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О $\sigma_{\mathfrak{I}}$-СУБНОРМАЛЬНЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

А.Н. Скиба

Гомельский государственный университет имени Франциска Скорины

ON $\sigma_{\mathfrak{I}}$-SUBNORMAL SUBGROUPS OF FINITE GROUPS

A.N. Skiba

Francisk Skorina Gomel State University

Аннотация. В данной работе: G – конечная группа; $\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$ – некоторое разбиение множества всех простых чисел $\mathbb{P}$, где $0 \in I$; $\mathfrak{I}$ – класс конечных σ_0-групп, который замкнут относительно расширений, эпиморфных образов и подгрупп и который содержит все конечные разрешимые σ_0-группы. Группа G называется: (i) $\sigma_{\mathfrak{I}}$-примарной, если G является конечной σ_i-группой для некоторого $i \in I$ и $G \in \mathfrak{I}$, если $i = 0$; (ii) $\sigma_{\mathfrak{I}}$-нильпотентной, если G является прямым произведением $\sigma_{\mathfrak{I}}$-первичных групп. Подгруппа A группы G называется $\sigma_{\mathfrak{I}}$-субнормальной в G, если существует цепь подгрупп $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ такая, что либо $A_{i-1} \trianglelefteq A_i$, либо $A_i / (A_{i-1})_{A_i}$ является $\sigma_{\mathfrak{I}}$-примарной для всех $i = 1,\ldots,t$. В данной работе мы изучаем $\sigma_{\mathfrak{I}}$-нильпотентные группы, $\sigma_{\mathfrak{I}}$-субнормальные подгруппы и соотношения между $\sigma_{\mathfrak{I}}$-нильпотентностью и $\sigma_{\mathfrak{I}}$-субнормальностью в группах. В частности, мы доказываем, что группа G является $\sigma_{\mathfrak{I}}$-нильпотентной в том и только в том случае, когда все ее подгруппы $\sigma_{\mathfrak{I}}$-субнормальны.

Ключевые слова: *конечная группа*, $\sigma_{\mathfrak{I}}$*-примарная группа*, $\sigma_{\mathfrak{I}}$*-нильпотентная группа*, $\sigma_{\mathfrak{I}}$*-разрешимая группа*, $\sigma_{\mathfrak{I}}$*-субнормальная подгруппа*.

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Abstract. In this paper: G is a finite group; $\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$ is some partition of the set of all primes $\mathbb{P}$, where $0 \in I$; $\mathfrak{I}$ is a class of finite σ_0-groups which is closed under extensions, epimorphic images and subgroups and which contains all finite soluble σ_0-groups. A group G is said to be: (i) $\sigma_{\mathfrak{I}}$-primary provided G is a finite σ_i-group for some $i \in I$ and $G \in \mathfrak{I}$ if $i = 0$; (ii) $\sigma_{\mathfrak{I}}$-nilpotent if G is the direct product of $\sigma_{\mathfrak{I}}$-primary groups. A subgroup A of G is said to be $\sigma_{\mathfrak{I}}$-subnormal in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary for all $i = 1,\ldots,t$. In this paper, we study $\sigma_{\mathfrak{I}}$-nilpotent groups, $\sigma_{\mathfrak{I}}$-subnormal subgroups and relations between $\sigma_{\mathfrak{I}}$-nilpotency and $\sigma_{\mathfrak{I}}$-subnormality in the groups. In particular, we prove that a group G is $\sigma_{\mathfrak{I}}$-nilpotent if and only if all its subgroups are $\sigma_{\mathfrak{I}}$-subnormal.

Keywords: *finite group*, $\sigma_{\mathfrak{I}}$*-primary group*, $\sigma_{\mathfrak{I}}$*-nilpotent group*, $\sigma_{\mathfrak{I}}$*-soluble group*, $\sigma_{\mathfrak{I}}$*-subnormal subgroup*.

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1 Introduction

Throughout this paper, all groups are finite, and G always denotes a finite group.

In what follows,

$$\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$$

is some partition of the set of all primes $\mathbb{P}$, where $0 \in I$; $\mathfrak{I}$ is a class of σ_0-groups which is closed under extensions, epimorphic images and subgroups and which contains all soluble σ_0-groups. Moreover, $\pi \subseteq \mathbb{P}$, $\pi' = \mathbb{P} \backslash \pi$ and

$$\Pi \subseteq \sigma, \quad \Pi' = \sigma \setminus \Pi.$$

***Definition* 1.1.** We say that a group G is:

(i) $\sigma_{\mathfrak{I}}$-*primary* provided G is a σ_i-group for some $i \in I$ and $G \in \mathfrak{I}$ if $i = 0$;

(ii) $\sigma_{\mathfrak{I}}$-*nilpotent* if G is the direct product of $\sigma_{\mathfrak{I}}$-primary groups;

(iii) $\sigma_{\mathfrak{I}}$-*soluble* if every chief factor of G is $\sigma_{\mathfrak{I}}$-primary.

