Известия Гомельского государственного университета
имени Ф. Скорины, № 6 (117), 2019

УДК 512.542

On the π-decomposable norm of a finite group

V.M. SELKIN, A.A. POBILOVSKY, S.I. ZINCHUK, D.A. YROSHENKO

Let G be a finite group and $\pi \subseteq P$. Then G is called: *π-decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$; *meta-$\pi$-decomposable* if G is an extension of a π-decomposable group by a π-decomposable group. We use $\mathfrak{N}_\pi$ to denote the class of all finite π-decomposable groups. Let $\mathrm{N}_\pi(G)$ be the intersection of the normalizers of the π-decomposable residuals of all subgroups of G, that is, $\mathrm{N}_\pi(G) = \bigcap_{H \le G} N_G(H^{\mathfrak{N}_\pi})$. We say that $\mathrm{N}_\pi(G)$ is the *π-decomposable norm* of G. We study the basic properties of the π-decomposable norm of G. In particular, we prove that G is meta-π-decomposable if and only if $G/\mathrm{N}_\pi(G)$ is meta-π-decomposable.

Kaywords: finite group, π-decomposable group, π-soluble group, π-decomposable residual of a group, π-decomposable norm of a group.

Пусть G является конечной группой и $\pi \subseteq P$. Тогда группа G называется: *π-разложимой*, если $G = O_\pi(G) \times O_{\pi'}(G)$; мета-*$\pi$-разложимой* если G является расширением π-разложимой группы по π-разложимой группе. Мы используем $\mathfrak{N}_\pi$ для обозначения класса всех конечных π-разложимых групп. Пусть $\mathrm{N}_\pi(G)$ пересечение нормализаторов π-разложимых радикалов подгрупп группы G: $\mathrm{N}_\pi(G) = \bigcap_{H \le G} N_G(H^{\mathfrak{N}_\pi})$. Мы говорим, что $\mathrm{N}_\pi(G)$ является *π-разложимой нормой группы* G. Мы изучаем базисные свойства π-разложимой норма группы G. В частности, мы доказали, что G является мета-π-разложимой тогда и только тогда, когда $G/\mathrm{N}_\pi(G)$ является мета-π-разложимой группой.

Ключевые слова: конечная группа, π-разложимая группа, π-разрешимая группа, π-разложимый радикал группы, π-разложимая норма группы.

1. Introduction. Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, P is the set of all primes, $\pi \subseteq P$ and $\pi' = P \setminus \pi$.

The group G is said to be: *π-decomposable* (respectively *p-decomposable*) if $G = O_\pi(G) \times O_{\pi'}(G)$ (respectively $G = O_p(G) \times O_{p'}(G)$); *meta-$\pi$-decomposable* if G is an extension of a π-decomposable group by a π-decomposable group. We use $\mathfrak{N}_\pi$ to denote the class of all π-decomposable groups; $\mathfrak{N}_{\{p\}}$ is the class of all p-decomposable groups.

Various classes of π-decomposable and meta-π-decomposable groups have been studied in many papers and, in particular, in the recent papers [1]–[6]. In this paper, we consider some new properties and applications of such groups.

If $1 \in \mathfrak{F}$ is the class of groups, then $G^{\mathfrak{F}}$ is the $\mathfrak{F}$*-residual* of G, that is, the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. In particular, $G^{\mathfrak{N}}$ is the nilpotent residual of G; $G^{\mathfrak{N}_\pi}$ is the π-decomposable residual of G.

Recall that the *norm* $N(G)$ of G is the intersection of the normalizers of all subgroups of G. This concept was introduced by R. Baer [7] (see also [8]) and the norm and the generalized norm of a group has been studied by many authors. In particular, in the recent paper [9] the following analogues of the subgroup $N(G)$ were introduced: (i) $S(G) = \bigcap_{H \le G} N_G(H^{\mathfrak{N}})$; (ii) let $1 = S_0(G) \le S_1(G) \le \cdots \le S_n(G) \le \cdots$, where $S_{i+1}(G)/S_i(G) = S(G/S_i(G))$ for all $i = 0, 1, 2, \ldots$. Then $S_\infty(G) = S_n(G)$, where let n is the smallest n such that $S_n(G)) = S_{n+1}(G))$.

The basic properties and some applications of the subgroups $S(G)$ and $S_\infty(G)$ were considered in [9]. In this paper we consider the following generalizations of the subgroups $S(G)$ and $S_\infty(G)$.

