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О $\sigma_{\Im}$-СВОЙСТВАХ КОНЕЧНЫХ ГРУПП I

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ON $\sigma_{\Im}$-PROPERTIES OF FINITE GROUPS I

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Аннотация. В данной работе все группы конечны, G всегда обозначает конечную группу. $\sigma_{\Im}$-Парой называется всякая пара $\sigma, \Im$, где $\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I\}$ – некоторое разбиение множества всех простых чисел и $\Im$ – полный (в смысле Виландта) класс σ_0-групп, содержащий все разрешимые σ_0-группы. Группа G называется: (i) $\sigma_{\Im}$-примарной, если G является σ_i-группой для некоторого $i \in I$ и $G \in \Im$, если $i = 0$; (ii) $\sigma_{\Im}$-нильпотентной, если каждый главный фактор H/K группы G является $\sigma_{\Im}$-центральным в G, т. е. полупрямое произведение $(H/K) \times (G/C_G(H/K))$ является $\sigma_{\Im}$-примарным; (iii) $\sigma_{\Im}$-разрешимой, если каждый главный фактор группы G является $\sigma_{\Im}$-примарным. Под $\sigma_{\Im}$-свойством группы мы понимаем любое из ее свойств, зависящее от $\sigma_{\Im}$ и не подразумевающее никаких ограничений на $\sigma_{\Im}$. В данной работе мы разрабатываем новый аспект теории $\sigma_{\Im}$-свойств, связанный с приложениями $\sigma_{\Im}$-пар. В частности, на множестве всех $\sigma_{\Im}$-пар мы определяем частичный порядок $\leq$ и доказываем, что если $\Sigma = \{\sigma^j_{\Im_j} \mid j \in J\}$ – множество всех $\sigma_{\Im}$-пар таких, что каждая группа в $\mathfrak{X}$ является $\sigma^j_{\Im_j}$-нильпотентной (соответственно, $\sigma^j_{\Im_j}$-разрешимой) для всех j, то относительно $\leq$ существует наименьший элемент в Σ.

Ключевые слова: *конечная группа*, $\sigma_{\Im}$*-свойство группы*, $\sigma_{\Im}$*-пара*, $\sigma_{\Im}$*-примарная группа*, *решетка* $\sigma_{\Im}$*-пар*.

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Abstract. In this paper all groups are finite, G always denotes a finite group. A $\sigma_{\Im}$-pair is any pair $\sigma, \Im$, where $\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I \subseteq \mathbb{N}\}$ is some partition of the set of all primes and $\Im$ is a complete (in the sence of Wielandt) class of σ_0-groups which contains all soluble σ_0-groups. A group G is said to be: (i) $\sigma_{\Im}$-primary provided G is a σ_i-group for some $i \in I$ and $G \in \Im$ if $i = 0$; (ii) $\sigma_{\Im}$-nilpotent if every chief factor H/K of G is $\sigma_{\Im}$-central in G, that is, the semidirect $(H/K) \times (G/C_G(H/K))$ is $\sigma_{\Im}$-primary; (iii) $\sigma_{\Im}$-soluble if every chief factor of G is $\sigma_{\Im}$-primary. By a $\sigma_{\Im}$-property of a group we mean any of its properties, that depends on $\sigma_{\Im}$ and which does not imply any restrictions on $\sigma_{\Im}$. In this paper, we develop a new aspect of the theory of $\sigma_{\Im}$-properties related to applications of $\sigma_{\Im}$-pairs. In particular, on the set of all $\sigma_{\Im}$-pairs we define a partial order $\leq$ and we prove that if $\Sigma = \{\sigma^j_{\Im_j} \mid j \in J\}$ is the set of all $\sigma_{\Im}$-pairs such that every group in $\mathfrak{X}$ is $\sigma^j_{\Im_j}$-nilpotent (respectively, $\sigma^j_{\Im_j}$-soluble) for all j, then with respect to $\leq$ there exists the least element in Σ.

Keywords: *finite group*, $\sigma_{\Im}$*-property of a group*, $\sigma_{\Im}$*-pair*, $\sigma_{\Im}$*-primary group*, *the lattice of* $\sigma_{\Im}$*-pairs*.

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Introduction

Throughout this paper, all groups are finite and G always denotes a finite group; $\pi(G)$ is the set all primes dividing the order $|G|$ of G. We say, following Wielandt [1], that a class of groups $\mathfrak{F}$ is *complete* provided $\mathfrak{F}$ is closed under taking extensions, epimorphic images and subgroups.

***Definition* 0.1.** (i) A $(\sigma, \Im)$-*pair* (or, for brevity, a $\sigma_\Im$-*pair*) is any pair $\sigma, \Im$, where

$$\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I\}$$

is some partition of the set of all primes $\mathbb{P}$ and $\Im$ is a complete class of σ_0-groups which contains all soluble σ_0-groups.

(ii) We will also use the term a $\sigma_\Im$-*pair* as a general name for pairs of this kind, and if $\sigma^* = \{\sigma_i^* \mid i \in I\}$ is any partition of $\mathbb{P}$, then we use $\sigma^*(G)$, following [2], [3], to denote the set $\{\sigma_i^* \mid \sigma_i^* \cap \pi(G) \neq \varnothing\}$.

(iii) By a $\sigma_\Im$-*property* of a group we mean any of its properties, that depends on $\sigma_\Im$ and which does not imply any restrictions on $\sigma_\Im$.

In particular, a σ-property of a group is any of its properties, that depends on the partition σ of $\mathbb{P}$ and which does not imply any restrictions on σ [4]–[7].

The study of the $\sigma_\Im$-properties goes back to the author's reviews [2], [8] and the recent papers [9], [10].

In this paper, we develop a new aspect of the theory of $\sigma_\Im$-properties related to applications of $\sigma_\Im$-pairs.

***Definition* 0.2.** We say that a group G is: (i) $\sigma_\Im$-*primary* provided G is a σ_i-group for some $i \in I$ and $G \in \Im$ if $i = 0$;

(ii) $\sigma_\Im$-*nilpotent* if every chief factor H/K of G is $\sigma_\Im$-central in G, that is, the semidirect $(H/K) \times (G/C_G(H/K))$ is $\sigma_\Im$-primary;

(iii) $\sigma_\Im$-*soluble* if every chief factor of G is $\sigma_\Im$-primary.