***Definition* 1.2.** We say that a subgroup A of G is $\sigma_{\mathfrak{I}}$-*subnormal* in G if there is a subgroup chain

$$A = A_0 \le A_1 \le \cdots \le A_t = G \qquad (1.1)$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary for all $i = 1, \ldots, t$.

***Remark* 1.3.** (i) In what follows, we always omit the symbol $\mathfrak{I}$ in all definitions and notations in the case when $\mathfrak{I}$ is the class of all σ_0-groups.

Thus we say, for example, that G is [1]–[3]: (i) σ-primary if G is a σ_i-group for some $i \in I$; (ii) σ-nilpotent if G is the direct product of σ-primary groups; (iii) σ-soluble if every chief factor of G is σ-primary.

We also say that a subgroup A of G is σ-subnormal in G [1]–[3] if G has a subgroup chain (1.1) such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

(ii) If $2 \notin \sigma_0$, then every σ_0-group is soluble by the Feit-Thompson's theorem, so $\mathfrak{I}$ coinsides with the class of all σ_0-groups since every soluble σ_0-group belongs to $\mathfrak{I}$ by the definition.

In the literature, one can find several special versions of $\sigma_{\mathfrak{I}}$-subnormality. The next results describe the first two of them.

Recall that a subgroup A of G is called $\mathfrak{F}$-*subnormal in G in the sense of Kegel* [4] or K-$\mathfrak{I}$-*subnormal* in G [5, 6.1.4] if there is a subgroup chain

$$A = A_0 \le A_1 \le \cdots \le A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ belongs to $\mathfrak{F}$ for all $i = 1, \ldots, t$.

***Theorem* 1.4.** *If a class of groups $\mathfrak{F}$ is closed under extensions, epimorphic images and subgroups, then the set $\mathcal{L}_{\mathfrak{F}sn}(G)$ $(G \ne 1)$, of all K-$\mathfrak{F}$-subnormal subgroups of G, coinsides with the set of all $\sigma_{\mathfrak{I}}$-subnormal subgroups of G, where $\mathfrak{I} = \mathfrak{F}$ and*

$$\sigma = \{\sigma_0, \{p_1\}, \ldots, \{p_n\}, \ldots\}, \quad \sigma_0 = \bigcup_{X \in \mathfrak{I}} \pi(X),$$

and

$$\{p_1, \ldots, p_n, \ldots\} = \mathbb{P} \setminus \sigma_0.$$

Let

$$\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$$

be some partition of $\mathbb{P}$, where $0 \in I$. Let $\sigma_0 \in \Pi \subseteq \sigma$ and $\mathfrak{I}$ be a class of σ_0-groups which is closed under extensions, epimorphic images and subgroups. Then G is said to be $\Pi_{\mathfrak{I}}$-*primary* [6] provided either $G \in \mathfrak{I}$ or G is a σ_i-group for some $\sigma_i \in \Pi \setminus \{\sigma_0\}$. A subgroup A of G is said to be $\Pi_{\mathfrak{I}}$-*subnormal* in G [6] if there is a subgroup chain

$$A = A_0 \le A_1 \le \cdots \le A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, t$.

Our next observation shows that the $\Pi_{\mathfrak{I}}$-subnormality is also a special case of the $\sigma_{\mathfrak{I}}$-subnormality.

***Theorem* 1.5.** *A subgroup A is $\Pi_{\mathfrak{I}}$-subnormal in G if and only if A is $\sigma_{\mathfrak{I}}$-subnormal in G, where*

$$\sigma = (\Pi \setminus \{\sigma_0\}) \cup \{\{\sigma_0^*\}\} \cup \{\{p_1\}, \ldots, \{p_n\}, \ldots\},$$
$$\sigma_0^* = \bigcup_{X \in \mathfrak{I}} \pi(X),$$

and $\{p_1, \ldots, p_n, \ldots\} = \mathbb{P} \setminus (\sigma_0^* \cup \bigcup_{\sigma_i \in \Pi \setminus \{\sigma_0\}} \sigma_i)$.

A group G is said to be: (i) π-*decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$; (ii) π-*special* [3], [7] if

$$G = F(G) \times O_{\pi'}(G) = O_{p_1}(G) \times \cdots \times O_{p_t}(G) \times O_{\pi'}(G),$$

where $p_1, \ldots, p_t \in \pi$.

In order to consider further useful examples, we first introduce the following three special partitions of the set $\mathbb{P}$:

(i) $\sigma = \sigma^1 := \{\{2\}, \{3\}, \{5\}, \ldots\}$;

(ii) $\sigma = \sigma^\pi := \{\pi, \pi'\}$;

(iii) $\sigma = \sigma^{1,\pi'} := \{\{p_1\}, \ldots, \{p_n\}, \ldots, \pi'\}$, where $\pi = \{p_1, \ldots, p_n, \ldots\}$.