Definition 1.1. Let $\mathrm{N}_{\pi(G)}$ be the intersection of the normalizers of the π-decomposable residuals of all subgroups of G, that is, $\mathrm{N}_\pi(G) = \bigcap_{H \le G} N_G(H^{\mathfrak{N}_\pi})$. We say that $\mathrm{N}_\pi(G)$ is the *π-decomposable norm* of G. If $\pi = \{p\}$, we write $\mathrm{N}_p(G)$ instead of $\mathrm{N}_{\{p\}}(G)$ and say that $\mathrm{N}_p(G)$ is the *p-decomposable norm* of G.

Definition 1.2. Let

$$1 = \mathrm{N}_\pi^0(G) \le \mathrm{N}_\pi^1(G) \le \mathrm{N}_\pi^2(G) \le \cdots \le \mathrm{N}_\pi^n(G) \le \cdots,$$

where

$$\mathrm{N}_\pi^{i+1}(G) / \mathrm{N}_\pi^i(G) = \mathrm{N}_\pi(G / \mathrm{N}_\pi^i(G)),$$

for all $i = 0,1,2,\ldots$. And let n be the smallest n such that $\mathrm{N}_\pi^n = \mathrm{N}_\pi^{n+1}$. Then we write $\mathrm{N}_\pi^\infty(G) = \mathrm{N}_\pi^n(G)$ and say that $\mathrm{N}_\pi^\infty(G)$ is the *π-decomposable hypernorm* of G.

Obviously, $\mathrm{N}_\pi(G)$ and $\mathrm{N}_\pi^\infty(G)$ are characteristic subgroups of G.

Before continuing, consider the following example.

Example 1.3. (i) Let $G = P \rtimes (Q \rtimes R)$, where $Q \rtimes R$ is a non-abelian group of order 6 and P is a simple $F_5(Q \rtimes R)$-module which is faithful for $Q \rtimes R$. Let $\sigma = \{\{2,5\},\{2,5\}'\}$. Then G every proper non-π-decomposable subgroup H of G is either of the form $V \rtimes Q^x$, where $V \le P$, or of the form $(Q \rtimes R)^y$ for some $x, y \in G$. In the former case we have $H^{\mathfrak{N}_\pi} = V$ and $N_G(V) = PQ^x = PQ$. In the second case we have $((Q \rtimes R)^y)^{\mathfrak{N}_\pi} = Q^y$ and $N_G(Q^y) = (Q \rtimes R)^y$. Moreover,

$$\bigcap_{y \in G} (Q \rtimes R)^y = (Q \rtimes R)_G \le C_G(P) = P$$

and so $\mathrm{N}_\pi(G) = 1 = \mathrm{N}_\pi^\infty(G)$.

(ii) Let G and σ are the same as in (ii). Let $A = G \times C_2$. Let $B = (Q \rtimes R)C_2$, where C_2 is a group of order 2. Then $B^{\mathfrak{N}_\pi} = Q$, $C_2 \le \mathrm{N}_\pi(A)$ and $N_A(Q) = B < A$. Hence $1 < \mathrm{N}_\pi(A) = C_2 = \mathrm{N}_\pi^\infty(G) < G$.

Our main goal here is to prove the following results.

Theorem 1.4. *For any group G, the subgroup $\mathrm{N}_\pi^\infty(G)$ is σ-separable.*

Theorem 1.5. *The group G is meta-π-decomposable if and only if $G / \mathrm{N}_\pi(G)$ is meta-π-decomposable.*

Theorem 1.6. *Suppose that G is p-soluble and all elements of G of order p are in $\mathrm{N}_p(G)$. If $p > 2$, then $l_p(G) \le 1$.*

2. Proofs of the results. First we prove the following facts about the subgroups $\mathrm{N}_\pi(G)$ and $\mathrm{N}_\pi^\infty(G)$.

Lemma 2.1. *If E is a subgroup of G, then $\mathrm{N}_\pi(G) \cap E \le \mathrm{N}_\pi(E)$.*

Proof. First observe that

$$\mathrm{N}_\pi(G) = \bigcap_{H \le G} N_G(H^{\mathfrak{N}_\pi}) \le \bigcap_{H \le E} N_G(H^{\mathfrak{N}_\pi}),$$

so

$$E \cap \mathrm{N}_\pi(G) = E \cap \bigcap_{H \le G} N_G(H^{\mathfrak{N}_\pi}) = \bigcap_{H \le E} N_G(H^{\mathfrak{N}_\pi}) = \mathrm{N}_\pi(E).$$

The lemma is proved.

