УДК 512.542

КОНЕЧНЫЕ ГРУППЫ С МОДУЛЯРНОЙ ПОДГРУППОЙ ШМИДТА

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ON FINITE GROUPS WITH MODULAR SCHMIDT SUBGROUP

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Пусть G конечная группа. Тогда G называется группой Шмидта, если G не является нильпотентной, а все ее собственные подгруппы нильпотентны. Подгруппа M группы G называется модулярной в G, если M является модулярным элементом (в смысле Куроша) решетки $L(G)$ всех подгрупп группы G, т. е., (i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ для всех $X \leq G, Z \leq G$ таких что $X \leq Z$, (ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ для всех $Y \leq G, Z \leq G$ таких что $M \leq Z$. В работе доказывается, что если каждая подгруппа Шмидта A группы G с $A \leq G'$ являятся модулярной в G, тогда G является разрешимой группой, и если каждая погруппа Шмидта группы G является модулярной в G, тогда коммутант G' является нильпотентой группой.

***Ключевые слова**: конечная группа, модулярная подгруппа, группа Шмидта, коммутант подгруппы, нильпотентная группа.*

Let G be a finite group. Then G is called a Schmidt group if G is not nilpotent but every proper subgroup of G is nilpotent. A subgroup M of G is called modular in G if M is a modular element (in the sense of Kurosh) of the lattice $L(G)$ of all subgroups of G, that is, (i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and (ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$. In this paper, we prove that if every Schmidt subgroup A of G with $A \leq G'$ is modular in G, then G is soluble, and if every Schmidt subgroup of G is modular in G, then the derived subgroup G' is nilpotent.

***Keywords**: finite group, modular subgroup, Schmidt group, derived subgroup, nilpotent group.*

1 Main notations

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\pi(G)$ is the set of all primes dividing the order $|G|$ of G.

A subgroup M of G is called *modular* in G if M is a modular element (in the sense of Kurosh [1, p. 43]) of the lattice $L(G)$ of all subgroups of G, that is,

(i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and

(ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

Modular subgroups have a significant impact on the structure of a group (see, for example, [1]–[7]).

Recall that G is said to be $\mathfrak{F}$*-critical*, $\mathfrak{F}$ is a class of groups, if G is not in $\mathfrak{F}$ but all proper subgroups of G are in $\mathfrak{F}$ [8, p. 517]; G is said to be a *Schmidt group* provided G is $\mathfrak{N}$-critical, where $\mathfrak{N}$ is the class of all nilpotent groups.

A large number of publications are related to the study of the influence on the structure of the group of its critical subgroups, in particular, Schmidt subgroups. It was proved, for example, that if every Schmidt subgroup of G is subnormal, then $G' \leq F(G)$ [9], [10], [12]. Later, this result was generalized in the paper [11], where in particular it was proved that if every Schmidt subgroup of G is σ-subnormal in G [13], then G' is σ-nilpotent [13], 14].

In this paper we prove the following result in this line researches.

Theorem 1.1. (i) *If every Schmidt subgroup A of G with $A \leq G'$ is modular in G, then G is soluble.*

(ii) *If every Schmidt subgroup of G is modular in G, then the derived subgroup G' is nilpotent.*

2 Proof of Theorem 1.1

A normal subgroup A of G is said to be *hypercyclically embedded* in G [1, p. 217] if either $A = 1$ or $A \neq 1$ and every chief factor of G below A is cyclic. We use $Z_{\mathfrak{U}}(G)$ to denote the product of all normal hypercyclically embedded subgroups of G. It is clear that a normal subgroup A of G is hypercyclically embedded in G if and only if $A \leq Z_{\mathfrak{U}}(G)$. .

The following two lemmas collect the properties of modular subgroups which we use in our proofs.

***Lemma* 2.1** [1, Theorem 5.2.5]. *If M is a modular subgroup of G, then*

$$M^G / M_G \leq Z_{\mathfrak{U}}(G / M_G).$$

***Lemma* 2.2** [1, p. 201]. *Let A, B and N be subgroups of G, where A is modular in G and N is normal in G.*

(1) *If* B *is modular in* G, *then* $\langle A,B\rangle$ *is modular in* G.

(2) AN/N *is modular in* G/N.

(3) *N is modular in G.*

(4) *If* $A\le B$, *then A is modular in* B.