***Definition* 0.3.** We say that a subgroup A of G is $\sigma_\Im$-*subnormal* in G if there is a subgroup chain

$$A = A_0 \le A_1 \le \cdots \le A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_\Im$-primary for all $i = 1, \ldots, t$.

***Remark* 0.4.** (i) In what follows, we always omit the symbol $\Im$ in all definitions and notations in the case when $\Im$ is *the class of all* σ_0-*groups* (note, in passing, that in view of the Burnside's $p^a q^b$-theorem and the Feit – Thompson theorem, we always deal with such a situation when either $|\sigma_0| \le 2$ or $2 \notin \sigma_0$).

Thus we say, in this case, that G is [5]: σ-primary if G is a σ_i-group for some $i \in I$; σ-nilpotent if every chief factor H/K of G is σ-central in G; σ-soluble if every chief factor of G is σ-primary.

We say also that a subgroup A of G is σ-subnormal in G [5] if there is a subgroup chain $A = A_0 \le A_1 \le \cdots \le A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

(ii) In view of Part (i), we will always suppose that $2 \in \sigma_0$.

(iii) Let $G \neq 1$ and $\sigma(G) = \{\sigma_{i_1}, \ldots, \sigma_{i_t}\}$. Then G is $\sigma_\Im$-nilpotent if and only if $G = H_1 \times \cdots \times H_t$, where H_j is a Hall σ_{i_j}-subgroup of G for all $j = 1, \ldots, t$ and $H_k \in \Im$ for $i_k = 0$ (see Theorem 0.19 (ix) and Lemma 1.1 in [9].

***Definition* 0.5.** We say that: (i) a collection of groups $\mathfrak{X}$ is $\sigma_\Im$-nilpotent ($\sigma_\Im$-soluble) provided every group in $\mathfrak{X}$ is $\sigma_\Im$-nilpotent (respectively, $\sigma_\Im$-soluble);

(ii) a set of subgroups $\mathfrak{X}$ of G is $\sigma_\Im$-subnormal in G provided every subgroup $A \in \mathfrak{X}$ is $\sigma_\Im$-subnormal in G.

***Example* 0.6.** (i) In the limiting case, when $\sigma = \{\mathbb{P}\}$, $\sigma_0 = \mathbb{P}$ and all soluble groups are contained in $\Im$.

A group G is $\sigma_\Im$-primary (respectively, $\sigma_\Im$-nilpotent, $\sigma_\Im$-soluble) if and only if $G \in \Im$.

A subgroup A of G is $\sigma_\Im$-subnormal in G if and only if there is a subgroup chain $A = A_0 \le A_1 \le \cdots \le A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i} \in \Im$ for all $i = 1, \ldots, n$.

(ii) If the second limiting case, when $\sigma = \sigma^1 := \{\{2\}, \{3\}, \{5\}, \ldots\}$, $\sigma_0 = \{2\}$ and so $\Im$ is the class of all 2-groups.

A group G is σ^1-primary (respectively, σ^1-nilpotent, σ^1-soluble) if and only if G is a p-group for some prime p (respectively, G is nilpotent, soluble). A subgroup A of G is σ^1-subnormal in G if and only if A is subnormal in G.

Finally, in view of Remark 0.4 (i), a group G is $\sigma_\Im$-primary (respectively, $\sigma_\Im$-nilpotent, $\sigma_\Im$-soluble) if and only if G is a p-group for some prime p (respectively, G is nilpotent, soluble). A subgroup A of G is $\sigma_\Im$-subnormal in G if and only if A is subnormal in G.

(iii) Let $|\sigma| = 2$ and $\sigma = \sigma^{\pi, \pi'} := \{\sigma_0, \sigma_1\}$, then G is $\sigma_\Im^{\pi, \pi'}$-primary (respectively, $\sigma_\Im^{\pi, \pi'}$-soluble, $\sigma_\Im^{\pi, \pi'}$-nilpotent) if and only if G is a $\Im$-group or a σ_1-group (respectively, every chief factor of G is an $\Im$-group or a σ_1-group, $G = H_0 \times H_1$, where

$H_0 \in \Im$ and H_1 is a Hall σ_1-subgroup of G by Remark 0.3 (iii)).

A subgroup A if G $\sigma_{\Im}^{\pi,\pi'}$-subnormal in G if and only if there is a chain $A = A_0 \le A_1 \le \cdots \le A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is an $\Im$-group or a σ_1-group for all $i = 1, \ldots, n$.

In particular, G is $\sigma^{\pi,\pi'}$-primary (respectively, $\sigma^{\pi,\pi'}$-soluble, $\sigma^{\pi,\pi'}$-nilpotent) if and only if G is a σ_0-group or a σ_1-group (respectively, G is σ_0-se-parable, G is σ_0-decomposable, that is, $G = O_{\pi}(G) \times O_{\pi'}(G)$).

A subgroup A if G $\sigma^{\pi,\pi'}$-subnormal in G if and only if there is a chain $A = A_0 \le A_1 \le \cdots \le A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is a σ_0-group or a σ_1-group for all $i = 1, \ldots, n$.

(iv) Now, let $\sigma = \sigma^{1,\pi'} := \{\sigma_0, \{p_1\}, \{p_2\}, \ldots\}$, where $\sigma_0 = \pi'$ and $\pi = \{p_1, p_2, \ldots\}$.

Then G is $\sigma^{1,\pi'}{}_{\Im}$-primary (respectively, $\sigma^{1,\pi'}{}_{\Im}$-soluble, $\sigma^{1,\pi'}{}_{\Im}$-nilpotent) if and only if G is an $\Im$-group or a p-group for some $p \in \pi$ (respectively, every chief factor of G is an $\Im$-group or p-group for some $p \in \pi$, $G = F(G) \times H$, where $F(G)$ is a nilpotent Hall π-subgroup of G and $H \in \Im$ by Remark 0.3 (iii)).