Lemma 2.2. *If N is a normal subgroup of G, then $\mathrm{N}_\pi(G)N / N \le \mathrm{N}_\pi(G / N)$.*

Proof. For any subgroup H / N of G / N we have $(H / N)^{\mathfrak{N}_\pi} = H^{\mathfrak{N}_\pi} N / N$. Then for every $x \in \mathrm{N}_\pi(G)$ we have $x \in N_G(H^{\mathfrak{N}_\pi})$, so $x \in N_G(H^{\mathfrak{N}_\pi} N)$ and hence xN normalizes $(H / N)^{\mathfrak{N}_\pi}$. Thus $\mathrm{N}_\pi(G)N / N \le \mathrm{N}_\pi(G / N)$. The lemma is proved.

Lemma 2.3. *If* N *is a normal subgroup of* G *and* $N \le \mathfrak{N}_\pi^\infty(G)$, *then* $\mathfrak{N}_\pi^\infty(G/N) = \mathfrak{N}_\pi^\infty(G)/N$.

Proof. Since $N \le \mathfrak{N}_\pi^\infty(G)$, for some i we have $N \le \mathfrak{N}_\pi^i(G)$. Let $N^i/N = \mathfrak{N}_\pi^i(G/N)$ for all $i = 1, 2, \ldots$, and let $N^\infty/N = \mathfrak{N}_\pi^\infty(G/N)$. First we claim that $N^1 \le \mathfrak{N}_\pi^{i+1}(G)$. Let $H/\mathfrak{N}_\pi^i(G)$ be any subgroup of $G/\mathfrak{N}_\pi^i(G)$ and $x \in N^1$. Then xN normalizes $(H/N)^{\mathfrak{N}_\pi} = H^{\mathfrak{N}_\pi}N/N$, that is, $(H^{\mathfrak{N}_\pi})^x N/N = H^{\mathfrak{N}_\pi}N/N$ and so $(H^{\mathfrak{N}_\pi})^x N = H^{\mathfrak{N}_\pi}N$, which implies that $(H^{\mathfrak{N}_\pi})^x \mathfrak{N}_\pi^i(G) = H^{\mathfrak{N}_\pi}\mathfrak{N}_\pi^i(G)$ since $N \le \mathfrak{N}_\pi^i(G)$. But then $(H^{\mathfrak{N}_\pi})^x \mathfrak{N}_\pi^i(G)/\mathfrak{N}_\pi^i(G) = H^{\mathfrak{N}_\pi}\mathfrak{N}_\pi^i(G)/\mathfrak{N}_\pi^i(G)$ and so $x\mathfrak{N}_\pi^i(G)$ normalizes $(H/\mathfrak{N}_\pi^i(G))^{\mathfrak{N}_\pi}$. Hence $x\mathfrak{N}_\pi^i(G) \in \mathfrak{N}_\pi^{i+1}(G)/\mathfrak{N}_\pi^i(G) = \mathfrak{N}_\pi(G/\mathfrak{N}_\pi^i(G))$. Thus $N^1 \le \mathfrak{N}_\pi^{i+1}(G)$. Moreover, if $N^n \le \mathfrak{N}_\pi^{i+n}(G)$, then similarly we can show that $N^{n+1} \le \mathfrak{N}_\pi^{i+n+1}(G)$, so $N^\infty \le \mathfrak{N}_\pi^\infty(G)$.

Conversely, $\mathfrak{N}_\pi^1(G) \le N^1$ by Lemma 2.2. And if for some n we have $\mathfrak{N}_\pi^n(G) \le N^n$, then for every $x \in \mathfrak{N}_\pi^{n+1}(G)$ and for every subgroup $H/\mathfrak{N}_\pi^n(G)$ of $G/\mathfrak{N}_\pi^n(G)$ we as above get that $(H^{\mathfrak{N}_\pi})^x \mathfrak{N}_\pi^n(G) = H^{\mathfrak{N}_\pi}\mathfrak{N}_\pi^n(G)$ and so $(H^{\mathfrak{N}_\pi})^x N^n = H^{\mathfrak{N}_\pi}N^n$ and hence $x \in N^{n+1}$. Thus $\mathfrak{N}_\pi^{n+1}(G) \le N^{n+1}$ and so $\mathfrak{N}_\pi^\infty(G) \le N^\infty$. Hence $\mathfrak{N}_\pi^\infty(G) = N^\infty$. The lemma is proved.

