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On the absence, non-uniqueness, and blow-up of classical solutions of mixed problems for some semilinear hyperbolic equations

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For semilinear hyperbolic equations of the form $\prod_{i=1}^{n}\left(\frac{\partial}{\partial t}-a_i\frac{\partial}{\partial x}+b_i\right)u(t,x)=f(t,x,u(t,x))$ given in the first quadrant and/or in a half-strip, we consider mixed problems with local boundary conditions, for which we study the issues related to the absence, non-uniqueness, and blow-up of classical solutions.
Keywords: absence of a solution, non-uniqueness of a solution, blow-up of a solution, mixed problem, Cauchy problem, matching conditions.

Для полулинейных гиперболических уравнений вида $\prod_{i=1}^{n}\left(\frac{\partial}{\partial t}-a_i\frac{\partial}{\partial x}+b_i\right)u(t,x)=f(t,x,u(t,x))$ заданных в первом квадранте и/или в полуполосе, мы рассматриваем смешанные задачи с локальными граничными условиями, для которых изучаем вопросы, связанные с отсутствием, неединственностью и разрушением классических решений.
Ключевые слова: отсутствие решения, неединственность решения, разрушение решения, смешанная задача, проблема Коши, условия соответствия.

1. Introduction. The present paper is a continuation of our work [1]. Here we will study mixed problems for the following nonlinear equations

$$\prod_{i=1}^{n}\left(\frac{\partial}{\partial t}-a_i\frac{\partial}{\partial x}+b_i\right)u(t,x)=f(t,x,u(t,x)),\quad (t,x)\in Q, \tag{1}$$

where $n\in\mathbb{N}$, a_i and b_i $(i=1,2,\ldots,n)$ are real numbers, Q is a set which will be specified later, and f is a function defined on the set $\overline{Q}\times\mathbb{R}$. We assume that $a_i\neq 0$ $(i=1,2,\ldots,n)$ in order for the curve $t=0$ to be noncharacteristic for Eq. (1). Without loss of generality, we also suppose $a_i\leq a_j$ if $i<j$. Let us denote $\Gamma=\overline{Q}\setminus\{(0,x)\mid x\in\mathbb{R}\}$, $p^{(+)}=\operatorname{card}\big(\{i\mid i\in\mathbb{N}\cap[1,n]\wedge a_i>0\}\big)$, and $p^{(-)}=\operatorname{card}\big(\{i\mid i\in\mathbb{N}\cap[1,n]\wedge a_i<0\}\big)$. Let $l\in\mathbb{R}\cup\{-\infty\}$ and $r\in\mathbb{R}\cup\{+\infty\}$, $l<r$. We denote

$$n^{(-)}=\begin{cases}p^{(-)}, & l=-\infty,\\ 0, & l\neq-\infty,\end{cases}\qquad n^{(+)}=\begin{cases}p^{(+)}, & r=+\infty,\\ 0, & r\neq+\infty.\end{cases}$$

The meaning of the quantities $p^{(-)}$ and $p^{(+)}$ ($n^{(-)}$ and $n^{(+)}$) is how many conditions can (should) be set on the boundary $x=l$ and $x=r$, respectively, so that a mixed problem for Eq. (1) is well-posed.

Equation (1) is equipped with the initial conditions

$$\frac{\partial^i u}{\partial t^i}(0,x)=\varphi_i(x),\quad x\in\Gamma,\quad i=0,1,\ldots,n-1. \tag{2}$$

and the boundary conditions

$$\mathcal{B}_i^{(-)}[u](t,l)=\mu_i^{(-)}(t),\quad t\in[0,\infty),\quad i=1,\ldots,n^{(-)}, \tag{3}$$

and

$$\mathcal{B}_i^{(+)}[u](t,r)=\mu_i^{(+)}(t),\quad t\in[0,\infty),\quad i=1,\ldots,n^{(+)}, \tag{4}$$

where $\mathcal{B}_i^{(\pm)}$ $(i=0,1,\ldots,n^{(\pm)})$ are differential operators, which have the form $\mathcal{B}_i^{(\pm)}=\mathcal{P}_i^{(\pm)}(\partial_t,\partial_x)$, where $\mathcal{P}_i^{(\pm)}$ are some polynomials such that $\mathcal{P}_i^{(\pm)}\not\equiv 0$. If l is infinite, then the conditions (3) are not specified. Again, if r is infinite, then the conditions (4) are not given[1]. The quantity l is always infinite if $n^{(-)}=0$, and the quantity r is always infinite if $n^{(+)}=0$. Now we can specify that the set Q can be expressed in the following formula