(5) *If* φ *is an isomorphism of G onto* $\overline{G}$, *then* A^{φ} *is modular in* $\overline{G}$.

(6) *If* $N\le B$ *and* B/N *is modular in G, then B is modular in G.*

(7) *If A is a maximal subgroup of G, then* $|G/A_G|$ *divides* pq *for some primes* $p\ne q$.

***Lemma* 2.3** [15, III, Satz 5.2] or [16, VI, Theorem 24.2]. *If G is a Schmidt group, then* $\pi(G)=\{p,q\}$ *for some primes* $p\ne q$ *and* $G=P\rtimes Q$, *where* $P=G^{\mathfrak{N}}$ *is a Sylow p-subgroup of* G *and* $Q=\langle x\rangle$ *is a cyclic Sylow q-subgroup of G. Moreover,* $\langle x^q\rangle\le\Phi(G)$ *and* P *is of exponent* p *or exponent* 4 *if P is a non-abelian* 2*-group.*

Recall that if *A* and *B* are subgroups of *G* such that $G=AB$, then B is said to be a supplement to *A* in *G*. If B is a supplement to *A* in *G* but $AL<G$ for every proper subgroup L of B, then B is called a minimal supplement to *A* in *G*.

Proof of Theorem 1.1. (i) Assume that this assertion is false and let *G* be a counterexample of minimal order.

(1) *If E is a proper subgroup of G, then E is soluble. Hence* $G'=G$.

Let *A* be any Schmidt subgroup of *E* such that $A\le E'$. Then $E'\le G'$, so *A* is modular in *G* by hypothesis. Therefore *A* is modular in E by Lemma 2.2 (4). Hence the hypothesis holds for E, therefore E is soluble by the choice of *G*. Finally, note that $G'=G$ since otherwise G' is soluble and so *G* is soluble, contrary to the choice of *G*.

(2) *If* N *is a minimal normal subgroup of G, then* G/N *is soluble.*

If G/N is nilpotent it is evident, it is evident. Now assume that G/N is not nilpotent, and let E/N be any Schmidt subgroup of G/N. Let H be a minimal supplement to N in E. Then

$$H/(H\cap N)\simeq HN/N=E/N$$

is a Schmidt group and $H\cap N\le\Phi(H)$ by [16, III, Lemma 11.3]. Let $\Phi=\Phi(H)$ and A be a Schmidt subgroup of H. Then $A\le G'=G$ by Claim (1), so *A* is modular in G by hypothesis.

Lemma 2.3 implies that

$$(H/(H\cap N))/\Phi(H/(H\cap N))=$$
$$=(H/(H\cap N))/(\Phi/(H\cap N))\simeq H/\Phi=P\rtimes Q,$$

where P is a Sylow *p*-subgroup and Q is a Sylow *q*-subgroup of H/Φ with $|Q|=q$, for some primes $p\ne q$. It follows, again by Lemma 2.3, that $A=A_p\rtimes A_q$, where $A=(A_q)^A$. Then $A_q\nleq\Phi$, since Φ is nilpotent. Therefore $\Phi A_q/\Phi$ is a Sylow *q*-subgroup of H/Φ and so

$$(\Phi A_q/\Phi)^{H/\Phi}=(A_q)^H\Phi/\Phi=H/\Phi.$$

Hence $(A_q)^H=H$, so $E=HN=(A_q)^H N$. By Lemma 2.2 (1), $(A_q)^H=A^H$ is modular in G and hence $E/N=(A_q)^H N/N$ is modular in G/R by Lemma 2.2 (1) (2). Therefore the hypothesis holds for G/N, so the choice of G implies that we have (2).

(3) *G is soluble.*

In view of Claims (1) and (2), it is enough to show that G is not non-abelian simple group. Assume that this is false and let *A* be any Schmidt subgroup of *G*. By hypothesis, *A* is modular in *G* since $G=G'$ by Claim (1). On the other hand, *G* is a non-abelian simple group. Hence $A_G=1$. But then $1<A^G$ and every chief factor of *G* below A^G is cyclic by Lemma 2.1. Hence *G* is not non-abelian simple group, a contradiction. Thus we have Claim (3). Therefore Statement (i) holds.