Proof of Theorem 1.4. It is enough to show that $\mathfrak{N}_\pi(G)$ is π-separable. Let $X = \mathfrak{N}_\pi(G)$. Then the group X has the following property: the π-decomposable residual of every subgroup of X is normal in X. We show that every group with such a property is π-separable. Assume that this is false and let X be a counterexample of minimal order. Let M be a maximal subgroup of X and let $N = M^{\mathfrak{N}_\pi}$ be the π-decomposable residual of M. Then N is normal in G. If $N \ne 1$, then X/N and N are π-separable since the hypothesis evidently holds for X/N and N and so in this case X π-separable by the choice of X. Therefore every maximal subgroup M of X is π-decomposable and so G is minimal non-π-decomposable group. Then G is a Schmidt group by the Belonogov result [10] and so soluble. This contradiction completes the proof of the result.

Proof of Theorem 1.5. It is enough to show that if $G/\mathfrak{N}_\pi(G)$ is meta-π-decomposable, then also G is meta-π-decomposable. Assume that this is false and let G be a counterexample of minimal order. Then $\mathfrak{N}_\pi(G) \ne 1$.

Let R be a minimal normal subgroup of G. Then $R\mathfrak{N}_\pi(G)/R \le \mathfrak{N}_\pi(G/R)$ by Lemma 2.2. Moreover,

$$G/R\mathfrak{N}_{\pi X}(G);\ \ (G/\mathfrak{N}_\pi(G)/(R\mathfrak{N}_\sigma(G)/\mathfrak{N}_\pi(G)) \in \mathfrak{N}_\pi$$

since the class of all meta-π-decomposable groups is a homomorph. Therefore the hypothesis holds for G/R, so the choice of G implies that G/R is meta-π-decomposable. Hence

$$(G/R)^{\mathfrak{N}_\pi} = G^{\mathfrak{N}_\pi}R/R;\ \ G^{\mathfrak{N}_\pi}/(G^{\mathfrak{N}_\pi} \cap R),$$

is π-decomposable. Therefore $R \le G^{\mathfrak{N}_\pi}$ and $G^{\mathfrak{N}_\pi}/R$ is π-decomposable. If G has a minimal normal subgroup $N \ne R$, then $G^{\mathfrak{N}_\pi}/L$ is also π-decomposable and so $G^{\mathfrak{N}_\pi}$; $G^{\mathfrak{N}_\pi}/(R \cap L)$ is π-decomposable and so G is meta-π-decomposable, contrary to the choice of G. Therefore R is the unique minimal normal subgroup of G, so $R \le \mathfrak{N}_\pi(G)$ since $\mathfrak{N}_\pi(G) \ne 1$. It is clear also that R ’ $\Phi(G)$ and so for some maximal subgroup M of G we have $G = RM$ and $M_G = 1$. Moreover, $M^{\mathfrak{N}_\pi} \ne 1$ since G is not meta-π-decomposable and R is π-decomposable in view of Theorem 1.4 and the inclusion $R \le \mathfrak{N}_\pi(G)$. Now observe that $R, \le N_G(M^{\mathfrak{N}_\pi})$ and so $M^{\mathfrak{N}_\pi}$ is normal in G. Hence $M_G \ne 1$. This contradiction completes the proof of the result.

Proof of Theorem 1.6. Assume that this theorem is false and let G be a counterexample of minimal order.

(1) *For every proper subgroup* E *of* G *we have* $l_p(E) \le 1$.

Let x be an element of E of order p. Then $x \in \mathrm{N}_p(G) \cap E \le \mathrm{N}_p(E)$ by Lemma 2.1. Therefore the hypothesis holds for E, so $l_p(E) \le 1$ by the choice of G.

(2) $O^{p'}(G) = G$, *so for some normal subgroup* V *of* G *we have* $|G:V| = p$.

Assume that $O^{p'}(G) < G$. Then $l_p(O^{p'}(G)) \leq 1$ by Claim (1) and so $O^{p'}(G)/O_{p',p}(O^{p'}(G))$ is a p'-group, where $O_{p',p}(O^{p'}(G))$ is characteristic in $O^{p'}(G)$ and hence normal in G. But then $G/O^{p',p}(G)$ is a p'-group and so $l_p(G) \leq 1$. This contradiction completes the proof of the fact that $O^{p'}(G) = G$ and so we have (2) since G is p-soluble.