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[1] This differs from widely used understanding in the literature of the conditions (3) and/or (4) as conditions at the infinity when l and/or r is infinite, i. e., $u(t,\pm\infty)=\lim\limits_{x\to\pm\infty}u(t,x)$.

$$Q = (0,\infty)\times(l,r), \tag{5}$$

which is obviously true if $l = -\infty$ and/or $r = +\infty$. Note that the case $l = -\infty \wedge r = +\infty$ leads to the absence of boundary conditions, and, therefore, the problem (1)–(4) degenerates into the Cauchy problem (1)–(2)[2], where $Q = (0,\infty)\times\mathbb{R}$. We have considered the particular cases of the Cauchy problem (1)–(2): 1) The numbers a_i are different, $b_i = 0$ $(1,2,\ldots,n)$ [2]. 2) The numbers a_i are equal to a, and the numbers b_i are equal to b [3].

2. Nonexistence of solutions. In this section, we consider the issues related to the nonexistence of the mixed problem (1)–(4). The most obvious case when this happens is when the matching conditions are not met.

Let us consider the following case:

$$\mathcal{B}_i^{(\pm)} = \frac{\partial^{j^{(\pm)}(i)}}{\partial x^{j^{(\pm)}(i)}}, \tag{6}$$

where $j^{(\pm)} : \mathbb{N}\cap[1,n^{(\pm)}] \mapsto S^{(\pm)}$ is a bijective mapping, where $S^{(\pm)} \subset \mathbb{N}$. Without loss of generality, we assume that $j^{(\pm)}(i_1) < j^{(\pm)}(i_2)$ if $i_1 < i_2$. Let

$$\tilde{n} = \max(j^{(-)}(n^{(-)}), j^{(+)}(n^{(+)}), n).$$

According to the definition of a classic solution, it should be found in the class $C^{\tilde{n}}(\overline{Q})$.

Let us differentiate the initial conditions (2) with respect to the variable x and obtain

$$\frac{\partial^{i+j}u}{\partial x^j \partial t^i}(0,x) = D^j\varphi_i(x), \quad x\in\Gamma, \quad i = 0,1,\ldots,n-1, \quad j = 1,2,\ldots,\tilde{n}-i. \tag{7}$$

In a similar way, differentiating the boundary conditions (2) and (4) with respect to the variable t, we have

$$\frac{\partial^{k+j^{(-)}(i)}u}{\partial t^k \partial x^{j^{(-)}(i)}}(t,l) = D^k\mu_i^{(-)}(t), \quad t\in[0,\infty), \quad i = 1,2,\ldots,n^{(-)}, \quad k = 1,2,\ldots,\tilde{n} - j^{(-)}(i) \tag{8}$$

and

$$\frac{\partial^{k+j^{(+)}(i)}u}{\partial t^k \partial x^{j^{(+)}(i)}}(t,r) = D^k\mu_i^{(+)}(t), \quad t\in[0,\infty), \quad i = 1,2,\ldots,n^{(+)}, \quad k = 1,2,\ldots,\tilde{n} - j^{(+)}(i) \tag{9}$$

Since $u \in C^{\tilde{n}}(\overline{Q})$, the formulas (7)–(9) imply the following set of equations

$$D^j\varphi_i(l) = D^k\mu_s^{(-)}(0), \quad j+i = k+j^{(-)}(s) \le \tilde{n}, \tag{10}$$

$$D^j\varphi_i(r) = D^k\mu_s^{(+)}(0), \quad j+i = k+j^{(-)}(s) \le \tilde{n}. \tag{11}$$

In the case of infinite l and/or r, the corresponding condition (10) and/or (11) is not required. Therefore, it is better to write them as follows

$$l \text{ is finite and } D^j\varphi_i(l) = D^k\mu_s^{(-)}(0), \quad j+i = k+j^{(-)}(s) \le \tilde{n}, \tag{12}$$

$$r \text{ is finite and } D^j\varphi_i(r) = D^k\mu_s^{(+)}(0), \quad j+i = k+j^{(-)}(s) \le \tilde{n}. \tag{13}$$

It allows us to formulate the following statement.