(v) Finally, let $\sigma_0 = \{2,3,5\}$ and $\Im$ be the class of all σ_0-groups G such that every non-abelian composition factor of G is isomorphic to the alternating group A_5. Then the class $\Im$ is closed under extensions, epimorphic images and subgroups and it contains all soluble σ_0-groups. Now let $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$. Then the group A_5 is $\sigma_{\Im}$-primary and the group A_6 is σ-primary and so σ-nilpotent. On the other hand, $A_6 \notin \Im$, so A_6 is not $\sigma_{\Im}$-primary and hence A_6 is not $\sigma_{\Im}$-soluble.

***Remark* 0.7.** The *usefulness* and *universality* of the above defined concepts are primarily due to the fact that, in view of Example 0.6 (i), any family of groups $\mathfrak{X}$ is $\sigma^{*}_{\Im^{*}}$-nilpotent (respectively, $\sigma^{*}_{\Im^{*}}$-soluble) and any set of subgroups of G is $\sigma^{**}_{\Im^{**}}$-subnormal in G, for some $\sigma^{*}_{\Im^{*}}$-pair and $\sigma^{**}_{\Im^{**}}$-pair.

In view of Remark 0.7, the following questions are quite natural.

***Problem* 0.8.** *Let $\mathfrak{X}$ be an arbitrary collection of groups.*

(i) *Is there*, *in some sense*, *a "minimal" $\sigma_{\Im}$-pair such that $\mathfrak{X}$ is $\sigma_{\Im}$-nilpotent* (*respectively*, *$\sigma_{\Im}$-soluble*)? And if it is so,

(ii) *what then is this minimal $\sigma_{\Im}$-pair?*

***Problem* 0.9.** *Let $\mathfrak{X}$ be an arbitrary collection of groups.*

(i) *Is there*, *in some sense*, *a "minimal" partition σ of $\mathbb{P}$ such that $\mathfrak{X}$ is σ-nilpotent* (*respectively*, *σ-soluble*)? And if it is so,

(ii) *what then is this minimal partition σ*?

***Problem* 0.10.** *Let $\mathfrak{X}$ be a set of subgroups of G.*

(i) *Is there*, *in some sense*, *a "minimal" $\sigma_{\Im}$-pair such that $\mathfrak{X}$ is $\sigma_{\Im}$-subnormal in G*? And if it is so,

(ii) *what then is this minimal $\sigma_{\Im}$-pair*?

***Problem* 0.11.** *Let $\mathfrak{X}$ be a set of subgroups of G.*

(i) *Is there*, *in some sense*, *a "minimal" partition σ of $\mathbb{P}$ such that $\mathfrak{X}$ is σ-subnormal in G*? And if it is so,

(ii) *what then is this minimal partition σ*?

In this paper, we give a partial answer to these questions.

Finally, note that the results of this work go back to some observations in [11], and partially, they are published in the preprint [12] and in the paper [13].

In [14], interesting computational aspects of the theory of σ-properties were studied. In particular, it was proved that in a permutation group of degree n, in polynomial time $t = t(n)$, one can find the smallest partition σ of $\mathbb{P}$ for which the group G is σ-nilpotent (σ-soluble).

1 The lattices of $\sigma_{\Im}$-pairs

We use $\sigma_{\Im}(\mathbb{P})$ to denote the set of all $\sigma_{\Im}$-pairs. In particular, $\sigma(\mathbb{P})$ denotes the set of all partitions of $\mathbb{P}$.

***Definition* 1.1.** For any two $\sigma_{\Im}$-pairs $\sigma^{*}_{\Im^{*}}$ and $\sigma^{**}_{\Im^{**}}$, where $\sigma^{*} = \{\sigma^{*}_{i} \mid i \in I^{*}\}$ and $\sigma^{**} = \{\sigma^{**}_{j} \mid j \in I^{**}\}$, we write $\sigma^{*}_{\Im^{*}} \le \sigma^{**}_{\Im^{**}}$ provided $\Im^{*} \subseteq \Im^{**}$ and for any $i \in I^{*} \setminus \{0\}$ there exists $j \in I^{**}$ such that $\sigma^{*}_{i} \subseteq \sigma^{**}_{j}$.

In particular, we write $\sigma^{*} \le \sigma^{**}$ provided for any $i \in I^{*}$ there exists $j \in I^{**}$ such that $\sigma^{*}_{i} \subseteq \sigma^{**}_{j}$.

***Remark* 1.2.** In fact, $\sigma^{*}_{\Im^{*}} \le \sigma^{**}_{\Im^{**}}$ if and only if $\Im^{*} \subseteq \Im^{**}$ and $\sigma^{*} \le \sigma^{**}$.

Indeed, since

$$\sigma^{*}_{0} = \pi(\Im^{*}) \subseteq \pi(\Im^{**}) = \sigma^{**}_{0},$$

so from $\sigma^{*}_{\Im^{*}} \le \sigma^{**}_{\Im^{**}}$ it follows that $\sigma^{*} \le \sigma^{**}$.

Let $\Im^{*} = \mathfrak{G}$ be the class of all groups and $\Im^{**} = \mathfrak{G}_2$ the class of all 2-groups.

***Proposition* 1.3.** *$\langle \sigma_{\Im}(\mathbb{P}), \le \rangle$ is a partially ordered set with greatest element $\{\mathbb{P}\}_{\Im^{*}}$ and least element $\{\{2\}, \{3\}, \{5\}, \ldots\}_{\Im^{**}}$. In particular, $\langle \sigma(\mathbb{P}), \le \rangle$*

is a partially ordered set with greatest element $\{\mathbb{P}\}$ *and least element* $\{\{2\},\{3\},\{5\},\ldots\}$.

Proof. It is clear that $\leq$ is reflexive and, in view of Remark 1.2, transitive.

Now, note that if

$$\sigma^{*}_{\mathfrak{I}^{*}} \leq \sigma^{**}_{\mathfrak{I}^{**}} \leq \sigma^{*}_{\mathfrak{I}^{*}},$$

then $\mathfrak{I}^{*} \subseteq \mathfrak{I}^{**} \subseteq \mathfrak{I}^{*}$, so $\mathfrak{I}^{*} = \mathfrak{I}^{**}$.