***Example* 1.6.** (i) In the limiting case, when $\sigma = \{\mathbb{P}\}$, $\sigma_0 = \mathbb{P}$, all soluble groups are contained in $\mathfrak{I}$, and every group $G \in \mathfrak{I}$ is $\sigma_{\mathfrak{I}}$-primary and so $\sigma_{\mathfrak{I}}$-nilpotent.

(ii) In the second limiting case, when $\sigma = \sigma^1$, $\sigma_0 = \{p\}$ for some prime p and so $\mathfrak{I}$ is the class of all p-groups. Thus, G is $\sigma_{\mathfrak{I}}$-primary (respectively, $\sigma_{\mathfrak{I}}$-nilpotent, $\sigma_{\mathfrak{I}}$-soluble) if and only if G is a q-group for some prime q (respectively, G is nilpotent, soluble).

A subgroup A of G is $\sigma_{\mathfrak{I}}$-subnormal in G if and only if A is subnormal in G.

(iii) Let $\sigma = \{\sigma_0, \sigma_1\} = \sigma^\pi$, where $\sigma_0 = \pi'$ and $\sigma_1 = \pi$.

Then G is $\sigma_{\mathfrak{I}}$-primary (respectively, $\sigma_{\mathfrak{I}}$-nilpotent) if and only if G is either an $\mathfrak{I}$-group or a σ_1-group (respectively, $G = H_0 \times H_1$, where $H_0 \in \mathfrak{I}$ and H_1 is a Hall σ_1-subgroup of G by Lemma 2.1 below). Moreover, G is $\sigma_{\mathfrak{I}}$-soluble if and only if every chief factor of G is either an $\mathfrak{I}$-group or σ_1-group.

If $2 \in \sigma_1$, then G is $\sigma_{\mathfrak{I}}$-primary (respectively, $\sigma_{\mathfrak{I}}$-nilpotent, $\sigma_{\mathfrak{I}}$-soluble) if and only if G is σ-primary (respectively, σ-nilpotent, σ-soluble). Moreover, in this case a subgroup A of G is $\sigma_{\mathfrak{I}}$-subnormal in G if and only if A is σ-subnormal in G.

(iv) Now, let

$$\sigma = \{\sigma_0, \{p_1\}, \ldots, \{p_n\}, \ldots, \} = \sigma^{1,\pi'},$$

where $\sigma_0 = \pi'$ and

$$\pi = \{p_1, \ldots, p_n, \ldots\}.$$

First assume that $2 \in \pi'$. Then G is $\sigma_{\mathfrak{I}}$-primary (respectively, $\sigma_{\mathfrak{I}}$-nilpotent, $\sigma_{\mathfrak{I}}$-soluble) if and only if G is a $\mathfrak{I}$-group or a p-group for some prime $p \in \pi$ (respectively, $G = F(G) \times H$, where $F(G)$ is a Hall π-subgroup and $H \in \mathfrak{I}$ by Lemma 2.1 below).

Finally, if $2 \in \pi$, then G is $\sigma_{\mathfrak{I}}$-primary (respectively, $\sigma_{\mathfrak{I}}$-nilpotent, $\sigma_{\mathfrak{I}}$-soluible) if and only if G is σ-primary (respectively, σ-nilpotent, σ-soluble).

Let $\mathfrak{F}$ be a class of groups containing all identity groups. Then $\mathfrak{F}$ is said to be:

(i) a *formation* if the following two conditions hold: (1) $\mathfrak{F}$ is closed under taking subdirect products, that is, $G/(N \cap M) \in \mathfrak{F}$ whenever $N, M \trianglelefteq G$ and $G/N, G/M \in \mathfrak{F}$, and (2) $\mathfrak{F}$ is closed under taking homomorphic images, that is, $G/N \in \mathfrak{F}$ whenever $N \trianglelefteq G$ and $G \in \mathfrak{F}$;

(ii) *hereditary* (Mal'cev [8]) if $G \in \mathfrak{F}$ whenever $G \leq A \in \mathfrak{F}$;

(iii) *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$;

(iv) a *Fitting class* if the following two conditions hold: (1) $\mathfrak{F}$ is normally hereditary, that is, $G \in \mathfrak{F}$ whenever $G \trianglelefteq A \in \mathfrak{F}$, and (2) $\mathfrak{F}$ is closed under taking products of normal subgroups, that is, $G \in \mathfrak{F}$ whenever $G = AB$, where $A, B \trianglelefteq G$ and $A, B \in \mathfrak{F}$.

If $K \leq H \leq G$, where $K, H \trianglelefteq G$, then we say that H/K is a *normal factor* or a *normal section* of G.