Theorem 1. *If the matching conditions (12) and (13) fail for given functions* φ_i $(i = 0,1,\ldots,n-1)$ *and* $\mu_j^{(\pm)}$ $(j = 1,2,\ldots,n^{(\pm)})$, *then, for any smoothness of these functions, the mixed problem (1)–(4) and (6) does not have a classical solution defined on the set* $\overline{Q}$.

Note that

$$\frac{\partial^n u}{\partial t^n}(t,l) = D^n\mu_1^{(-)}(t), \quad t\in[0,\infty), \tag{14}$$

if

$$j^{(-)}(1) = 0 \tag{15}$$

and l is finite. On the other side, from Eq. (1) we can derive

$$\frac{\partial^n u}{\partial t^n}(t,x) = \mathcal{P}(\partial_t,\partial_x)[u](t,x) + f(t,x,u(t,x)), \quad (t,x)\in\overline{Q}, \tag{16}$$

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[2] In the Cauchy problem (1)–(2), the values l and r can be finite. In this case $Q = \mathrm{Conv}\{(0,l),(0,r),((r-l)/(a_n-a_1),(la_n-ra_1)/(a_n-a_1))\}$ if $a_1 \ne a_n$, or $Q = \{(t,x) \mid x+a_1t \in (l,r)\}$ if $a_1 = a_n$.

where $\mathcal{P}$ is some polynomial. To find the form of the polynomial $\mathcal{P}$, we need the following lemma.

Lemma 1. *The following equality is true:*

$$\prod_{i=1}^{n}\left(T - a_i X + b_i\right) = \sum_{\substack{i,j=0,1,\ldots,n \\ i+j\le n}} p_{i,j}^{(n)} T^i X^j, \tag{17}$$

where

$$\begin{aligned} p_{1,0}^{(1)} &= 1, \quad p_{0,1}^{(1)} = -a_1, \quad p_{0,0}^{(1)} = b_1, \\ p_{i,j}^{(m)} &= \mathcal{J}(i+j\le m-1, b_m p_{i,j}^{(m-1)}) + \mathcal{J}(i+j\le m \wedge i-1\ge 0 \wedge j\ge 0, b_m p_{i-1,j}^{(m-1)}) + \\ &+ \mathcal{J}(i+j\le m \wedge i\ge 0 \wedge j-1\ge 0, -a_m p_{i,j-1}^{(m-1)}), \quad 2\le m\le n \end{aligned} \tag{18}$$

where $\mathcal{J}(condition, j_2)$ gives j_2 if *condition* evaluates to true, and 0 if it evaluates to false.

Proof. For $n=1$ the formula (17) with (18) is obviously correct. For $m>1$ the transition formula (18) follows from the expression

$$\left(\sum_{\substack{i,j=0,1,\ldots,m-1 \\ i+j\le m-1}} p_{i,j}^{(m-1)} T^i X^j\right)(T - a_m X + b_m) =$$

$$= \sum_{\substack{i,j=0,1,\ldots,m-1 \\ i+j\le m-1}} p_{i,j}^{(m-1)} T^{i+1} X^j - \sum_{\substack{i,j=0,1,\ldots,m-1 \\ i+j\le m-1}} a_m p_{i,j}^{(m-1)} T^i X^{j+1} + \sum_{\substack{i,j=0,1,\ldots,m-1 \\ i+j\le m-1}} b_m p_{i,j}^{(m-1)} T^i X^j.$$

The lemma is proved.

Therefore, the polynomial $\mathcal{P}$ has the form

$$\mathcal{P}(\partial_t, \partial_x) = -\sum_{\substack{i=0,1,\ldots,n-1 \\ j=0,1,\ldots,n \\ i+j\le n}} p_{i,j}^{(n)} \partial_t^i \partial_x^j, \tag{19}$$

where the coefficients $p_{i,j}^{(n)}$ are given by the formulas (18).

Note that the relations (7) allow us to calculate the quantities $\partial_t^i \partial_x^j u(0,l)$. From the expressions (7), (14), and (16) we obtain

$$D^n \mu_1^{(-)}(0) = f(0,l,\varphi(l)) - \sum_{\substack{i,j=0,1,\ldots,n-1 \\ j=0,1,\ldots,n \\ i+j\le n}} p_{i,j}^{(n)} D^j \varphi_i(l). \tag{20}$$

It allows us to formulate the following statement.