On the other hand, if $i, i_1 \in I^{*}$ and $j \in I^{**}$ such that

$$\sigma^{*}_{i} \subseteq \sigma^{**}_{j} \subseteq \sigma^{*}_{i_1},$$

then $\sigma^{*}_{i} = \sigma^{*}_{i_1} = \sigma^{**}_{j}$, so for each $i \in I^{*}$ there is $j \in I^{**}$ such that $\sigma_{i} = \sigma^{*}_{j}$. Similarly, for each $j \in I^{**}$ there is $i \in I^{*}$ such that $\sigma^{**}_{j} = \sigma^{*}_{i}$. Hence $\sigma^{*} = \sigma^{**}$, so $\leq$ is antisymmetric.

Thus, $\langle \sigma_{\mathfrak{I}}(\mathbb{P}), \leq \rangle$ is a partially ordered set with greatest element $\{\mathbb{P}\}_{\mathfrak{I}^{*}}$ and least element $\{\{2\},\{3\},\{5\},\ldots\}_{\mathfrak{I}^{**}}$ (see Example 1.6 (i) (ii)).

The proposition is proved.

***Proposition* 1.4.** *Let* $\{\sigma^{j}_{\mathfrak{I}_j} \mid j \in J\}$ *be any set of* $\sigma_{\mathfrak{I}}$*-pairs, where* $\sigma^{j} = \{\sigma^{j}_{k_j} \mid k_j \in K_j\}$ *for all* $j \in J$. *Let* $\mathfrak{I} := \bigcap_{j\in J} \mathfrak{I}_j$ *and let* $\{\bigcap_{j\in J} \sigma^{j}_{k_j}\}$ *be the set of all intersections of the form* $\bigcap_{j\in J} \sigma^{j}_{k_j}$. *Then the following statements hold*:

(i) $\sigma := \bigcap_{j\in J} \sigma^{j}$ *and* $\mathfrak{I}$ *form a* $\sigma_{\mathfrak{I}}$*-pair (denoted by* $\bigcap_{j\in J} \sigma^{j}_{\mathfrak{I}_j}$*).*

In particular, for any two $\sigma_{\mathfrak{I}}$*-pairs* $\sigma^{*}_{\mathfrak{I}^{*}}$ *and* $\sigma^{**}_{\mathfrak{I}^{**}}$, *where* $\sigma^{*} = \{\sigma^{*}_{i} \mid i \in I^{*}\}$ *and* $\sigma^{**} = \{\sigma^{**}_{j} \mid j \in I^{**}\}$,

$$\sigma^{*}_{\mathfrak{I}^{*}} \cap \sigma^{**}_{\mathfrak{I}^{**}}$$

is a $\sigma_{\mathfrak{I}}$*-pair, where*

$$\sigma = \{\sigma^{*}_{i} \cap \sigma^{**}_{j} \mid i \in I^{*}, j \in I^{**}\}$$

and $\mathfrak{I} = \mathfrak{I}^{*} \cap \mathfrak{I}^{**}$;

(ii) $\sigma_{\mathfrak{I}} := \bigcap_{j\in J} \sigma^{j}_{\mathfrak{I}_j}$ *is the greatest of the lower bounds for* $\{\sigma^{j}_{\mathfrak{I}_j} \mid j \in J\}$ *in* $\sigma_{\mathfrak{I}}(\mathbb{P})$. *In particular,* $\sigma_{\mathfrak{I}}(\mathbb{P})$ *is a complete lattice*;

(iii) *if* $\mathfrak{X}$ *is* $\sigma^{j}_{\mathfrak{I}_j}$*-nilpotent* (*respectively,* $\sigma^{j}_{\mathfrak{I}_j}$*-soluble*) *for all* $j \in J$, *then* $\mathfrak{X}$ *is* $\sigma_{\mathfrak{I}}$*-nilpotent* (*respectively,* $\sigma_{\mathfrak{I}}$*-soluble*);

(iv) *if* $\mathfrak{X} \subseteq L(G)$ *and* $\mathfrak{X}$ *is* $\sigma^{j}_{\mathfrak{I}}$*-subnormal in* G *for all* $j \in J$, *then* $\mathfrak{X}$ *is* $\sigma_{\mathfrak{I}}$*-subnormal in* G.

Proof. See the proof of Proposition 1.6.7 in [12].

***Corollary* 1.5.** *Let* $\{\sigma^{j} \mid j \in J\}$ *be any set partitions of* $\mathbb{P}$, *where* $\sigma^{j} = \{\sigma^{j}_{k_j} \mid k_j \in K_j\}$ *for all* $j \in J$. *Let* $\{\bigcap_{j\in J} \sigma^{j}_{k_j}\}$ *be the set of all intersections of the form* $\bigcap_{j\in J} \sigma^{j}_{k_j}$. *Then the following statements hold*:

(i) $\sigma := \bigcap_{j\in J} \sigma^{j}$ *is a partition of* $\mathbb{P}$ (*denoted by* $\bigcap_{j\in J} \sigma^{j}$).

In particular, for any two partition $\sigma_{\mathfrak{I}}$*-pairs* σ^{*} *and* σ^{**} *of* $\mathbb{P}$, *where* $\sigma^{*} = \{\sigma^{*}_{i} \mid i \in I^{*}\}$ *and* $\sigma^{**} = \{\sigma^{**}_{j} \mid j \in I^{**}\}$,

$$\sigma^{*} \cap \sigma^{**}$$

is a partition of $\mathbb{P}$, *where*

$$\sigma = \{\sigma^{*}_{i} \cap \sigma^{**}_{j} \mid i \in I^{*}, j \in I^{**}\};$$

(ii) $\sigma := \bigcap_{j\in J} \sigma^{j}$ *is the greatest of the lower bounds for* $\{\sigma^{j} \mid j \in J\}$ *in* $\sigma(\mathbb{P})$. *In particular,* $\sigma(\mathbb{P})$ *is a complete lattice*;

(iii) *if* $\mathfrak{X}$ *is* σ^{j}*-nilpotent* (*respectively,* σ^{j}*-soluble for all* $j \in J$, *then* $\mathfrak{X}$ *is* σ*-nilpotent* (*respectively,* σ*-soluble*);

(iv) *if* $\mathfrak{X} \subseteq L(G)$ *and* $\mathfrak{X}$ *is* σ^{j}*-subnormal in* G *for all* $j \in J$, *then* $\mathfrak{X}$ *is* σ*-subnormal in* G.