***Definition* 1.7.** (i) If G is $\sigma_{\mathfrak{I}}$-nilpotent, where

$$\sigma = \{\sigma_0, \sigma_1\} = \sigma^{\pi}$$

and $\sigma_0 = \pi'$, $\sigma_1 = \pi$, then we say that G is $(\pi, \mathfrak{I})$*-decomposable.*

(ii) If G is $\sigma_{\mathfrak{I}}$-nilpotent, where

$$\sigma = \{\sigma_0, \{p_1\}, \ldots, \{p_n\}, \ldots, \} = \sigma^{1,\pi'}$$

and $\sigma_0 = \pi'$, $\pi = \{p_1, \ldots, p_n, \ldots\}$ then we say that G is $(1, \mathfrak{I})$*-special.*

***Definition* 1.8.** (i) If a subgroup A of G is $\sigma_{\mathfrak{I}}$-subnormal in G, where $\sigma = \{\sigma_0, \sigma_1\} = \sigma^{\pi}$ and $\sigma_0 = \pi'$, $\sigma_0 = \pi$, then we say that A is $(\pi, \mathfrak{I})$*-subnormal in G.*

(ii) If a subgroup A of G is $\sigma_{\mathfrak{I}}$-subnormal in G, where

$$\sigma = \{\sigma_0, \{p_1\}, \ldots, \{p_n\}, \ldots, \} = \sigma^{1,\pi'}$$

and

$$\sigma_0 = \pi', \quad \pi = \{p_1, \ldots, p_n, \ldots\},$$

then we say that A is $(1, \mathfrak{I})$*-subnormal in G.*

***Definition* 1.9.** Let H/K be a normal factor of G. Then we say that H/K is:

(i) $\sigma_{\mathfrak{I}}$*-central in G* if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is $\sigma_{\mathfrak{I}}$-primary;

(ii) $(\pi, \mathfrak{I})$*-central in G* if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is either an $\mathfrak{I}$-group or is a π-group;

(iii) $(1, \mathfrak{I})$*-central in G* if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is either an $\mathfrak{I}$-group or a p-group for some prime $p \in \pi$.

Applications of the $\sigma_{\mathfrak{I}}$-nilpotent groups and $\sigma_{\mathfrak{I}}$-subnormal subgroups are based on our three results as follows.

***Theorem* 1.10.** (i) *The class* $\mathfrak{N}_{\sigma_{\mathfrak{I}}}$*, of all* $\sigma_{\mathfrak{I}}$*-nilpotent groups, is both a hereditary saturated formation and a Fitting class.*

(ii) *If G is* $\sigma_{\mathfrak{I}}$*-nilpotent, then every subgroup of G is* $\sigma_{\mathfrak{I}}$*-subnormal in G.*

***Theorem* 1.11.** *Any two of the following conditions are equivalent*:

(i) G is $\sigma_{\mathfrak{I}}$-nilpotent.

(ii) *Every subgroup of G is* $\sigma_{\mathfrak{I}}$*-subnormal in G.*

(iii) *Every maximal subgroup of G is* $\sigma_{\mathfrak{I}}$*-subnormal in G.*

(iv) *G has a* $\sigma_{\mathfrak{I}}$*-subnormal Hall* σ_i*-subgroup for every* $\sigma_i \in \sigma(G)$.

(v) *Every chief factor of G is* $\sigma_{\mathfrak{I}}$*-central in G.*

(vi) *every factor of some normal series of G is* $\sigma_{\mathfrak{I}}$*-central in G.*

In view of Example 1.6 (iii) (iv) we get from Theorem 1.11 the following results.

***Corollary* 1.12.** *A group G is* $(\pi, \mathfrak{I})$*-decomposable if and only if every subgroup of G is* $(\pi, \mathfrak{I})$*-subnormal in G.*

***Corollary* 1.13** (Kegel [4]). *A group G is* $(1, \mathfrak{I})$*-special if and only if every subgroup of G is* $(1, \mathfrak{I})$*-subnormal in G*.

In the case when $\mathfrak{I}$ is the class of all σ_0-groups we get from Corollaries 1.12 and 1.13 the following new results.

***Corollary* 1.14.** *A group G is* π*-decomposable if and only if for every subgroup A of G there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ *or* $A_i / (A_{i-1})_{A_i}$ *is either a* π*-group or a* π'*-group*.

***Corollary* 1.15.** *A group G is* π*-special if and only if for every subgroup A of G there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ *or* $A_i / (A_{i-1})_{A_i}$ *is a* π'*-group*.

We get also from Theorem 1.11 the following results.

***Corollary* 1.16.** *A group G is* $(\pi, \mathfrak{I})$*-decomposable if and only if every chief factor of G is* $(\pi, \mathfrak{I})$*-central in G*.

***Corollary* 1.17.** *A group G is* $(\pi, \mathfrak{I})$*-special if and only if every chief factor of G is* $(1, \mathfrak{I})$*-central in G*.