From now on, we suppose that every Schmidt subgroup of *G* is modular in *G*. We show that in this case the derived subgroup G' is nilpotent. Assume that this is false and let *G* be a counterexample of minimal order.

(4) *If* V *is a proper subgroup of G, then* $V'\le F(V)$ (see the proof of Claim (2)).

(5) *If* N *is a minimal normal subgroup of G, then* $(G/N)'\le F(G/N)$ (see the proof of Claim (1)).

(6) $R\nleq\Phi(G)$ *and for some prime* p *we have* $R=C_G(R)=O_p(G)=F(G)$. *Moreover,* $|R|>p$ *and for some maximal subgroup* M *of G we have* $G=R\rtimes M$.

First note that for some prime p we have $R\le O_p(G)$ by Claim (3). Claim (5) implies that the derived subgroup

$$(G/R)'=G'R/R\simeq G'/(G'\cap R)$$

of G/R is nilpotent. Suppose that *G* has a minimal normal subgroup $L\ne R$. Then $G'/(G'\cap L)$ is nilpotent. But then

$$G'\simeq G'/1=G'/((G'\cap R)\cap(G'\cap L))=$$
$$=G'/(R\cap L)$$

is nilpotent, contrary to the choice of G. Therefore R is the unique minimal normal subgroup of *G* and $R\le G'$. Moreover, $R\nleq\Phi(G)$ since otherwise G' is nilpotent by [8, Chapter A, Lemma 13.2]. Hence $R=C_G(R)=O_p(G)$ by [8, A, 15.6(2)]. Finally, note that R is not cyclic since otherwise the group

$$G/C_G(R)=G/R=G/F(G)$$

is cyclic, a contradiction. Hence we have (6).

(7) $M\simeq G/R$ *is nilpotent. Hence* R *is a Sylow p-subgroup of G.*

Assume that M is not nilpotent and let H be any Schmidt subgroup of M. Then H is modular in *G*. It is clear also that $H_G = 1$, so $H^G \le Z_{\mathfrak{U}}(G)$ by Lemma 2.1, that is, every chief factor of G below H^G is cyclic. But $R \le H^G$ by Claim (6) and hence R is cyclic, contrary to Claim (6). This contradiction shows that $M \simeq G/R$ is nilpotent. Then $O_p(M)R \le O_p(G) = R$. Hence $O_p(M) = 1$, so Claim (7) holds.

(8) M *is a Miller-Moreno group* (*that is*, M *is not abelian but every proper subgroup of* M *is abelian*). *Moreover*, M *is a q-group for some prime* $q \ne p$.

First note that M is a Hall p'-subgroup of G by Claims (6) and (7).

Now, let S be any maximal subgroup of M. Then $RS/F_{\sigma}(RS)$ is abelian by Claim (4). In view of Claims (6) and (7), $R = (RS)'$ and hence $S \simeq RS/R$ is abelian. Therefore the choice of G implies that M is either a Schmidt group or a minimal non-abelian group of prime power order q^a. But in the former case we have $|G:M| = p = |R|$ by Lemma 2.2 (7), contrary to Claim (6). Thus we have (6).

Final contradiction for (*ii*). In view of Claim (8), $Z(M) \cap \Phi(M) \ne 1$. Let Z be a subgroup of order q in $Z(M) \cap \Phi(M)$ and let $E = RZ$. Then E is not nilpotent by Claim (6). On the other hand, $R = R_1 \times \cdots \times R_t$, where R_k is a minimal normal subgroup of E for all $k = 1, \ldots, t$ by Mashcke's theorem. Hence for some i the subgroup $R_i \rtimes Z$ is not nilpotent, so this subgroup contains a Schmidt subgroup A of the form $A = A_p \rtimes Z$.

Suppose that $A < E$. Then $|A_p| < |R|$ and $A_G = 1$. Hence $1 < A^G \le Z_{\mathfrak{U}}(G)$ by Lemma 2.1. But then $R \le A^G$ and so R is cyclic by Lemma 2.1, contrary to Claim (6). Therefore $A = E$, so $R = A_p$ and Z acts irreducibly on R. Since $Z \le \Phi(M)$, every maximal subgroup of M acts irreducibly on R, which implies that every maximal subgroup of M is cyclic. Hence $q = 2$ and so $|R| = p$, contrary to Claim (6). □

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Поступила в редакцию 12.09.19.