Theorem 2. *Let l be finite and $j^{(-)}(1) = 0$. If the matching condition (20) fails for given functions φ_i ($i = 0,1,\ldots,n-1$) and $\mu_1^{(-)}$, then, for any smoothness of these functions, the mixed problem (1)–(4) and (6) does not have a classical solution defined on the set $\overline{Q}$.*

The following theorem can be proved in a similar way.

Theorem 3. *Let r be finite and $j^{(+)}(1) = 0$. If the matching condition*

$$D^n \mu_1^{(+)}(0) = f(0,r,\varphi(r)) - \sum_{\substack{i,j=0,1,\ldots,n-1 \\ j=0,1,\ldots,n \\ i+j\le n}} p_{i,j}^{(n)} D^j \varphi_i(r) \tag{21}$$

fails for given functions φ_i ($i = 0,1,\ldots,n-1$) and $\mu_1^{(+)}$, then, for any smoothness of these functions, the mixed problem (1)–(4) and (6) does not have a classical solution defined on the set $\overline{Q}$.

3. Nonuniqueness of solutions. In this section, we consider the problem (1)–(4) in the following case

$$\begin{gathered} f(t,x,u) := u^\alpha, \quad 0<\alpha<1, \quad \mu_i^{(\pm)} = 0, \quad i = 1,2,\ldots,n^{(\pm)}, \quad \varphi_j = 0, \quad j = 0,1,\ldots,n-1, \\ \mathcal{B}_i^{(-)}[(t,x)\mapsto g(t)](t,l) = 0, \quad i = 1,2,\ldots,n^{(-)}, \quad \mathcal{B}_j^{(+)}[(t,x)\mapsto g(t)](t,r) = 0, \quad j = 1,2,\ldots,n^{(+)}, \\ b_i = 0, \quad i = 1,2,\ldots,n. \end{gathered} \tag{22}$$

Obviously, the problem (1)–(4) has the trivial solution $u \equiv 0$. To find non-trivial solutions, consider the following ansatz

$$u(t,x) = \beta t^\gamma, \quad (t,x)\in\overline{Q}, \tag{23}$$

where β and γ are some real numbers. Note that the function u of the form (23) will satisfy the conditions (2)–(4) with (22). We substitute the ansatz (23) into Eq. (1) with (22) and obtain

$$\text{FactorialPower}(\gamma, n)\beta t^{\gamma-n} = \beta^\alpha t^{\gamma\alpha}, \tag{24}$$

where we have denoted

$$\mathrm{FactorialPower}(\gamma,n)=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-n+1)},$$

where Γ is the gamma function. The equality (24) leads to the system of equations

$$\gamma-n=\gamma\alpha,\quad \mathrm{FactorialPower}(\gamma,n)\beta=\beta^{\alpha},$$

which has a solution

$$\gamma=\frac{n}{1-\alpha},\quad \beta=\mathrm{FactorialPower}\left(\frac{n}{1-\alpha},n\right)^{\frac{1}{\alpha-1}}$$

Consequently, one nontrivial solution to the problem (1)–(4) and (22) has the form

$$u_p(t,x)=\mathrm{FactorialPower}\left(\frac{n}{1-\alpha},n\right)^{\frac{1}{\alpha-1}}t^{\frac{n}{1-\alpha}},\quad (t,x)\in\overline{Q}. \tag{25}$$

Furthermore, we can easily show that the 'glued' solution [4, p. 14–15]

$$u_{p;s}(t,x)=\begin{cases}0, & t\in[0,s),\\ u_p(t-s,x), & t\in[s,\infty),\end{cases}$$

with parameter $s>0$ also satisfies the problem (1)–(4) and (22). Thus, we have built an infinite set of nontrivial classical solutions to the problem (1)–(4) and (22).

It allows us to formulate the following statement.