(3) $O_{p'}(G) = 1$. *Hence* $C_G(O_p(G)) \leq O_p(G)$.

Assume that $O_{p'}(G) \neq 1$. Then for every element $a/O_{p'}(G)$ of $G/O_{p'}(G)$ order p, there is an element x of G of order p such that $a/O_{p'}(G) = x/O_{p'}(G)$. Then $x \in \mathrm{N}_p(G)$, so $a/O_{p'}(G) = x/O_{p'}(G) \in \mathrm{N}_p(G/O_{p'}(G))$ by Lemma 2.2. Therefore the hypothesis holds for $G/O_{p'}(G)$ and so $l_p(G/O_{p'}(G)) \leq 1$ by the choice of G. But then $l_p(G) \leq 1$, a contradiction. Hence $O_{p'}(G) = 1$, so $O_{p',p}(G) = O_p(G)$. Therefore we have $C_G(O_p(G)) \leq O_p(G)$ since G is p-soluble by hypothesis.

(4) G *is* q*-nilpotent for every prime* $q \neq p$.

Assume that this is false and let A be a minimal non-q-nilpotent subgroup of G. Then A is a q-closed Schmidt group by [11, IV, Satz 5.4] and, by [12, V, Theorem 26.1], for a Sylow q-subgroup Q of A we have $Q = A^{\mathfrak{N}}$. It is clear that, in fact, $Q = A^{\mathfrak{N}_{\{p\}}}$ and so $\Omega_1(O_p(G)) \leq N_G(Q)$. Then $Q \leq C_G(\Omega_1(O_p(G)))$ and so $Q \leq C_G(O_p(G))$ by [11, IV, Satz 5.12], contrary to (3). Hence we have (4).

The final contradiction. From Claim (4) we get that G is p-closed. But then $l_p(G) = 1$, contrary to the choice of G. THis final contradiction completes the proof of the result.

References

1. Guo W. On σ-supersoluble groups and one generalization of CLT-groups / W. Guo, A.N. Skiba // J. Algebra. – 2018. – V. 512 (1). – P. 92–108.

2. Skiba, A.N. On σ-subnormal and σ-permutable subgroups of finite groups / A.N. Skiba // J. Algebra. – 2015. – V. 436. – P. 1–16.

3. Skiba, A.N. Some characterizations of finite σ-soluble $P\sigma T$-groups / A.N. Skiba // J. Algebra. – 2018. – V. 495 (1). – P. 114–129.

4. Chi Z. On Σ_t^σ-closed classes of finite groups / Z. Chi, A.N. Skiba // Ukrainian Math. J. – 2018. – № 12. – P. 1707–1716.

5. Chi, Z. On n-multiply σ-local formations of finite groups / Z. Chi, V.G. Safonov, A.N. Skiba // Commun. Algebra – 2019. – № 47 (1). – P. 957–968.

6. Chi, Z. A generalization of the Kramer's theory / Z. Chi, A.N. Skiba // Acta Math. Hungarica. – 2019. – V. 158, is. 1. – P. 87–99.

7. Baer, R. Der Kern eine charkteristishe Untergruppe / R. Baer // Compos. Math. – 1935. – V. 1. – P. 254–283.

8. Baer, R. Norm and hypernorm / R. Baer // Publ. Math. Debrecen. – 1956. – V. 4. – P. 347–350.

9. Shen, Z. On the norm of the nilpotent residuals of all subgroups of a finite group / Z. Shen, W. Shi, G. Qian // J. Algebra. – 2012. – V. 352. – P. 290–298.

10. Belonogov, V.A. Finite groups all of whose 2-maximal subgroups are π-decomposable / V.A. Belonogov // Trudi Instituta Matematiki i Mekhaniki Uro RAN. – 2014. – V. 20 (2). – P. 29–43.

11. Huppert, B. Endliche Gruppen I / B. Huppert. – Berlin-Heidelberg-New York : Springer-Verlag, 1967. – 793 p.

12. Shemetkov, L.A. Formations of finite groups / L.A. Shemetkov. – Moscow : Nauka, 1978. – 272 p.

Гомельский государственный
университет им. Ф. Скорины

Поступила в редакцию 18.10.2019