2 The $\mathfrak{X}$-nilpotent and the $\mathfrak{X}$-soluble $\sigma_{\mathfrak{I}}$-pairs

We write $\mathfrak{N}_{\sigma_{\mathfrak{I}}}$ (respectively, $\mathfrak{S}_{\sigma_{\mathfrak{I}}}$) to denote the class of all $\sigma_{\mathfrak{I}}$-nilpotent (respectively, $\sigma_{\mathfrak{I}}$-soluble) groups; $\mathfrak{N}$ (respectively, $\mathfrak{S}$) is the class of all nilpotent (respectively, soluble) groups.

***Definition* 2.1.** (i) If $\Sigma = \{\sigma^{j}_{\mathfrak{I}_j} \mid j \in J\}$ is the set of all $\sigma_{\mathfrak{I}}$-pairs such that $\mathfrak{X} \subseteq \mathfrak{N}_{\sigma^{j}_{\mathfrak{I}_j}}$ (respectively, $\mathfrak{X} \subseteq \mathfrak{S}_{\sigma^{j}_{\mathfrak{I}_j}}$) for all $j \in J$, then we write $\Sigma_{n,\mathfrak{I}}(\mathfrak{X})$ (respectively, $\Sigma_{s,\mathfrak{I}}(\mathfrak{X})$) to denote the $\sigma_{\mathfrak{I}}$-pair $\bigcap_{j\in J} \sigma^{j}_{\mathfrak{I}_j}$.

(ii) If $\Sigma = \{\sigma^{j} \mid j \in J\}$ is the set of all partitions of $\mathbb{P}$ such that $\mathfrak{X} \subseteq \mathfrak{N}_{\sigma^{j}}$ (respectively, $\mathfrak{X} \subseteq \mathfrak{S}_{\sigma^{j}}$) for all $j \in J$, then we write $\Sigma_{n}(\mathfrak{X})$ (respectively, $\Sigma_{s}(\mathfrak{X})$) to denote the partition $\bigcap_{j\in J} \sigma^{j}$ of $\mathbb{P}$.

By Proposition 1.4 (iii), $\Sigma_{n,\mathfrak{I}}(\mathfrak{X}) \in \Sigma$ (respectively, $\Sigma_{s,\mathfrak{I}}(\mathfrak{X}) \in \Sigma$) and $\Sigma_{n,\mathfrak{I}}(\mathfrak{X})$ (respectively, $\Sigma_{s,\mathfrak{I}}(\mathfrak{X})$) is the smallest element in Σ.

By Corollary 1.5 (iii), $\Sigma_n(\mathfrak{X}) \in \Sigma$ (respectively, $\Sigma_s(\mathfrak{X}) \in \Sigma$) and $\Sigma_n(\mathfrak{X})$ (respectively, $\Sigma_s(\mathfrak{X})$) is the smallest element in Σ.

***Definition* 2.2.** (i) We say that $\Sigma_{n,\mathfrak{J}}(\mathfrak{X})$ is the $\mathfrak{X}$*-nilpotent* $\sigma_{\mathfrak{J}}$-pair and $\Sigma_{s,\mathfrak{J}}(\mathfrak{X})$ is the $\mathfrak{X}$*-soluble* $\sigma_{\mathfrak{J}}$-pair.

(ii) We say that $\Sigma_n(\mathfrak{X})$ is the $\mathfrak{X}$*-nilpotent* partition and $\Sigma_s(\mathfrak{X})$ is the $\mathfrak{X}$*-soluble* partition of $\mathbb{P}$.

It is clear that $\Sigma_{n,\mathfrak{J}}(\varnothing) = \sigma^1 = \Sigma_{s,\mathfrak{J}}(\varnothing)$ and $\Sigma_n(\varnothing) = \sigma^1 = \Sigma_s(\varnothing)$.

If $\mathfrak{X} = \{G\}$, we write $\Sigma_{n,\mathfrak{J}}(G)$, $\Sigma_{s,\mathfrak{J}}(G)$ and $\Sigma_n(G)$, $\Sigma_s(G)$.

If $G \notin \mathfrak{N}_{\sigma_{\mathfrak{J}}}$ but every proper subgroup of G belongs to $\mathfrak{N}_{\sigma_{\mathfrak{J}}}$, then G is called an $\mathfrak{N}_{\sigma_{\mathfrak{J}}}$*-critical* or a *minimal non-*$\sigma_{\mathfrak{J}}$*-nilpotent* group. If G is not nilpotent but every proper subgroup of G is nilpotent, then G is said to be a *Schmidt group*.

***Lemma* 2.3.** *Suppose that G is a minimal non-$\sigma_{\mathfrak{J}}$-nilpotent group, then G is either a Schmidt group or a non-abelian simple minimal non-$\mathfrak{J}$-group.*

Proof. Let $\sigma = \{\sigma_i \mid i \in I.$ It is clear that G is not a $\sigma_{\mathfrak{J}}$-primary group, so G is not a σ_i-group for all $i \neq 0$.

First assume that G is a σ_0-group. Then G is a minimal non-$\mathfrak{J}$-group, since every proper subgroup of G is $\sigma_{\mathfrak{J}}$-nilpotent and so it is $\sigma_{\mathfrak{J}}$-primary. Moreover, G is a non-abelian simple group since $\mathfrak{J}$ is a complete class of groups and so it is closed under extensions.

Now assume that G is not a σ_0-group, so G is not a σ_i-group for all $i \in I$. In view of Remark 0.4(iii), a group is σ-nilpotent if and only if G is σ_i-decomposable for all $i \in I$. Therefore, for some i, G is a minimal non-σ_i-decomposable group and so G is a Schmidt group by [15]. □

***Corollary* 2.4.** *If G is a minimal non-σ-nilpotent group, then G is a Schmidt group.*

Now, returning to Problem 0.8, we see that:

(i) a "minimal" $\sigma_{\mathfrak{J}}$-pair for which the family $\mathfrak{X}$ is $\sigma_{\mathfrak{J}}$-nilpotent (respectively, $\sigma_{\mathfrak{J}}$-soluble) exists and $\sigma_{\mathfrak{J}}$ coincides with $\Sigma_{n,\mathfrak{J}}(\mathfrak{X})$ (respectively, with $\Sigma_{s,\mathfrak{J}}(\mathfrak{X})$);

(ii) a "minimal" partition σ of $\mathbb{P}$ for which the family $\mathfrak{X}$ is σ-nilpotent (respectively, σ-soluble) exists and σ coincides with $\Sigma_n(\mathfrak{X})$ (respectively, with $\Sigma_s(\mathfrak{X})$).

