). If L covers (W, G), then $LW = G = PW$, so $L = P$, a contradiction. Hence L avoids (W, G). Thus $L \le W$. This contradiction completes the proof of Theorem C.

Proof of Theorem **A.** First we show that (1) implies (2). Let A be any subgroup of $F^*(G)$ and (K, H) a maximal pair of G. Since G is supersoluble, $F^*(G) = F(G)$ by Lemma 1.8, so $A \cap H \le F(G) \cap H \le F(H)$. Hence by induction we may assume that $H = G$. If $A \subseteq K$, then $A = A \cap K = A \cap G$, that is, A avoids $(K, G) = (K, H)$. Suppose that $A \not\subseteq K$ and $K_G \ne 1$. Since $AK_G/K_G \le F(G)K_G/K_G \le F(G/K_G)$, by induction, $(K_G A/K_G)(K/K_G) = G/K_G$. Hence $AK = G$, that is, A covers (K, G). Hence we can assume that $K_G = 1$. In this case G is primitive. Therefore $F(G)$ is a minimal normal subgroup of G and so $|F(G)| = p$ is prime. Hence either $A = 1$ or $A = F(G)$. Therefore A covers or avoids $(K, G) = (K, H)$. Thus, (2) is a consequence of (1).

(2) $\Rightarrow$ (3) It is evident.

Finally, we shall prove the implication (3) $\Rightarrow$ (1). By Theorem C, $F^{*}(G) \le Z_{U}(G)$. Hence G is supersoluble by Lemma 1.9.

REFERENCES

1. *Guo, W.* Finite groups with systems of Σ-embedded subgroups / W. Guo, A.N. Skiba // Science China. Mathematics. – 2011. – 54. – P. 1–18.
2. *Huppert, S.* Finite Groups III / S. Huppert, N. Blackburn. – Berlin-New York : Springer-Verlag, 1982.
3. *Doerk, K.* Finite Soluble Groups / K. Doerk, T. Hawkes. – Berlin-New York : Walter de Gruyter, 1992.
4. *Ballester-Bolinches, A.* Classes of Finite groups / A. Ballester-Bolinches, L.M. Ezquerro. – Dordrecht : Springer-Verlag, 2006.
5. *Guo, W.* The Theory of Classes of Groups / W. Gou. – Beijing-New York-Dordrecht-Boston-London : Science Press-Kluwer Academic Publishers, 2000.
6. *Shemetkov, L.A.* The generalized X-hypercentre of a finite group / L.A. Shemetkov, A.N. Skiba // Proc. Fr. Skorina Gomel State University. – 2008. – № 6. – P. 128–130.
7. *Wielandt, H.* Subnormal subgroups and permutation groups: Lectures given at the Ohio State University / H. Wielandt. – Columbus, Ohio, 1971.
8. Bercovich, Y. Indices of elements and normal structure of finite groups / Y. Bercovich, L. Kazarin // J. Algebra. – 2005. – 283 (1). – P. 564–583.
9. *Shemetkov, L.A.* Formations of finite groups / L.A. Shemetkov. – M. : Nauka, 1978.
10. *Skiba, A.N.* A characterization of the hypercyclically embedded subgroups of finite groups / A.N. Skiba // Journal of Pure and Apllied Algebra (2010) (in Press), (available at:doi:10.1016/j.jpaa.2010.04.017).
11. *Huppert, B.* Endliche Gruppen I / B. Huppert. Berlin-Heidelberg-New York : Springer-Verlag, 1967.
12. *Robinson, D.* A course in the theory of groups / D. Robinson. – New York-Heidelberg-Berlin : Springer-Verlag, 1982.
13. *Gorenstein, D.* Finite Groups / D. Gorenstein. – New York-Evanston-London: Harper & Row Publishers, 1968.
14. *Shemetkov, L.A.* On the $\mathcal{XF}$-hypercentre of finite groups / L.A. Shemetkov, A.N. Skiba // J. Algebra. – 2009. – 322. – P. 2106–2117.
15. *Wang, Y.* c-normality of groups and its properties / Y. Wang // J. Algebra. – 1996. – 180. – P. 954–965.
16. *Srinivasan, S.* Two sufficient conditions for supersolvability of finite groups / S. Srinivasan // Israel J. Math. – 1980. – 35. – P. 210–214.
17. *Ballester-Bolinches, A.* Finite groups with some c-normal minimal subgroups A. Ballester-Bolinches, Y. Wang // J. Pure Appl. Algebra. – 2000. – 153. – P. 121–127.
18. *Ballester-Bolinches, A.* c-supplemented subgroups of finite groups / A. Ballester-Bolinches, Y. Wang, X.Y. Guo // Glasgow Math. J. – 2000. – 42. – P. 383–389.
19. *Wang, Y.* Finite groups with c-supplemented minimal subgroups / Y. Wang, Y. Li, J. Wang // Algebra Colloquium. – 2003. – 10 (3). – P. 413–425.
20. *Wei, H.* On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups, II / H. Wei, Y. Wang, Y. Li // Comm. Algebra. – 2003. – 31. – P. 4807–4816.
21. *Wang, Y.* A generalization of Kramer's theorem and its applications / Y. Wang, H. Wei, Y. Li // Bull. Australian Math. Soc. – 2002. – 65. – P. 467–475.
22. *Wei, H.* On c-supplemented maximal and minimal subgroups of Sylow subgroups of finite groups / H. Wei, Y. Wang, Y. Li // Proc. Amer. Math. Soc. – 2004. – 132 (8). – P. 2197–2204.
23. Wei, H. On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups / H. Wei // Comm. Algebra. – 2001. – 29. – P. 2193–2200.
24. *Ballester-Bolinches, A.* On complemented subgroups of finite groups / A. Ballester-Bolinches, X.Y. Guo // Arch. Math. – 1999. – 72. – P. 161–166.
25. *Ballester-Bolinches, A.* On minimal subgroups of finite groups / A. Ballester-Bolinches, M. C. Pedraza-Aguilera // Acta Math. Hungar. – 1996. – 73. – P. 335–342.
26. *Guo, X.Y.* On p-Nilpotency of Finite Groups with Some Subgroups c-supplemented / X.Y. Guo // Algebra Colloquium. – 2003. – 10 (3). – P. 259–256.
27. *Wang, Y.* Finite Groups with Some Subgroups of Sylow Subgroups c-supplemented / Y. Wang // J. Algebra. – 2000. – 224. – P. 464–78.
28. *Radaman, M.* On c-normaliti of certain subgroups of prime power order of finite groups / M. Radaman, M. Azzat Mohamed, A.A. Heliel // Arch. Math. – 2005. – 85. – P. 203–210.

Поступила в редакцию 17.05.12.