In the case when $\mathfrak{I}$ is the class of all σ_0-groups we get also from Theorem 1.11 the following known result.

***Corollary* 1.18** (W. Guo, A.N. Skiba [9]). *Any two of the following conditions are equivalent*:

(i) *G is σ-nilpotent*.

(ii) *Every chief factor of G is σ-central in G*.

(iii) *G has a σ-subnormal Hall* σ_i*-subgroup for every* $\sigma_i \in \sigma(G)$.

(iv) *Every subgroup of G is σ-subnormal in G*.

(v) *Every maximal subgroup of G is σ-subnormal in G*.

We use $O^{\Pi}(G)$ to denote the subgroup of *G* generated by all its Π'-subgroups, where $\Pi' = \sigma \setminus \Pi$; $O_{\Pi}(G)$ is the product of all normal Π-subgroups of *G*. We also use $G^{\sigma_{\mathfrak{I}}}$ to denote the $\sigma_{\mathfrak{I}}$*-nilpotent residual* of *G*, that is, the intersection of all normal subgroups *N* of *G* with $\sigma_{\mathfrak{I}}$-nilpotent G / N.

***Theorem* 1.19.** *Let A*, *K and N be subgroups of G*, *where A is* $\sigma_{\mathfrak{I}}$*-subnormal and N is normal in G*.

(i) $A \cap K$ *is* $\sigma_{\mathfrak{I}}$*-subnormal in K*.

(ii) *If K is a* $\sigma_{\mathfrak{I}}$*-subnormal subgroup of A*, *then K is* $\sigma_{\mathfrak{I}}$*-subnormal in G*.

(iii) AN / N *is* $\sigma_{\mathfrak{I}}$*-subnormal in* G / N.

(iv) *If* $N \leq K$ *and* K / N *is* $\sigma_{\mathfrak{I}}$*-subnormal in* G / N, *then K is* $\sigma_{\mathfrak{I}}$*-subnormal in G*.

(v) *If* $K \leq A$ *and A is* $\sigma_{\mathfrak{I}}$*-primary*, *then K is* $\sigma_{\mathfrak{I}}$*-subnormal in G*.

(vi) *The* $\sigma_{\mathfrak{I}}$*-nilpotent residual* $A^{\mathfrak{N}_{\sigma_{\mathfrak{I}}}}$ *of A is subnormal in G*.

(vii) *If A is* $\sigma_{\mathfrak{I}}$*-nilpotent, then* A^G *is* $\sigma_{\mathfrak{I}}$*-nilpotent and* $\sigma(A) = \sigma(A^G)$.

(viii) *If* $\sigma(|G : A|) \subseteq \Pi$, *then* $O^{\Pi}(A) = O^{\Pi}(G)$.

(ix) *If A is a σ-Hall subgroup of G*, *then A is normal in G*.

2 Proofs of the results

***Proof of Theorem* 1.4.** First assume that *A* is *K*-$\mathfrak{I}$-subnormal in *G* and

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G,$$

where either $A_{i-1} \trianglelefteq A_i$ or

$$A_i / (A_{i-1})_{A_i} \in \mathfrak{I}$$

for all $i = 1, \ldots, t$, then either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary. Hence *A* is $\sigma_{\mathfrak{I}}$-subnormal in *G*.

Now assume that *A* is $\sigma_{\mathfrak{I}}$-subnormal in *G* and let

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G,$$

where either $A_{i-1} \trianglelefteq A_i$, or $A_i / (A_{i-1})_{A_i} \in \mathfrak{I}$, or

$$A_i / (A_{i-1})_{A_i}$$

is a *p*-group for some prime *p*. But if $A_i / (A_{i-1})_{A_i}$ is a *p*-group, then A_{i-1} is subnormal in A_i. Therefore there is a subgroup chain

$$A = B_0 \leq B_1 \leq \cdots \leq B_n = G$$

such that either $B_{i-1} \trianglelefteq B_i$ or

$$B_i / (B_{i-1})_{B_i} \in \mathfrak{I}$$

for all $i = 1, \ldots, n$, so *A* is *K*-$\mathfrak{I}$-subnormal in *G*. □

***Proof of Theorem* 1.5.** First suppose that *A* is $\Pi_{\mathfrak{I}}$-subnormal in *G* and let

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G,$$

where $A_{i-1} \trianglelefteq A_i$, or

$$A_i / (A_{i-1})_{A_i} \in \mathfrak{I},$$

or $A_i / (A_{i-1})_{A_i}$ is a σ_j-group for some

$$\sigma_j \in \Pi \setminus \{\sigma_0\}.$$

But in the last two cases, $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary and so *A* is $\sigma_{\mathfrak{I}}$-subnormal in *G*.