Our next goal is to give an answer to the second part of Problem 0.8.

***Definition* 2.5.** Let $\mathfrak{X}$ be a (possibly empty) family of groups. Then we define the sets of groups $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$ and $S^0 = S^0(\mathfrak{X})$ as follows:

(i) $A \in S$ if and only if A is a Schmidt group and there is a group $G \in \mathfrak{X}$ and a section H/K of G such that $A \simeq H/K$;

(ii) $A \in S^*$ if and only if A is a non-abelian simple group and there is a group $G \in \mathfrak{X}$ and a section H/K of G such that $A \simeq H/K$.

(iii) $A \in S^0$ if and only if $A \in S$ and $\pi(A) \cap \pi(S^*) \neq \varnothing$.

If $\mathfrak{X} = \{G\}$, we write $S(G)$ (respectively, $S^*(G)$, $S^0(G)$).

It is clear that if $\mathfrak{X}$ is nilpotent (respectively, soluble), then $S(\varnothing) = \varnothing$ (respectively, $S^*(\varnothing) = \varnothing$). In particular, $S(\varnothing) = \varnothing = S^*(\varnothing)$.

***Definition* 2.6.** (i) For any set of Schmidt groups $S = \{A_k \mid k \in K\}$ and any set of non-abelian simple groups S^* we define:

(i) a partition $\sigma_{S,S^*} = \{\sigma_i \mid i \in I = \{0\} \cup I\}$ of $\mathbb{P}$, where $\sigma_0 = \pi(S^* \cup S^0)$ and for any σ_i such that $\sigma_i \cap \sigma_0 = \varnothing$ and $p \neq q \in \sigma_i$ there exist groups $A_{k_1}, \ldots, A_{k_t}$ in S such that $p \in \pi(A_{k_1})$, $q \in \pi(A_{k_t})$ and $\pi(A_{k_{j-1}}) \cap \pi(A_{k_j}) \neq \varnothing$ for $j = 2, \ldots, t$;

(ii) a partition $\sigma_{S^*} = \{\sigma_j \mid j \in J = \{0\} \cup J\}$ of $\mathbb{P}$, where $\sigma_0 = \pi(S^*)$ and $\sigma_j = \{p\}$, where p is a prime, for all $j \neq 0$;

(iii) a partition $\sigma_S = \{\sigma_k \mid k \in K$ of $\mathbb{P}$, where for any σ_i such that $|\sigma_k| > 1$ and $p, q \in \sigma_i$ there exist groups $A_{k_1}, \ldots, A_{k_t}$ in S such that $p \in \pi(A_{k_1})$, $q \in \pi(A_{k_t})$ and $\pi(A_{k_{j-1}}) \cap \pi(A_{k_j}) \neq \varnothing$ for $j = 2, \ldots, t$;

(iv) a complete set of groups $\mathfrak{J}_{S^*}$ such that it is the intersection of all complete sets of groups containing $S^* \cup S^0$.

***Lemma* 2.7.** *Let $\mathfrak{X}$ be a non-soluble family of groups. Let $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$ and $S^0 = S^0(\mathfrak{X})$. Then σ_{S,S^*}, $\mathfrak{J}_{S^0}$ and σ_{S^*}, $\mathfrak{J}_{S^*}$ form $\sigma_{\mathfrak{J}}$-pairs.*

Proof. First note that $\sigma_0 = \pi(S^* \cup S^0) = \pi(\mathfrak{J}_{S^*})$ since the class of all $\pi(S^* \cup S^0)$-groups is a complete set of subgroups and so, in fact, we have only to show that every soluble σ_0-group G belongs to $\mathfrak{J}_{S^*}$.

Let $\mathfrak{J}^*$ be any complete set of groups containing $S^* \cup S^0$. Then every chief factor of G belongs to $\mathfrak{J}^*$ since $\sigma_0 \subseteq \pi(\mathfrak{J}^*)$ and $\mathfrak{J}^*$ is hereditary, so G belongs to $\mathfrak{J}^*$ since $\mathfrak{J}^*$ is closed with respect to the extension of its groups. Hence $G \in \mathfrak{I}_{S^*}$.

It can be proved similarly, that σ_{S^*} and $\mathfrak{I}_{S^*}$ form a $\sigma_{\mathfrak{I}}$-pair. □

***Example* 2.8.** (i) By definition, in the case when $S = \varnothing$, we have $\sigma_S = \{\{2\}, \{3\}, \{5\}, \ldots\}$.

(ii) If $A = A_p \rtimes A_q$ is a Schmidt group and $S = \{A\}$, then $\sigma_S = \{\{p, q\}, \{r\}, \{s\}, \ldots\}$, where $\{r, s, \ldots\} \cap \{p, q\} = \varnothing$.

(iii) Let S be the set of all Schmidt groups. Then $\sigma_S = \{\mathbb{P}\}$. Indeed, let $\sigma_S = \{\sigma_k \mid k \in K$ and let $p \in \sigma_j$. Then for every prime $q \neq p$ there exists a Schmidt group A such that $\pi(A) = \{p, q\}$, so $q \in \sigma_j$. Hence, in fact, $\mathbb{P} \subseteq \sigma_j$.

***Lemma* 2.9** [16, Proposition 2.7]. *The class* $\mathfrak{S}_{\sigma_{\mathfrak{I}}}$ *is complete.*

***Theorem* 2.10.** *Let* $\mathfrak{X}$ *be a non-soluble family of groups and let* $S^* = S^*(\mathfrak{X})$. *Let* $\sigma^* = \sigma_{S^*}$ *and* $\mathfrak{I}^* = \mathfrak{I}_*$. *Then the following statements hold.*

(i) *If* $\mathfrak{X}$ *is* $\sigma_{\mathfrak{I}}$*-soluble, then* $\sigma^*_{\mathfrak{I}^*} \leq \sigma_{\mathfrak{I}}$.