Theorem 4. *The problem (1)–(4) and (22) has an infinite number of global classical solutions defined on* $\overline{Q}$ *and no unique local classical solution.*

4. Blow-up of solutions. In this section, we consider the problem (1)–(4) in the case of non-negative nonlinearities, namely:

$$\begin{gathered} f(t,x,u):=g(u),\quad g\in C([0,\infty)),\quad g([0,\infty))\subseteq[0,\infty),\\ \mu_i^{(\pm)}=0,\quad i=1,2,\ldots,n^{(\pm)},\quad \varphi_0=\mathrm{const},\quad \varphi_j=0,\quad j=1,\ldots,n-1,\\ \mathcal{B}_i^{(-)}[(t,x)\mapsto g(t)](t,l)=0,\quad i=1,2,\ldots,n^{(-)},\\ \mathcal{B}_j^{(+)}[(t,x)\mapsto g(t)](t,r)=0,\quad j=1,2,\ldots,n^{(+)},\\ b_i=0,\quad i=1,2,\ldots,n. \end{gathered} \tag{26}$$

If $g(0)=0$ and $\varphi_0=0$, then the problem (1)–(4) and (26) has a trivial solution $u\equiv 0$, which, obliviously, does not have the blow-up. So, we assume $\varphi_0=0$ and seek nontrivial solution to the problem(1)–(4) and (26), which blows up in a finite time. To do this, we consider an ansatz of the form

$$u(t,x)=u(t),\quad (t,x)\in\overline{Q}, \tag{27}$$

which leads us to the following ordinary differential equation

$$D^n u(t)=g(u(t)), \tag{28}$$

with the initial conditions

$$D^i u(0)=0,\quad i=0,1,\ldots,n-1. \tag{29}$$

Let us check [5] the necessary condition for the existence of nontrivial solutions of Eq. (28) with the conditions (29). It lies in the fact that the integral

$$\int_0^{\delta} g(x)x^{\frac{2-n}{n-1}}dx$$

must converge for some delta $\delta>0$. Since $g\in C([0,\infty))$, we have $g(x)\le M$ for all $x\in[0,1]$, where M is some positive number. Thus,

$$\int_0^1 g(x)x^{\frac{2-n}{n-1}}dx\le M\int_0^1 x^{\frac{2-n}{n-1}}dx=M(n-1),\quad n\ge 2.$$

Consequently, the Cauchy problem (28), (29) has non-trivial solutions.

It is known [5] that the problem (28), (29) has a positive solution u, which blows up in a finite time if $n\ge 2$ and the integral

$$\int_0^{\infty}(h(x))^{\frac{1}{1-n}}dx$$

converges, where

$$h(x) = x^{\frac{n-2}{n}} \left(\int_0^x (x-z)^{n-2} g(z) z^{\frac{n-2}{1-n}} dz \right)^{\frac{n-1}{n}}.$$

Since the function u of the form (27) satisfies the conditions (2)–(4) with (26) and $\varphi_0 = 0$, we can formulate the following theorem.

Theorem 5. *Let us assume* $n \geq 2$ *and* $\varphi_0 = 0$. *The mixed*[3] *problem (1)–(4) and (26) has a nontrivial positive solution, i. e.,* $u(t,x) > 0$ *for all* $(t,x) \in Q \setminus \Gamma$, *which blows up in a finite time, i. e., there exists* $T_* > 0$ *such that* $\lim\limits_{t \to T_* - 0} u(t,x) = \infty$, *if the integral*

$$\int_0^\infty x^{\frac{2-n}{n^2-n}} \left(\int_0^x (x-z)^{n-2} g(z) z^{\frac{n-2}{1-n}} dz \right)^{-\frac{1}{n}} dx$$

converges.

If $n = 1$, then we can use the Osgood test[4] for the blow-up in a finite time of a nontrivial solution of the Cauchy problem (28), (29) from the paper [6]. It has the form

$$g \in C^{0,1}_{\mathrm{loc}}([0,\infty)), \quad \mathrm{Image}(g) \subseteq [0,\infty), \quad g \text{ is nondecreasing,} \tag{30}$$

$$T_* = \int_{\varphi_0}^{\infty} \frac{dz}{g(z)} < \infty. \tag{31}$$

It allows us to formulate the following theorem.

Theorem 6. *Let us assume* $n = 1$. *The mixed problem (1)–(4) and (26) has a nontrivial solution, which blows up in a finite time if the conditions (30) and (31) are satisfied. The blow-up time* T_* *is determined by the formula (31).*

5. Conclusions. In this article, we have derived the sufficient conditions for the absence, non-uniqueness, and blow-up of classical solutions of mixed and initial value problems for some semilinear hyperbolic equations.

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[3] In the case of the Cauchy problem, we must additionally require that at least one of quantities r and l be infinite or the value $r - l$ be big enough.

[4] https://www.ub.edu/probabilitats-seminaribcn/Curs2011/leon.pdf.