Finally, if *A* is $\sigma_{\mathfrak{I}}$-subnormal in *G* and

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G,$$

where $A_i/(A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary for some *i*, then $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary (see Part (i)) and so every $\sigma_{\mathfrak{I}}$-subnormal is also $\Pi_{\mathfrak{I}}$-subnormal in *G*. □

***Lemma* 2.1.** *Let $G \neq 1$ and $\sigma(G)=\{\sigma_{i_1},\ldots,\sigma_{i_t}\}$. Then G is $\sigma_{\mathfrak{I}}$-nilpotent if and only if*

$$G = H_1 \times \cdots \times H_t,$$

where H_j is a Hall σ_{i_j}-subgroup of G for all $j=1,\ldots,t$ and $H_k \in \mathfrak{I}$ for $i_k = 0$.

Proof. Indeed, assume that $G = A_1 \times \cdots \times A_n$, where A_i is $\sigma_{\mathfrak{I}}$-primary for all $i=1,\ldots,n,$ and let H_j be the product of all factors A_l such that A_l is a σ_{i_j}-group. Then H_j is a Hall σ_{i_j}-subgroup of *G* for all *j* and, clearly, $H_k \in \mathfrak{I}$ for $i_k = 0$ since the $\mathfrak{I}$ is closed under extensions and epimorphic images. In particular, $G = H_1 \times \cdots \times H_t$. □

***Proof of Theorem* 1.10.** In view of Lemma 2.1, this theorem can be proved by the direct checking.

***Proof of Theorem* 1.19.** Assume that this theorem is false and let *G* be a counterexample of minimal order. Then $1 \neq A \neq G$ since, in the cases $A=1$ and $A=G$, the Statemets (i)–(ix) trivially hold for *G*.

By hypothesis, there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_r = G$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{I}}$-primary for all $i=1,\ldots,r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$ since $A \neq G$.

(i) Consider the chain

$$K_0 = K \cap A_0 \leq K \cap A_1 \leq \cdots \leq K \cap A_r = K.$$

If A_{i-1} is normal in A_i, then $K \cap A_{i-1}$ is normal in $K \cap A_i$.

Suppose that $A_i/(A_{i-1})_{A_i}$ is an $\mathfrak{I}$-group. Then $(A_i \cap K)(A_{i-1})_{A_i}/(A_{i-1})_{A_i} \simeq (A_i \cap K)/((A_{i-1})_{A_i} \cap K)$ is an $\mathfrak{I}$-group, where $(A_{i-1})_{A_i} \cap K$ is normal in $A_i \cap K$ and so

$$(A_{i-1})_{A_i} \cap K \leq (K \cap A_{i-1})_{K \cap A_i}.$$

Hence $(K \cap A_i)/(K \cap A_{i-1})_{K\cap A_i}$ is an $\mathfrak{I}$-group. Finally, if $A_i/(A_{i-1})_{A_i}$ is a σ_j-group for some $j \neq 0$, then similarly we get that $(K \cap A_i)/(K \cap A_{i-1})_{K\cap A_i}$ is a σ_j-group. Therefore $(K \cap A_i)/(K \cap A_{i-1})_{K\cap A_i}$ is $\sigma_{\mathfrak{I}}$-primary for all $i=1,\ldots,r$, so $A \cap K$ is $\sigma_{\mathfrak{I}}$-subnormal in *K*.

(ii) Assume that *K* is $\sigma_{\mathfrak{I}}$-subnormal in *A* and let

$$K = K_0 \leq K_1 \leq \cdots \leq K_m = A,$$

where either $K_{i-1} \trianglelefteq K_i$ or $K_i/(K_{i-1})_{K_i}$ is $\sigma_{\mathfrak{I}}$-primary for all $i=1,\ldots,m$. Then the subgroup series

$$K = K_0 \leq K_1 \leq \cdots \leq K_m = A = A_0 \leq A_1 \leq \cdots \leq A_r = G$$

has the same property in *G* and so *K* is $\sigma_{\mathfrak{I}}$-subnormal in *G*.

(iii) Consider the chain

$$AN/N = A_0N/N \leq A_1N/N \leq \cdots \leq A_rN/N = G/N.$$

Assume that $A_{i-1}N/N$ is not normal in A_iN/N. Then $L = A_{i-1}$ is not normal in $T = A_i$ and so T/L_T is $\sigma_{\mathfrak{I}}$-primary. Hence

$$(T/L_T)/(L_T(T \cap N)/L_T) =$$
$$= (T/L_T)/((T \cap NL_T)/L_T) \simeq$$
$$T/(T \cap NL_T) \simeq TN/L_TN \simeq (TN/N)/(L_TN/N)$$

is $\sigma_{\mathfrak{I}}$-primary. But

$$L_TN/N \leq (LN/N)_{TN/N}.$$

Hence $(TN/N)/(LN/N)_{TN/N}$ is $\sigma_{\mathfrak{I}}$-primary, so AN/N is $\sigma_{\mathfrak{I}}$-subnormal in G/N.