(ii) $\mathfrak{X}$ *is* $\sigma^*_{\mathfrak{I}^*}$*-soluble.*

(iii) $\Sigma_{s,\mathfrak{I}}(\mathfrak{X}) = \sigma^*_{\mathfrak{I}^*} = \Sigma_{s,\mathfrak{I}}(S^*)$.

Proof. (i) Let $\sigma = \{\sigma_i \mid i \in I\}$ and

$$\sigma_{S^*} = \{\sigma^*_j \mid j \in J = \{0\} \cup J\}.$$

Assume that $\mathfrak{X}$ is $\sigma_{\mathfrak{I}}$-soluble. We show that $\sigma^*_{\mathfrak{I}^*} \leq \sigma_{\mathfrak{I}}$.

First we show that $\mathfrak{I}_{S^*} \subseteq \mathfrak{I}$. In view of Lemma 2.9, every group $G \in S^*$ is $\sigma_{\mathfrak{I}}$-soluble and so it is $\sigma_{\mathfrak{I}}$-primary. Hence G is a σ_i-group for some i. Assume that $i \neq 0$, then $2 \notin \sigma_i$, so G is soluble by the Feit-Thompson theorem on solubility groups of odd order, a contradiction. Therefore $i = 0$ and so $G \in \mathfrak{I}$. Hence $S^* \subseteq \mathfrak{I}$ which implies that $\mathfrak{I}_{S^*} \subseteq \mathfrak{I}$.

Finally, for every $j \in J \setminus \{0\}\}$ we have $|\sigma^*_j| = 1$, by definition of the partition σ_{S^*}, so for some r it follows that so $\sigma^*_j \subseteq \sigma_r$. Hence $\sigma^*_{\mathfrak{I}^*} \leq \sigma_{\mathfrak{I}}$.

(ii) Now we show that $\mathfrak{X}$ is $\sigma^*_{\mathfrak{I}^*}$-soluble.

Assume that some section H/K of some group $G \in \mathfrak{X}$ is not $\sigma^*_{\mathfrak{I}^*}$-soluble and let A be a minimal non-$\sigma^*_{\mathfrak{I}^*}$-soluble subgroup of H/K. Then A is a non-abelian simple group, so $A \in S^* \subseteq \mathfrak{I}^*$ and hence A is $\sigma^*_{\mathfrak{I}^*}$-soluble, a contradiction. Hence (ii) holds.

(iii) From Proposition 1.7 and Part (i) it follows that

$$\sigma^*_{\mathfrak{I}^*} \leq \Sigma_{s,\mathfrak{I}}(\mathfrak{X}).$$

On the other hand, from Part (ii) it follows that $\Sigma_{s,\mathfrak{I}}(\mathfrak{X}) \leq \sigma^*_{\mathfrak{I}^*}$. Hence $\Sigma_{s,\mathfrak{I}}(\mathfrak{X}) = \sigma^*_{\mathfrak{I}^*}$.

Finally, note that $S^* = S^*(\mathfrak{X}) = S^*(S^*)$, so we have (iii). □

***Theorem* 2.11** [13, Theorem 3.12]. *Let* $\mathfrak{X}$ *be a non-soluble family of groups and let* $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$. *Let* $\sigma^* = \sigma_{S,S^*}$ *and* $\mathfrak{I}^* = \mathfrak{I}_{S^*}$. *Then the following statements hold.*

(i) *If* $\mathfrak{X}$ *is* $\sigma_{\mathfrak{I}}$*-nilpotent, then* $\sigma^*_{\mathfrak{I}^*} \leq \sigma_{\mathfrak{I}}$.

(ii) $\mathfrak{X}$ *is* $\sigma^*_{\mathfrak{I}^*}$*-nilpotent.*

(iii) $\Sigma_{n,\mathfrak{I}}(\mathfrak{X}) = \sigma^*_{\mathfrak{I}^*} = \Sigma_{n,\mathfrak{I}}(S \cup S^*)$.

***Corollary* 2.12.** *Let* $\mathfrak{X}$ *be a soluble non-nilpotent family of groups and let* $S = S(\mathfrak{X})$. *Let* $\sigma^* = \sigma_S$. *Then the following statements hold*:

(i) *if* $\mathfrak{X}$ *is* σ*-nilpotent, then* $\sigma^* \leq \sigma$.

(ii) $\mathfrak{X}$ *is* σ^**-nilpotent.*

(iii) $\Sigma_n(\mathfrak{X}) = \sigma^* = \Sigma_n(S)$.

3 The (G, A)-subnormal $\sigma_{\mathfrak{I}}$-pair

***Definition* 3.1.** (i) Let A be a subgroup of G and let $\{\sigma^j_{\mathfrak{I}_j} \mid j \in J\}$ be the set of all of $\sigma_{\mathfrak{I}}$-pairs, where $\sigma^j = \{\sigma^j_{k_j} \mid k_j \in K_j\}$ for all $j \in J$, such that A is $\sigma^j_{\mathfrak{I}_j}$-subnormal in G for all $j \in J$. Then we write $\Sigma_{sn,\mathfrak{I}}(G, A)$ to denote the $\sigma_{\mathfrak{I}}$-pair $\bigcap_{j \in J} \sigma^j_{\mathfrak{I}_j}$.

(ii) Let A be a subgroup of G and let $\{\sigma^j \mid j \in J\}$ be the of partitions of $\mathbb{P}$, where $\sigma^j = \{\sigma^j_{k_j} \mid k_j \in K_j\}$ for all $j \in J$, such that A is σ^j-subnormal in G for all $j \in J$. Then we write $\Sigma_{sn}(G, A)$ to denote the partition $\bigcap_{j \in J} \sigma^j$ of $\mathbb{P}$.

By Proposition 1.7 (iv), $\Sigma_{sn,\mathfrak{I}}(G, A)$ is the smallest element in $\{\sigma^j_{\mathfrak{I}_j} \mid j \in J\}$, that is, $\Sigma_{sn,\mathfrak{I}}(G, A) \in \{\sigma^j_{\mathfrak{I}_j} \mid j \in J\}$ and $\Sigma_{sn,\mathfrak{I}}(G, A) \leq \sigma^j_{\mathfrak{I}_j}$ for all $j \in J$.