(iv) Consider the chain

$$K/N = K_0/N \leq K_1/N \leq \cdots \leq K_n/N = G/N$$

be a subgroup chain such that either K_{i-1}/N is normal in K_i/N or $(K_i/N)/(K_{i-1}/N)_{K_i/N}$ is $\sigma_{\mathfrak{F}}$-primary for all $i=1,\ldots,n$. Suppose that K_{i-1} is not normal in K_i. Then K_{i-1}/N is not normal in K_i/N, so

$$(K_i/N)/(K_{i-1}/N)_{K_i/N} =$$
$$= (K_i/N)/((K_{i-1})_{K_i}/N) \simeq K_i/(K_{i-1})_{K_i}$$

is $\sigma_{\mathfrak{F}}$-primary. Hence *K* is $\sigma_{\mathfrak{F}}$-subnormal in *G*.

(v) Since *A* is $\sigma_{\mathfrak{I}}$-primary, every subgroup of *A* is $\sigma_{\mathfrak{I}}$-subnormal in *A*. Thus this assertion is a corollary of Part (ii).

(vi) First we show that $V := A^{\mathfrak{N}_{\sigma_{\mathfrak{I}}}} \leq M_G$. Indeed, if *M* is normal in *G*, it is clear. Now assume that G/M_G is an $\mathfrak{I}$-group. Then from the isomorphism

$$AM_G/M_G \simeq A/(A \cap M_G) \in \mathfrak{I} \subseteq \mathfrak{N}_{\sigma_{\mathfrak{I}}}$$

it follows that $V \leq A \cap M_G$.

Finally, if G/M_G is a σ_j-group for some $j \neq 0$, then similarly we get that $V \leq M_G$.

The choice of *G* implies that *V* is subnormal in *M*, so *A* is subnormal in M_G. Therefore *V* is subnormal in *G*.

(vii) Let $E = NA$. Then $N = N_1 \times \cdots \times N_n$ for some minimal normal subgroups $N_1,\ldots,N_n$ of *E* and N_i is not $\sigma_{\mathfrak{I}}$-primary for all *i* by Lemma 2.1.

Assume that $E < G$. By Part (i), A is $\sigma_{\Im}$-subnormal in E, so due to the choice of G this means that $A \le C_E(N_i)$ for all i. Hence $A \le C_E(N)$.

Finally, assume that $NA = E = G$. Then $N \nleq M$ and hence G/M_G is not $\sigma_{\Im}$-primary, since $N \simeq NM_G/M_G$ is not $\sigma_{\Im}$-primary. This implies that M is normal in G and hence $N \cap M = 1$. Therefore $M \le C_G(N)$ and so $A \le C_G(N)$.

(viii) Since $A \le M$, it follows that $|M : A|$ and $|G : M|$ are Π-numbers. Moreover, A is $\sigma_{\Im}$-subnormal in M, so the choice of G implies that

$$O^{\Pi}(A) = O^{\Pi}(M).$$

Now we show that G/M_G is a Π-number. If M is normal in G, then $M_G = M$ and so $|G : M_G| = |G : M|$ is a Π-number. Now assume that G/M_G is σ_i-group for some $i \in I$. Then $|G/M|$ is a σ_i-number, so $\sigma_i \in \Pi$ since $|G : M|$ is a Π-number. Hence G/M_G is a Π-number.

Now let V be any Π'-subgroup of G. Then from $VM_G/M_G \le G/M_G$ and

$$VM_G/M_G \simeq V/(V \cap M_G)$$

it follows that VM_G/M_G is simultaneously a Π'-group and a Π-group, so VM_G/M_G is the identity group, that is, $V \le M_G$. Therefore every Π'-subgroup of G is contained in M_G, so

$$O^{\Pi}(G) \le O^{\Pi}(M_G) \le O^{\Pi}(M) \le O^{\Pi}(G).$$

Hence

$$O^{\Pi}(G) = O^{\Pi}(M) = O^{\Pi}(A).$$

(ix) By hypothesis, A is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$. Then A^x is a Hall Π-subgroup of G for every $x \in G$, so $A^x \cap A$ is a Hall Π-subgroup of A. Hence $|A : A^x \cap A|$ is a Π'-number and so $A = A^x \cap A$. Therefore $A \le A^x$, so $A = A^x$ for all $x \in G$. □

***Lemma* 2.2** [11, 3.29]. *Let R be an abelian minimal normal subgroup of G such that $G = RM$ for a maximal subgroup M of G. Then*

$$G/M_G \simeq R \rtimes (G/C_G(R)).$$

Now, we are in position to prove Theorem 1.11.

***Proof of Theorem* 1.11.** Since Conditions (i) and (iv) are equivalent by Lemma 2.1 and Theorem 1.19 (ix), (ii) $\Rightarrow$ (iii), and Conditions (v) and (vi) are equivalent by the Jordan – Hölder theorem for the chief series. it is enough to prove the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (vi), (iii) $\Rightarrow$ (i), (v) $\Rightarrow$ (i).

Let $G \ne 1$ and

$$\sigma(G) = \{\sigma_{i_1}, \ldots, \sigma_{i_t}\}.$$

(i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (vi).

Assume that G is $\sigma_{\Im}$-nilpotent, so $G = H_1 \times \cdots \times H_t$, where H_j is a Hall σ_{i_j}-subgroup of G for all $j = 1, \ldots, t$ and $H_k \in \Im$ for $i_k = 0$, by Lemma 2.1.

Suppose that some subgroup A of G is not $\sigma_{\Im}$-subnormal in G. Then $t > 1$ by Theorem 1.19 (v) and $1 < A < G$, so $A \le M$ for some maximal subgroup M of G and for some i, for $i = 1$ say, we have $H_1 \nleq M$.

It follows that

$$H_2 \cdots H_t \le M,$$

so $M/H_2 \cdots H_t$ is $\sigma_{\Im}$-subnormal in $G/H_2 \cdots H_t$ by induction since $G/H_2 \cdots H_t$ is $\sigma_{\Im}$-nilpotent by Theorem 1.10 (i). Therefore M is $\sigma_{\Im}$-subnormal in G by Theorem 1.19 (iv). In view of Theorem 1.10 (i), M is also $\sigma_{\Im}$-nilpotent, so A is $\sigma_{\Im}$-subnormal in M by induction on $|G|$. Therefore A is $\sigma_{\Im}$-subnormal in G Theorem 1.19 (ii). This contradiction shows that (i) $\Rightarrow$ (ii).

Now, consider the normal series

$$1 \le H_1 \le H_1 H_2 \le \cdots \le H_1 \cdots H_{t-1} \le H_1 \times \cdots \times H_t = G$$

of G. If H/K is a chief factor of G such that

$$H_1 \cdots H_{j-1} \le K < H \le H_1 \cdots H_j,$$

then H/K is a σ_{i_j}-group since

$$H_1 \cdots H_i / H_1 \cdots H_{j-1} \simeq H_j,$$

and

$$H_1 \cdots H_{j-1} \le C_G(H/K).$$

On the other hand, from

$$G = (H_1 \cdots H_j) \times H_{j+1} \cdots H_t$$

it follows that

$$H_{j+1} \cdots H_t \le C_G(H/K),$$

so $G/C_G(H/K)$ is a σ_{i_j}-group.

Moreover, if $i_j = 0$, then H/K and $G/C_G(H/K)$ belong to $\Im$, hence

$$(H/K) \rtimes (G/C_G(H/K)) \in \Im.$$

Hence G has a chief series whose all factors are $\sigma_{\Im}$-central in G, so Statement (vi) holds for G.

Now, we show that if every maximal subgroup of G is $\sigma_{\Im}$-subnormal in G, then G is $\sigma_{\Im}$-nilpotent. Assume that this is false. Then G is not $\sigma_{\Im}$-primary and, by Theorem 1.19 (iii), the hypothesis holds for G/R, so G/R is $\sigma_{\Im}$-nilpotent by induction on $|G|$. Therefore R is a unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Theorem 1.10 (i). Hence for some maximal subgroup M of G we have $G = RM$ and $M_G = 1$. It is clear that $M \ne 1$, so M

is not normal in G. Therefore $G \simeq G/M_G$ is $\sigma_{\mathfrak{Z}}$-primary, a contradiction. Hence (iii) $\Rightarrow$ (i).

Next we show that if every chief factor of G is $\sigma_{\mathfrak{Z}}$-central in G, then G is $\sigma_{\mathfrak{Z}}$-nilpotent. Assume that this is false and let R be a minimal normal subgroup of G. It is clear that the hypothesis holds for G/R, so G is $\sigma_{\mathfrak{Z}}$-nilpotent by induction. Therefore R is a unique minimal normal subgroup of G and $R \not\leq \Phi(G)$ by Theorem 1.10 (i). Hence for some maximal subgroup M of G we have $G = RM$ and $M_G = 1$, so $C_G(R) \leq R$ by Lemma 2.2. If R is non-abelian, $C_G(R) = 1$, hence $G \simeq G/C_G(R)$ is $\sigma_{\mathfrak{Z}}$-primary. Therefore $\sigma_{\mathfrak{Z}}$-nilpotent, a contradiction. Hence $C_G(R) = R$ is abelian group, so

$$G \simeq G/M_G \simeq R \rtimes (G/C_G(R))$$

is $\sigma_{\mathfrak{Z}}$-primary by Lemma 2.2, so G is $\sigma_{\mathfrak{Z}}$-nilpotent. Thus, (v) $\Rightarrow$ (i). □

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Информация об авторах

Скиба Александр Николаевич – д.ф.-м.н., профессор