We say that $\Sigma_{sn,\mathfrak{I}}(G, A)$ is the (G, A)*-subnormal* $\sigma_{\mathfrak{I}}$*-pair.*

By Corollary 1.8 (iv), $\Sigma_{sn}(G, A)$ is the smallest element in $\{\sigma^j \mid j \in J\}$, that is,

$$\Sigma_{sn}(G, A) \in \{\sigma^j \mid j \in J\}$$

and $\Sigma_{sn}(G, A) \leq \sigma^j$ for all $j \in J$.

We say that $\Sigma_{sn}(G, A)$ is the (G, A)*-subnormal partition of* $\mathbb{P}$.

Now, returning to Problem 0.9 (i) we see that a "minimal" $\sigma_{\mathfrak{I}}$-pair for which A is $\sigma_{\mathfrak{I}}$-subnormal in G exists and $\sigma_{\mathfrak{I}}$ coincides with $\Sigma_{sn,\mathfrak{I}}(G, A)$.

Thus, Proposition 1.7 (iv) gives an answer to the first part of Problem 0.9.

Now we shall give a partial answer to the second part of Problem 0.9.

We say that $A = A_0 < A_1 < \cdots < A_{t-1} < A_t = G$ is a maximal $\sigma_{\mathfrak{I}}$-subnormal chain of G provided A_i is a maximal $\sigma_{\mathfrak{I}}$-subnormal subgroup of A_{i+1} for all $i = 0, 1, \ldots, t-1$.

***Lemma* 3.2.** *A proper subgroup A of G is a maximal* $\sigma_{\mathfrak{I}}$*-subnormal subgroup of G if and only if either* $G / A_G = G / A$ *is a non-*$\sigma_{\mathfrak{I}}$*-primary simple group or A is a maximal subgroup of G and* G / A_G *is a* $\sigma_{\mathfrak{I}}$*-primary group.*

Proof. First suppose that A is a maximal $\sigma_{\mathfrak{I}}$-subnormal subgroup of G. Assume that A is normal in G and $G / A = G / A_G$ is a simple group. If G / A is a group of prime order, then A is a maximal subgroup of G and $G / A_G = G / A$ is a $\sigma_{\mathfrak{I}}$-primary group. Now assume that $G / A = G / A_G$ is a non-abelian simple group and let $A \leq M$, where m is a maximal subgroup of G. If $G / A_G = G / A$ is a $\sigma_{\mathfrak{I}}$-primary group, then G / M_G and is a $\sigma_{\mathfrak{I}}$-primary group and so M $\sigma_{\mathfrak{I}}$-subnormal in G. The maximality of A implies that $A = M$.

Finally, if either $G / A_G = G / A$ is a non-σ-primary simple group or A is a maximal non-normal subgroup of G and G / A_G is a σ-primary group, then A is a maximal σ-subnormal subgroup of G. □

***Theorem* 3.3.** *Let A be a* $\sigma_{\mathfrak{I}}$*-subnormal subgroup of G, where* $\sigma_{\mathfrak{I}} = \Sigma_{sn,\mathfrak{I}}(G, A)$. *Let*

$$A = A_0 < A_1 < \cdots < A_{t-1} < A_t = G$$

be a maximal $\sigma_{\mathfrak{I}}$*-subnormal chain of G with as minimal as possible set of i such that* $A_{i-1} \ntrianglelefteq A_i$, *and let*

$$X = \{A_i / (A_{i-1})_{A_i} \mid A_{i-1} \neq (A_{i-1})_{A_i}\}.$$

Then $\sigma_{\mathfrak{I}} = \Sigma_{n,\mathfrak{I}}(X)$.

Proof. Let $\sigma^*_{\mathfrak{I}^*} = \Sigma_{n,\mathfrak{I}}(\mathfrak{X})$.

If $\mathfrak{X} = \varnothing$, then A is subnormal in G and so $\sigma = \sigma^1$. On the other hand, $\sigma^* = \Sigma_{n,\mathfrak{I}}(\varnothing) = \sigma^1$.

Now assume that $\mathfrak{X} \neq \varnothing$. First we show that the subgroup A is $\sigma^*_{\mathfrak{I}^*}$-subnormal in G. Indeed, in a maximal $\sigma_{\mathfrak{I}}$-subnormal chain

$$A = A_0 < A_1 < \cdots < A_{t-1} < A_t = G$$

of G for every i such that $A_k \neq (A_{k-1})_{A_k}$ the factor $A_k / (A_{k-1})_{A_k}$ belongs to $\mathfrak{X}$, so this factor is $\sigma^*_{\mathfrak{I}}$-nilpotent and hence A is $\sigma^*_{\mathfrak{I}^*}$-subnormal. It follows that $\sigma_{\mathfrak{I}} \leq \sigma^*_{\mathfrak{I}^*}$.

In view of Lemma 3.2, for a maximal $\sigma_{\mathfrak{I}}$-subnormal chain $A = A_0 < A_1 < \cdots < A_{t-1} < A_t = G$ of G every factor $A_{i-1} / (A_{i-1})_{A_i}$, where i is such that $A_{i-1} \neq (A_{i-1})_{A_i}$, is $\sigma_{\mathfrak{I}}$-primary and so $\sigma_{\mathfrak{I}}$-nilpotent. Therefore $\mathfrak{X}$ is $\sigma_{\mathfrak{I}}$-nilpotent, so $\sigma^*_{\mathfrak{I}^*} \leq \sigma_{\mathfrak{I}}$ by the definition of

$$\sigma^*_{\mathfrak{I}^*} = \Sigma_{n,\mathfrak{I}}(\mathfrak{X}).$$

Therefore $\sigma_{\mathfrak{I}} = \sigma^*_{\mathfrak{I}^*}$. □

The following theorem is proved similarly.

***Theorem* 3.4.** *Let A be a* σ*-subnormal subgroup of G, where* $\sigma = \Sigma_{sn}(G, A)$. *Let*

$$A = A_0 < A_1 < \cdots < A_{t-1} < A_t = G$$

be a maximal σ*-subnormal chain of G with as minimal as possible set of i such that* $A_{i-1} \ntrianglelefteq A_i$, *and let*

$$X = \{A_i / (A_{i-1})_{A_i} \mid A_{i-1} \neq (A_{i-1})_{A_i}\}.$$

Then $\sigma = \Sigma_n(X)$.

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